Math 10120, Spring 2013

Exam I solutions

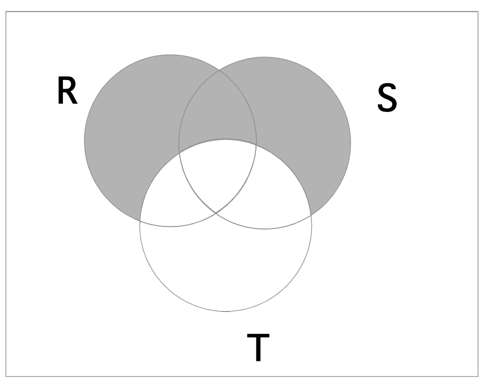
February 7 2013

1. Let $U = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 8, 12, 16, 20\}$ and $C = \{8, 10, 12, 14, 16\}$. Which of these sets is equal to $(A \cup B')' \cap C$?

   Solution: By DeMorgan's law, $(A \cup B')' = A' \cap (B')' = A' \cap B$, so $(A \cup B')' \cap C = A' \cap B \cap C$. So we are looking for the elements in B and C but not in A. These are 12 and 16. So

   $$(A \cup B')' \cap C = \{12, 16\}.$$

2. Which region of the Venn diagram below is shaded?



   Solution: The region includes everything in either R or S, except that part in T, so the region is described by

   $$(R \cup S) \cap T'.$$

3. R and S are subsets of a certain universal set U. If

$$n(R) = 20, \quad n(S) = 18, \quad n((S \cup R)') = 5 \quad \text{and} \quad n(U) = 35,$$

how many elements does $S \cap R$ have?

Solution: We know $n(R \cup S) = n(R) + n(S) - n(R \cap S)$. Since $n((S \cup R)') = 5$ and $n(U) = 35$, we have $n(R \cup S) = 30$. So $30 = 20 + 18 - n(R \cap S)$, and

$$n(R \cap S) = 20 + 18 - 30 = 8.$$

4. Out of 56 Notre Dame First Years who responded to a survey, 25 were registered in a language class, 15 in a science class, and 20 in a philosophy class. 10 were registered in both a language class and a science class, 5 in both a science and a philosophy class, and 7 in a language and a philosophy class. Three people were registered in all three. How many of the respondents were enrolled in **exactly one** of these types of classes?

Solution: Since 3 are registered in all three, and 7 are in a language and a philosophy class, we have that 4 are in language and philosophy but not science, and since 5 are in both a science and a philosophy class, we have that 2 are in science and philosophy but not language. This accounts for 9 students in philosophy: 4 in language and philosophy but not science, 2 are in science and philosophy but not language, and 3 in science, philosophy and language. That means there must be 11 in philosophy but not science and not language. (This is easily seen by filling in the regions of a three-set Venn diagram.)

By the same thinking, there must be 3 in science but not philosophy and not language, and 11 in language but not philosophy and not science.

So there are $11 + 3 + 11$ in exactly one type of class, or

$$25.$$

5. A club consisting of ten men and twelve women decide to make a brochure to attract new members. On the cover of the brochure, they want to have a picture of two men and two women from the club. How many pictures are possible (taking into account the order in which the

four people line up for the picture)? [Note: there are at least two ways to do this, so if you don't see your answer in terms of C's and P's, compute the numerical value to see which option it matches.]

Solution: First choose two women for the photo from among the 12 women, then choose 2 men from among the 10 men, then decide on the order in which the 4 chosen stand. This gives a total of

$$C(10, 2) \cdot C(12, 2) \cdot 4! = 71, 280.$$

6. How many four-letter words (including nonsense words) can be made from the letters of the word

$$EXAMINATION$$

if the letters that you use in the word cannot be repeated?

