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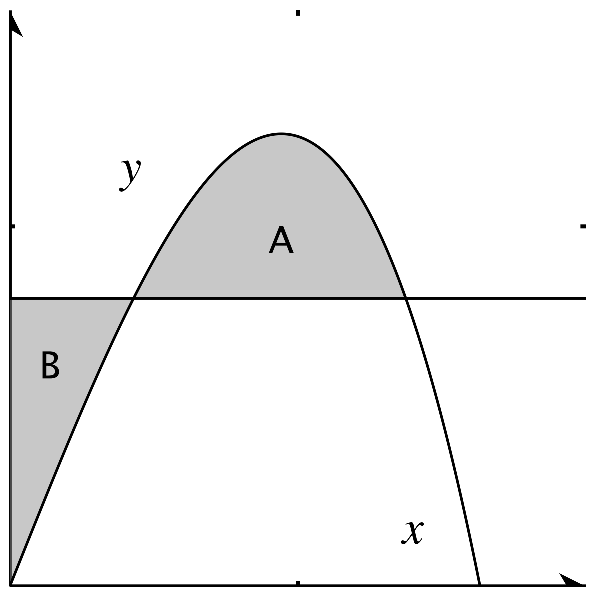
# Equal Areas

Consider the curve $y = 2x - 3x^3$ and the line $y = c$ in the first quadrant pictured below. Find the value of $c$ which makes both the shaded regions have equal areas.




**Solution:** Let $f(x) = 2x - 3x^3$ and let $(a, c)$ and $(b, c)$ with $0 < a < b$ give the points of intersection of $f$ and $y = c$. The areas desired are then given by

$$\int_0^a c - f(x)\,dx \qquad \text{and} \qquad \int_a^b f(x) - c\,dx.$$

Requiring the areas to be equal is equivalent to requiring the following integral to be zero

$$\int_0^b c - f(x)\,dx = \int_0^a c - f(x)\,dx - \int_a^b f(x) - c\,dx = 0.$$

Integrate the left hand side to get $\left[cx - x^2 + \frac{3}{4}x^4\right]_0^b = b(c - b + \frac{3}{4}b^3) = \frac{b}{4}(3c - 2b + c - f(b))$. Because $f(b) = c$ (by definition) we have $\frac{b}{4}(3c - 2b) = 0$. Thus $b = \frac{3}{2}c$ since $b > 0$. Now solve for $c$. Once again $f(b) = c$ so $0 = c - f(b) = c - f(\frac{3}{2}c) = c - 3c + \frac{3^4}{2^3}c^3 = 2c(\frac{9}{4}c + 1)(\frac{9}{4}c - 1)$ giving the solution $c = \frac{4}{9}$ since $c > 1$.