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Maximum Product

You have n positive real numbers $x_1, \ldots, x_n$ which sum to k, another positive real number. Consider their product, $x_1 \cdots x_n$. Show the maximum product is achieved precisely when $x_i = \frac{k}{n}$ for each $1 \le i \le n$. That is to say, $\prod x_i \le (\frac{k}{n})^n$.

Solution: The problem is about finding the maximum of the function $f(x_1, \ldots, x_n) = \prod x_i$ given the constraint $\sum x_i = k$.

Since each $x_i > 0$ the function f is always positive so we can instead ask for the maximum of $\log f$. Calculate

$$\begin{aligned}
\log f(x_1, \ldots, x_n) &= \log f(x_1, \ldots, x_{n-1}, k - \sum_1^{n-1} x_i) \\
&= \log \left[\left(k - \sum_1^{n-1} x_i \right) \prod_1^{n-1} x_i \right] \\
&= \log \left(k - \sum_1^{n-1} x_i \right) + \sum_1^{n-1} \log x_i
\end{aligned}$$

For each i from 1 to $n-1$ take the derivative of $\log f$ with respect to x_i

$$\begin{aligned}
\frac{\partial}{\partial x_i} \log f(x_1, \ldots, x_n) &= \frac{\partial}{\partial x_i} \left[\log \left(k - \sum_1^{n-1} x_i \right) + \sum_1^{n-1} \log x_i \right] \\
&= -\frac{1}{k - \sum_1^{n-1} x_i} + \frac{1}{x_i} \\
&= \frac{1}{x_i} - \frac{1}{x_n}
\end{aligned}$$

Set the derivative equal to zero to find the critical point $x_i = x_n$. Do this for each i to see the critical point for each x_i is achieved precisely when $x_1 = x_2 = \cdots = x_n$. Since all the variables sum to k we get $x_i = \frac{k}{n}$ for $1 \le i \le n$ as the point which maximizes f. Thus $f(x_1, \ldots, x_n) \le (\frac{k}{n})^n$ when $\sum x_i = k$.