Solution: There are 8 distinct letters, E, X, A, M, I, N, T, O. There are $P(8, 4)$ ways of picking out 4 of these letters, and arranging them in order. This leads to a count of

$$8 \cdot 7 \cdot 6 \cdot 5$$

7. To order a pizza, you have to first choose a style (from among classic, thin crust or deep dish), a sauce (from among red, white and barbecue) and then choose toppings (from among mushroom, pepperoni, sausage, green pepper, artichoke and seaweed). If you are required to choose **at least one** topping, how many different pizzas can you create?

Solution: 3 choices for a style and 3 for a sauce, so 9 choices before deciding on toppings. Without the restriction that at least one topping must be chosen, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ choices for toppings (for each of the 6 toppings, either choose it or not). 1 of these 64 is invalid (refusing all 6 toppings), so there are 63 valid choices for toppings given the restriction. So the total number of choices is 9×63 or

$$567$$

8. My bicycle lock uses a four-digit combination, each digit being between 0 and 9. At the moment I cannot remember the actual number, but I do remember that it either starts with a 9, or ends with 65 (in that

order), or perhaps both. How many such four-digit numbers are there? (Remember that a number may start with a zero).

Solution: Let A be the set of combinations that begin with a 9; $n(A) = 1000$ (ten choices for each of three remaining digits). Let B be the set of combinations that end with a 65; $n(B) = 100$ (ten choices for each of two initial digits). We want $n(A \cup B)$, which we know is $n(A) + n(B) - n(A \cap B)$. Since $A \cap B$ is all combinations that start with 9 and end with 65, $n(A \cap B) = 10$ (one spot to make a choice). So $n(A \cup B) = 1000 + 100 - 10$, or

$$1090$$

9. A partially eaten bag of M&M's contains 15 candies. 7 of them are red, 6 are white and 2 are blue. A sample of 4 M&M's is to be selected. How many such samples contain more blue M&M's than white?

   Solution: One possibility is that the sample has 2 blues and no whites; there is one way to choose the 2 blues, 1 way to choose the no whites, and $\binom{7}{2} = 21$ ways to choose the remaining 2, which must be red. So 21 possibilities in all here.

   Another possibility is that the sample has 2 blues and 1 white; there is one way to choose the 2 blues, 6 ways to choose the 1 white, and 7 ways to choose the remaining one, which must be red. So 42 possibilities in all here.

   A final possibility is that the sample has 1 blue and no whites; there are 2 ways to choose the 1 blue, 1 way to choose the no whites, and $\binom{7}{3} = 35$ ways to choose the remaining 3, which must be red. So 70 possibilities in all here.

   The total number of possibilities in this either-or-or experiment is $21 + 42 + 70$ or

$$133$$

10. There are 100 Senators in the U.S. Senate, two from each of the 50 states. A committee of six Senators is to be formed, such that no two are from the same state. In how many ways can this be done?

    Solution: First choose six states from which the six committee members will come; $C(50, 6)$ ways. Then, for each of the six states, choose which (of 2) senators to put on the committee, 2^6 ways. The total count is

$$C(50,6) \cdot 2^6$$

11. A family has nine chihuahuas and four dalmatians (yikes!). Answer the following questions; if your answer involves a $C(n,r)$ or $P(n,r)$ or $r!$, you must calculate the actual value numerically for full credit.

(a, 3 pts) In how many ways can the 13 dogs be fed in the evening, one after the other, if all the dalmatians have to be fed before all the chihuahuas?

Solution: First, arrange the dalmatians in order: 4! ways; then arrange the chihuahuas in order: 9! ways. Since this is a first-then experiment, this leads to a total of

$$4!9! = 8709120.$$

(b, 3 pts) In how many ways can the family pick either three chihuahuas or three dalmatians to take on a walk?

