The Random Graph is Simple

Donald Brower

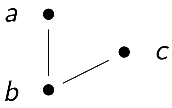
September 29, 2008

The Beginning

A Graph consists of a set V of vertices and a two place relation E on them which is

- Anti-reflexive: $\forall x(\neg xEx)$
- Symmetric: $\forall x \forall y(xEy \leftrightarrow yEx)$

Graphs can be visualized by drawing the vertices as dots and connecting pairs of vertices for which the relation E holds with an edge. E is called the *edge relation*.



The picture above represents a graph on three vertices with aEb, bEc and $\neg aEc$.

Random Graph

To construct the random graph, take a countably infinite set, say $\mathbb{N}$. For each pair of vertices $i, j \in \mathbb{N}$, with $i < j$ decide whether iEj holds by flipping a coin.

The resulting graph is the random graph.

We wish to show the random graph is unique, but this definition is not useful to work with.

Random Graph

To construct the random graph, take a countably infinite set, say $\mathbb{N}$. For each pair of vertices $i, j \in \mathbb{N}$, with $i < j$ decide whether iEj holds by flipping a coin.

The resulting graph is the random graph.

We wish to show the random graph is unique, but this definition is not useful to work with.

And, technically, this definition is not quite right.

Property P

Consider the following property a graph may have. Call it property P

> *For every pair of disjoint finite subsets A and B, there is a vertex outside of $A \cup B$ connected to everything in A and disconnected from everything in B.*

Note that no finite graph can have property P.

A back-and-forth argument shows a countable graph with this property is unique up to isomorphism. (Suppose graphs G and H have property P. Choose a vertex in G and in H. Choose a second vertex in G, use property P to choose a corresponding vertex in H. Repeat, except on even turns go from H to G.)

The Random Graph has property P

The random graph is a countable graph, so if it has property P, it is unique up to isomorphism.

Suppose A and B are disjoint finite subsets of $\mathbb{N}$. Let $k = |A|$ and $l = |B|$. Since A and B are finite there is a natural number n which bounds A and B above.

Consider an arbitrary element $m > n$. There are 2^{k+l} ways for it to be connected to the elements in A and B. Of these ways, exactly one is the desired choice. Thus the probability m is the desired vertex is $\frac{1}{2^{k+l}}$.

But if m isn't the desired vertex, we have an infinite number of choices above m to choose, and the probability that we don't find the desired vertex after s picks $(1 - \frac{1}{2^{k+l}})^s$ goes to 0 as $s \longrightarrow \infty$.

Thus the random graph has property P.

First Order theory of the R.G.

This talk is interested in the first order properties of the random graph. We ask, what first-order sentences are true in the random graph? If G is the random graph this is denoted as

$$T_{RG} = \{\phi : G \models \phi\}$$

And the set T_{RG} is called the theory of the random graph.

It turns out that T_{RG} can be characterized as the set of sentences true for almost all finite graphs.

The fact that any first order sentence is true in almost all or almost no finite graphs is amazing and is called a 0-1 law.

First order properties

But! What is a first order sentence?

I will leave that undefined. Suffice to say that a sentence is a finite sequence of symbols.

A first order statement is "No vertex is isolated":

$$\forall x \exists y (x \neq y \wedge xEy)$$

Another first order statement is "The graph has radius 2"

$$\exists x \forall y (y = x \vee xEy \vee \exists z (xEz \wedge zEy))$$

The statement is "The graph is connected" is not first order since a path connecting two vertices may be arbitrarily long (though finite).

However, the statement "The graph is connected, and the longest path has length n" is first order.

Property P, first order redux

Property P is not first order expressible since the size of the finite sets are not bounded.

But for each $k, l < \omega$ we can write the first order sentence

$$\begin{aligned} A_{k,l} := &\quad \text{Property } P \text{ holds for every disjoint } A \text{ and } B \\ &\quad \text{where } |A| = k \text{ and } |B| = l \\ = &\quad \forall x_1 \ldots x_k y_1 \ldots y_l \Big(\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_{i,j} x_i \neq y_j \\ &\quad \rightarrow \exists z \Big(\bigwedge_i zEx_i \wedge \bigwedge_j (z \neq y_j \wedge \neg zEy_j)\Big)\Big) \end{aligned}$$

And then possessing property P is equivalent to all the statements in the following set being true.

$$T_P = \{A_{k,l} : k, l < \omega\}$$

The theory of the random graph

The claim is that T_{RG} is equivalent to the theory

$$T^* = \{\forall x(\neg xEx), \quad \forall xy(xEy \leftrightarrow yEx)\} \cup T_P$$

In other words, every first order sentence in T_{RG} is entailed by merely knowing a graph has property P.

