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Integer Requirements

Do there exist integers k, l, i, j, m, n such that the following relations hold?

1. $im - n = l$
2. $n - mj = l$
3. m does not divide n
4. $\gcd(i - j, k) \neq 1$
5. $\gcd(k, l) = 1$

Solution: It is not possible. Adding together (1) and (2) gives $m(i - j) = 2l$, so m divides $2l$. In fact, we can show m divides l.

Claim. *m divides l.*

Proof. Suppose not. Then since m divides $2l$ we must have 2 dividing m. This means we can write $m = 2a$ for some integer a, and so $2l = m(i - j) = 2a(i - j)$ giving $l = a(i - j)$ which means a divides l. Let $b = \gcd(i - j, k) = \gcd(\frac{l}{a}, k)$. Then b divides both l and k and is not 1, so $\gcd(l, k) \geq b > 1$. But this contradicts requirement (5). Thus m divides l. □

Since m divides l we can write $l = mc$ for some integer c. Using requirement (1) we have $n - mj = l$ which implies $n = l + mj = mc + mj = m(c + j)$, and so m divides n. But that contradicts requirement (3).

Thus these requirements are inconsistent and cannot be satisfied.