Solution: $\binom{9}{3}$ ways to choose chihuahuas; $\binom{4}{3}$ ways to choose dalmatians; (order doesn't matter). Since this is an either-or experiment, this leads to a total of

$$\binom{9}{3} + \binom{4}{3} = 88.$$

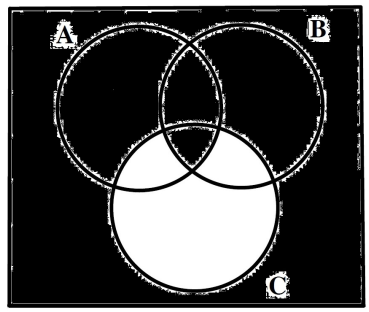
(c, 4 pts) Three dogs are allowed on the bed each night; at least two of them must be chihuahuas. How many different ways can these three lucky dogs be selected?

Solution: Either choose 2 chihuahuas and 1 dalmatian ($\binom{9}{2}\binom{4}{1}$ ways), or choose 3 chihuahuas and 0 dalmatians ($\binom{9}{3}$ ways), leading to a total of

$$\binom{9}{2}\binom{4}{1} + \binom{9}{3} = 228.$$

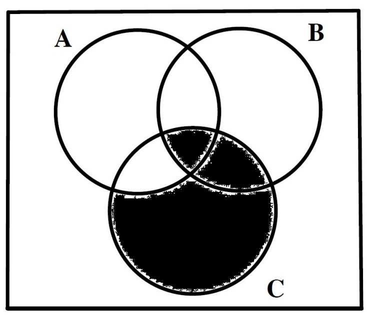
12. Shade **ONLY** the appropriate regions in the diagrams below (10 pts for all 3 correct, 8 for 2 correct, 6 for 1 correct):

(a)



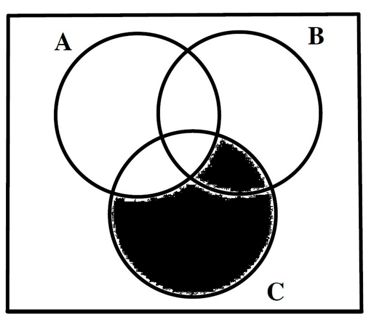
$(A \cap B) \cup C'$

(b)



$(A' \cup B) \cap C$

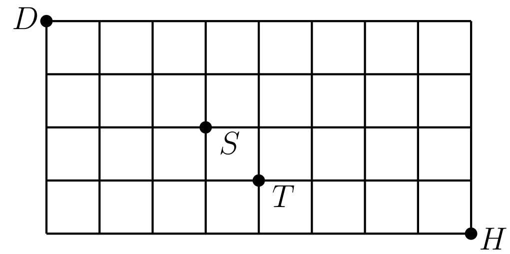
(c)



$C \cap A'$

13. Both parts of this problem refer to the following city map, where I have marked the houses of David (D), Harry (H), Sam (S) and Terri (T). For this problem, you may leave your answer in terms of mixtures of com-

binations and permutations (i.e. $C(n, r)$ and $P(n, r)$ for appropriate n and r) if you choose.



(a, 6 pts) In how many ways can David walk from his house (D) to Harry's house (H), in as few blocks as possible (12)? (Ignore "S" and "T" at this point.)

Solution: Since David needs to choose which 4 of the 12 blocks he walks are south (the rest must be east), the number of ways is

$$\binom{12}{4}.$$

(b, 4 pts) Harry wants to walk from his house (H) to David's (D), but on the way he has to visit either Terri's house (T) or Sam's house (S) to pick something up (he doesn't mind if his route takes him past both S and T). In how many ways can he make this trip in as few blocks as possible (12)?

Solution: Let A be the set of ways that Harry can walk to David's house, via Terri's. He has $\binom{5}{1}$ ways to get to Terri's house (choose 1 north block out of 5). He then has $\binom{7}{3}$ ways to get to David's house (choose 3 north block out of 7). So $n(A) = \binom{5}{1}\binom{7}{3} = 175$.

Let B be the set of ways that Harry can walk to David's house, via Sam's. He has $\binom{7}{2}$ ways to get to Sam's house (choose 2 north block out of 7). He then has $\binom{5}{2}$ ways to get to David's house (choose 2 north block out of 5). So $n(B) = \binom{7}{2}\binom{5}{2} = 210$.