Proof.

Suppose there is a sentence φ such that both $T^* \cup \{\varphi\}$ and $T^* \cup \{\neg\varphi\}$ are consistent. Then there are countable models M and N of T^* with

$$M \models \varphi \qquad \text{and} \qquad N \models \neg\varphi$$

But since both M and N are countable and have property P, $M \cong N$. Thus $N \models \varphi$. Contradiction. □

The Elementary Class of T_{RG}

Once we have a theory T we can look at the models of the theory.

In the case of T_{RG} it comes down to looking at all graphs with property P, by the previous slide.

One question we can ask is how many graphs are there with property P?

We already showed that there is only one such countable graph.

A theorem in logic says if a theory has an infinite model then it has a model of every larger cardinality. So, there are many models of T_{RG}. That is, there are many graphs with property P.

Cardinal Numbers

Cardinal numbers measure the size of a set.

Say $|X| = |Y|$ if there is a bijection between X and Y.

The cardinal numbers can be thought of as equivalence classes of sets under the relation $|X| = |Y|$.

Thus there are unboundedly many cardinal numbers, and they are linearly ordered. Some are familiar

$$0, 1, 2, 3, \ldots$$

After these, there are cardinals representing the sizes of infinite sets. The smallest is $\aleph_0$, which represents the size of a countable set. e.g. $|\mathbb{N}| = \aleph_0$.

The next larger cardinal is denoted $\aleph_1$. And then $\aleph_2$, and so on.

$$0, 1, 2, 3, \ldots, \aleph_0, \aleph_1, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$$

Spectrum

Let $I(T, \lambda)$ denote the number of models of T with cardinality λ.

We have

$$
\begin{aligned}
I(T_{RG}, 0) &= 0 \\
I(T_{RG}, 1) &= 0 \\
&\vdots \\
I(T_{RG}, \aleph_0) &= 1 \\
I(T_{RG}, \aleph_1) &= ? \quad (\geq 1)
\end{aligned}
$$

$I(T_{RG}, \aleph_1)$, The number of T_{RG} graphs of size $\aleph_1$

In the countable random graph every vertex has degree $\aleph_0$. Two ways to show this.

1. If not, then some vertex has degree n, and there is a first order sentence saying there is a vertex with degree n. But this sentence cannot be true for almost all graphs.
2. For any vertex v, use property P to build a collection of $\aleph_0$ neighbors.

Extending this idea to random graphs of cardinality $\aleph_1$, each vertex may have degree $\aleph_0$ or $\aleph_1$. So we can count how many of each any given graph has. And graphs with different counts cannot be isomorphic.

$I(T_{RG}, \aleph_1)$, cont.

We have the following possibilities.

degree $\aleph_0$	degree $\aleph_1$
$\aleph_1$	0
$\aleph_1$	1
$\vdots$	$\vdots$
$\aleph_1$	$\aleph_0$
$\aleph_1$	$\aleph_1$
$\aleph_0$	$\aleph_1$
$\vdots$	$\vdots$
0	$\aleph_1$

In total, $\aleph_0 + 3 + \aleph_0 = \aleph_0$ combinations.

Thus $I(T_{RG}, \aleph_1) \geq \aleph_0$

Unstability

But it is more subtle than just looking at the degrees of the vertices. We need to look at how the vertices in a model are hooked together.

In fact, a general purpose theorem tells us $I(T_{RG}, \aleph_1) = 2^{\aleph_1}$.

Definition (Shelah)

A theory T is unstable if there is a formula interpreting either the Independence Property or the Strict Order Property.

Theorem (Shelah)

If T is unstable then $I(T, \kappa) = 2^{\kappa}$ for all $\kappa > \aleph_0$.

