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Commuting Elements

Let G be a group, possibly non-abelian. Suppose there are three elements x, y, z in G with the following properties

- x has order m, where m is odd
- y and z both have order 2
- x and y commute with each other (that is $xy = yx$)
- z commutes with the product xy (that is $zxy = xyz$)

Can you show that z commutes with x and y individually. That is, show $xz = zx$ and $yz = zy$.

Can you answer the same question supposing m is even?

Solution: Since xy and z commute $(zxy)^k = z^k(xy)^k = z^k x^k y^k$, the latter equality comes from x and y commuting. Ditto for $(xyz)^k = x^k y^k z^k$. Since m is odd,

$$(zxy)^m = z^m x^m y^m = zy$$

And

$$(xyz)^m = x^m y^m z^m = yz$$

So $yz = (xyz)^m = (zxy)^m = zy$, which shows z and y commute. For x and z observe

$$zxy = xyz = xzy$$

so

$$zx = xz$$

And we are done.

If m is even then the argument above won't work since then $(zxy)^m = 1$.