R's and C's optimal mixed strategies

Math 10120, Spring 2013

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Review: finding R's optimal mixed strategy

R and C play game w. payoff matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, **all payoffs positive**

Here's what R does to find his optimal mixed (random) strategy $[r_1 \; r_2]$:

R finds the **minimum** value of

$$y_1 + y_2$$

subject to the constraints (one for each **column** of the payoff matrix)

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 &\geq 1 \\ a_{12}y_1 + a_{22}y_2 &\geq 1 \\ y_1 &\geq 0 \\ y_2 &\geq 0 \end{aligned}$$

R then sets $v = 1/(y_1 + y_2)$, $r_1 = vy_1$ and $r_2 = vy_2$

R's worst-case expected payoff in this case is v (given that C plays best possible counter strategy); no other mixed strategy for R gives a better worst-case expected payoff than v

Review: finding C's optimal mixed strategy

The same game, w. payoff matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, **all payoffs positive**

Here's what C does to find his optimal mixed (random) strategy $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$:

C finds the **maximum** value of

$$x_1 + x_2$$

subject to the constraints (one for each **row** of the payoff matrix)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq 1 \\ a_{21}x_1 + a_{22}x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

C then sets $v = 1/(x_1 + x_2)$, $c_1 = vx_1$ and $c_2 = vx_2$

C's worst-case expected payout in this case is v (given that R plays best possible counter strategy); no other mixed strategy for C gives a better worst-case expected payout than v

The fundamental fact about optimal mixed strategies

The v that R finds, and the v that C finds, are always **the same** v

R has a mixed strategy that on average gives him a payoff of v, no matter how C responds; any other strategy has the potential to lead to a lower payoff, if C chooses his counterstrategy carefully; so if R moves away to another mixed strategy, C may notice this after a while and take advantage to reduce R's payoff

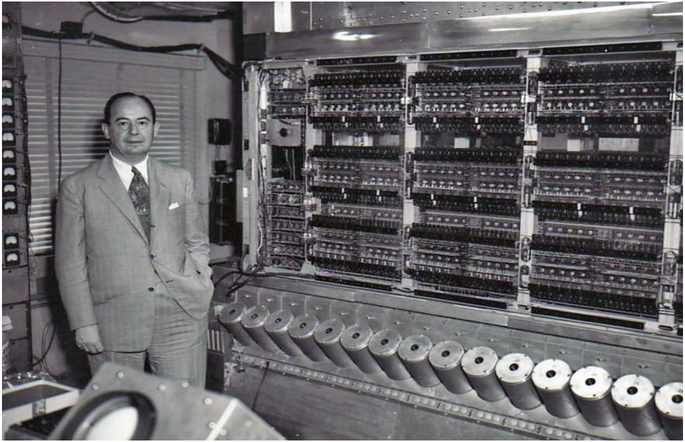
C has a mixed strategy that on average makes him pay out v, no matter how R responds; any other strategy has the potential to lead to a higher payout, if R chooses his counterstrategy carefully; so if C moves away to another mixed strategy, R may notice this after a while and take advantage to increase C's payout

Conclusion: It's in R's best interest to play his optimal mixed strategy, and in C's best interest to play his optimal mixed strategy; the game is **stable** under these strategies

John Von Neumann, 1903–1957

Showed (in 1928) that every two-player, zero-sum game is stable

John Von Neumann, 1903–1957



Von Neumann's 1951 computer (5k memory, 1 million times less than my laptop)
(Von Neumann also designed the first computer virus in 1952...)

What to do if some entries are non-positive

The linear programming problems that R and C set up depend on all entries of the payoff matrix being **positive**

Problem: How do we deal with negative entries?

Solution: Add a number N to each entry (the same number for each entry), to make them all positive (think of it as: the referee of the game picks C's pocket before the game begins, stealing N dollars, and gives the N dollars to R; so R's payoff is increased by N, no matter how the game is played, and C's pay out is increased by N)

This action doesn't change R and C's thinking about strategies

Once the value of the new, positive, game has been found, subtract N to get the value of the original game (think of it as: the referee fesses up after the game, and makes R give the N dollars back to C)

A typical feasible set in three dimensions