We want $n(A \cup B)$, which is $n(A) + n(B) - n(A \cap B)$. So we need to know $n(A \cap B)$. $A \cap B$ is the set of ways that Harry can walk to David's house, via both Terri's and Sam's. He has $\binom{5}{1}$ ways to get to

Terri's house (choose 1 north block out of 5). He then has 2 ways to get to Sam's (NW or WN). He then has $\binom{5}{2}$ ways to get to David's house (choose 2 north block out of 5). So $n(A \cap B) = \binom{5}{1}2\binom{5}{2} = 100$.

We get then that $n(A \cup B) = 175 + 210 - 100$ or

$$285.$$

14. Remember that a poker hand consists of a sample of 5 cards drawn from a deck of 52 cards. The deck consists of four suits (hearts, clubs, spades, diamonds), and within each suit there are 13 denominations: ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king (so in total there are four cards of each denomination, one from each suit). The order of the cards within the hand doesn't matter. For this problem, you may leave your answer in terms of mixtures of combinations and permutations (i.e. $C(n, r)$ and $P(n, r)$ for appropriate n and r) if you choose.

(a, 4 pts) How many poker hands are there that only include jacks, queens and kings?

Solution: There are 12 cards in total which are jacks, queens and kings, so the number of such hands is

$$\binom{12}{5}.$$

(b, 4 pts) How many poker hands are there that consist of three queens and two sevens?

Solution: First we choose the particular 3 queens: $\binom{4}{3}$ ways. Then we choose the particular 2 sevens: $\binom{4}{2}$ ways. The total number of ways is

$$\binom{4}{3}\binom{4}{2}.$$

(c, 2 pts) 3-of-a-kind is a poker hand that includes three cards of one particular denomination and two other cards of different denominations (so three queens, one seven and one ace is an example of 3-of-a-kind, but three queens and two sevens is not). How many different 3-of-a-kind poker hands are there?

Solution: First we choose the denomination which gives the triple, then we choose the actual three cards. Next, we choose the 2 denominations which each give a single card to the five, **note that the order in which we pick these two denominations does not matter**, and for each of these two denominations, choose the particular card from that denomination. This leads to a total of

$$13 \times \binom{4}{3} \times \binom{12}{2} \times 4 \times 4.$$

NOTE: Many people picked the first of the two singleton denominations, and then the second, giving an answer of $13 \times \binom{4}{3} \times 12 \times 4 \times 11 \times 4$. This gives an answer that is two times too big, because it puts an unwanted order on the last two cards. For example, it counts the hand (Ace hearts, Ace clubs, A diamonds, 7 spades, 3 clubs) as different from the hand (Ace hearts, Ace clubs, A diamonds, 3 clubs, 7 spades); but these are in fact the same.

15. Six married couples are going to be in a group picture, all lined up in a row. For each of these parts, you can give your answer using either $C(n, r)$, $P(n, r)$ or factorial notation, or you can give a numerical answer.

(a, 4 pts) In how many ways can the 12 people line up?

Solution: Twelve people need to be lined up, arbitrarily, so

$$12!$$

(b, 3 pts) In how many ways can they line up if everyone has to be standing next to their spouse?

Solution: First line up the six couples in order (6! ways), then within each couple choose an order for the pair (2 choices, 6 successive times, for a total of 2^6). This gives a total of

$$6!2^6.$$

(c, 3 pts) In how many ways can they line up if everyone has to be standing next to their spouse, with **EITHER** each husband always to the right of his wife **OR** each husband always to the left of his wife?

Solution: First, line up the six couples (6!) ways. Then either put all husbands to right of their wives, or put all husbands to left of their wives. So there are 2 ways to complete the line up, once the six couples have been arranged. This leads to a total of

$$6! \times 2.$$