Formulas

Before we used sentences. In a sentence, every variable is attached to some quantifier—either $\forall$ or $\exists$.

$$\exists x \forall y (y = x \vee xEy \vee \exists z (xEz \wedge zEy))$$

If at least one variable is not attached to a quantifier then it is called a formula.

$$\forall y (y = x \vee xEy \vee \exists z (xEz \wedge zEy))$$

Here the variable x is not attached to any quantifiers, so we have a formula $\varphi(x)$. We cannot ask whether a formula is true in a model since we do not know how to interpret the symbol 'x'.

Instead we can ask, which elements in a model M make the formula true? Write this as

$$\varphi(M) = \{a \in M : M \models \varphi(a)\} \subseteq M$$

Definable Sets

If M is a model, a subset $B \subset M^n$ of n-tuples is definable if there is a formula $\varphi(\bar{x}, \bar{y})$ and a parameter $\bar{b} \in M$ such that $B = \varphi(M, \bar{b}) = \{\bar{a} \in M^n : M \models \varphi(\bar{a}, \bar{b})\}$.

Example

The set of all triples of vertices which form a triangle is definable via the formula $\varphi(x_1, x_2, x_3) := \bigwedge_{i<j}(x_i E x_j)$.

Example

In $(\mathbb{R}, +, \cdot)$ the ordering $\leq$, i.e. all pairs (a, b) such that $a \leq b$, is definable via $\varphi(x, y) := \exists z(z^2 = y - x)$.

Structure of Definable Sets

The definable sets of a structure depend on the language used.

Shelah's idea is that we can count the number of models by looking at the complexity of the definable sets. Having definable sets which nest in intricate ways make it easy to mess up isomorphisms between models.

IP and SOP

Consider a single formula $\varphi(\bar{x})$. For our purposes it has complicated definable sets if it has either of these properties.

Definition

A formula $\varphi(\bar{x}, \bar{y})$ has the strict order property if there is a model M and such that φ defines a partial order on the set of n-tuples with an infinite chain.

Definition

A formula $\varphi(\bar{x}, \bar{y})$ has the independence property if there is a model M and elements $\bar{a}_1, \bar{a}_2, \ldots \in M$ such that for every finite subset $\Delta \subset \mathbb{N}$ there is a parameter $\bar{b}_\Delta \in M$ which picks out the set $\{a_i : i \in \Delta\}$.

$$M \models \varphi(\bar{a}_i, \bar{b}_\Delta) \qquad \text{iff} \qquad i \in \Delta$$

SOP Example

The main example of a theory having SOP is the theory of dense linear orders without endpoints: $(\mathbb{Q}, <)$.

From earlier slide, the theory of real-closed fields also has SOP: $(\mathbb{R}, 0, 1, +, \cdot)$.

Also, the theory of arithmetic has SOP: $(\mathbb{N}, 0, 1, +, \cdot)$.

The theory of algebraically closed fields does not have SOP. $(\mathbb{C}, 0, 1, +, \cdot)$.

The random graph does not have SOP, either.

Random Graph and IP

In the case of the random graph, the formula xEy has the independence property.

Let M be the random graph. Choose any sequence $a_1, a_2, \ldots \in M$ of vertices. Property P may not guarantee all of the needed b_Δ elements exist inside M since we need each element to be disconnected from an *infinite* number of a_i. But we can use it to construct an extension N of M which has the required elements.

Thus T_{RG} is unstable. Ergo, $I(T_{RG}, \aleph_1) = 2^{\aleph_1}$.

Map of Theories

	NSOP	SOP
IP	Random Graph $VS(\mathbb{F}_n)$ with bilinear form Alg. closed fields with automorphism	Arithmetic
NIP	Algebraically closed fields Vector spaces Abelian groups	Real-closed fields linear orders

For awhile model theory ignored the unstable theories for the stable ones. That is, theories with *NIP* and *NSOP*.

Pseudo-Finiteness

The sentences in T_{RG} are those which are true in almost all finite graphs.

We can do similar things for other collection of finite things, e.g. linear orders, fields.

In the case of fields, we get the theory of the *pseudo-finite fields*. Models K are characterized by (Ax)

- K is perfect
- K has exactly one algebraic extension of every degree
- every absolutely irreducible variety over K has a point in K