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An Integral Identity

Can you prove the following identity?

$$\int_0^1 \frac{1}{x^x} \mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{n^n}$$

Solution The inside of the integral can be rewritten in terms of the exponential function $\frac{1}{x^x} = x^{-x} = e^{-x \ln x}$. Using the series representation for e^x, we get

$$e^{-x \ln x} = \sum_{n=0}^{\infty} \frac{(-x \ln x)^n}{n!}$$

as a series representation for x^{-x}. We wish to integrate the series term by term. The following lemma will be helpful.

Lemma 1. *For every natural number $n > 0$ and $k \geq 0$,*

$$\int_0^1 x^n (\ln x)^k \, \mathrm{d}x = \frac{(-1)^k k!}{(n+1)^{k+1}}$$

Proof. Chose n, arbitrarily. Proof is by induction on k. If $k = 0$ then

$$\int_0^1 x^n \, dx = \frac{1}{n+1}.$$

Now suppose the identity holds for all $k' < k$. Using integration by parts,

$$\int_0^1 x^n (\ln x)^k \; dx = \left. \frac{x^{n+1} (\ln x)^k}{n+1} \right|_0^1 - \frac{k}{n+1} \int_0^1 x^n (\ln x)^{k-1} \, dx$$

The middle term evaluates to 0, so using the induction hypothesis on the right hand side integral we get the desired equality.

$$\begin{aligned}
\int_0^1 x^n (\ln x)^k \; dx &= -\frac{k}{n+1} \int_0^1 x^n (\ln x)^{k-1} \, dx \\
&= -\frac{k}{n+1} \frac{(-1)^{k-1} (k-1)!}{(n+1)^k} \\
&= \frac{(-1)^k k!}{(n+1)^{k+1}}
\end{aligned}$$

□

To show the initial identity we replace the integrand by its series expansion, and then integrate term by term, using the identity in the previous lemma to do the actual integration.

$$\begin{aligned}
\int_0^1 \frac{1}{x^x}\, dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x \ln x)^n}{n!}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^n \left(\ln x\right)^n\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^n}
\end{aligned}$$

Thanks to John Holmes for the central idea of integrating the series termwise.