Notes on the Banach-Tarski Paradox

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The Banach-Tarski paradox is not a logical paradox, but rather a counter intuitive result. It says that a ball can be broken into a finite number of pieces and only using rigid motions, be reassembled into two identical balls the exact same size as the original one. These notes follow the presentation given in Jech [2].

For $X, Y \subseteq \mathbb{R}^3$ write $X \approx Y$ if there are a finite decomposition of X into disjoint sets

$$X = X_1 \cup \cdots \cup X_m$$

and a decomposition of Y into the same number of disjoint sets

$$Y = Y_1 \cup \cdots \cup Y_m$$

such that X_i is congruent to Y_i for each $i \in \{1, \ldots, m\}$.

Theorem 1 (Banach-Tarski). *A closed ball U can be decomposed into two disjoint sets $U = X \cup Y$ such that $U \approx X$ and $U \approx Y$.*

This is similar to a prior result by Hausdorff.

Theorem 2 (Hausdorff). *A sphere S can be decomposed into disjoint sets*

$$S = A \cup B \cup C \cup Q$$

such that A, B, C, and $B \cup C$ are congruent to each other, and Q is countable.

Theorem 2 will be proved first and then used to prove theorem 1.

Since we are working on the sphere the relevant rigid motions are rotations. Let G be the free product of the groups $\{1, \phi\}$ and $\{1, \psi, \psi^2\}$ where ϕ has order 2 and ψ order 3. Choose two axes of rotation a_ϕ, a_ψ through the center of the ball U. By having ϕ represent a rotation of 180° about a_ϕ and ψ a rotation of 120° about a_ψ, we have an action of G on the sphere. Our goal is to find a suitable partition of G and then use this to find a partition of the sphere via the action of G. The first step is to show that there is a choice of axes such that G can be embedded into the group of rotations on the sphere.

Lemma 1. *The axes a_ϕ and a_ψ can be chosen such that the action of G on the sphere is faithful. That is, distinct elements of G will give distinct rotations.*

Proof. It suffices to find an angle θ between a_ϕ and a_ψ such that no nonidentity element of G represents the identity rotation. Take a_ψ to be the z-axis and a_ϕ to lie in the x–z plane at angle θ to a_ψ We can represent the rotations ϕ and ψ by the matricies

$$\psi = \begin{pmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\phi = \begin{pmatrix} -\cos\theta & 0 & \sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{2}$$

where $\lambda = \cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\mu = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

Choose θ so that $\cos\theta$ is irrational. We wish to show that if $\alpha \in G$ is not the identity then the corresponding action is not the identity.

If $\alpha = \sigma_m\sigma_{m-1}\cdots\sigma_2\sigma_1$ where each σ is either $\psi\phi$ or $\psi^2\phi$, then the action of each σ can be represented by one of the two matricies

$$\sigma = \begin{pmatrix} -\lambda\cos\theta & \mp\mu & \lambda\sin\theta \\ \pm\mu\cos\theta & -\lambda & \mp\mu\sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Choose the vector $K = (0,0,1)$. We have

$$\alpha \cdot K = \sigma_m\sigma_{m-1}\cdots\sigma_1 \cdot K = (\sin\theta P_m(\cos\theta), \sqrt{3}\sin\theta Q_m(\cos\theta), R_m(\cos\theta)),$$

where P_m, Q_m, and R_m are polynomials with rational coefficients. In fact,

$$P_1(x) = -\frac{1}{2} \qquad Q_1(x) = \mp\frac{1}{2} \qquad R_1(x) = x$$

and

$$\begin{aligned} P_{m+1}(x) &= -\lambda x P_m(x) \mp \frac{3}{2} Q_m(x) + \lambda R_m(x) \\ Q_{m+1}(x) &= \pm\frac{1}{2} x P_m(x) - \lambda Q_m(x) \pm \frac{1}{2} R_m(x) \\ R_{m+1}(x) &= (1-x^2)P_m(x) + xR_m(x). \end{aligned}$$

Since $\cos\theta$ is irrational it is not the root of any polynomial with rational coefficients. Thus $\alpha \cdot K \neq K$ since otherwise $R_m(\cos\theta) - 1 = 0$ would have a solution.

The cases where α has the forms $\phi\sigma_m\cdots\sigma_1$, $\sigma_m\cdots\sigma_1\psi^{\pm 1}$, and $\phi\sigma_m\cdots\sigma_1\psi^{\pm 1}$ follow from this case. □

Now we need to partition G into suitable subsets.

Lemma 2. *The group G can be decomposed into three disjoint sets $G = A \cup B \cup C$ such that*

$$\phi \cdot A = B \cup C, \qquad \psi \cdot A = B, \qquad \psi^2 \cdot A = C \tag{3}$$

Proof. Construct A, B, and C by induction on the length of elements in G. Begin with $1 \in A$, $\phi, \psi \in B$, and $\psi^2 \in C$. Partition longer elements according to the table.

	$\alpha \in A$	$\alpha \in B$	$\alpha \in C$
α begins with $\psi^{\pm 1}$	$\phi\alpha \in B$	$\phi\alpha \in A$	$\phi\alpha \in A$
α begins with ϕ	$\psi\alpha \in B$	$\psi\alpha \in C$	$\psi\alpha \in A$
α begins with ϕ	$\psi^2\alpha \in C$	$\psi^2\alpha \in A$	$\psi^2\alpha \in B$

It is easy to check that these sets are disjoint and satisfy the required properties. □

Now we can prove theorem 2 using this decomposition.

Proof of theorem 2. Put

$$Q = \{x \in S : x \text{ is a fixed point of some nonidentity } \rho \in G\}.$$

Since each rotation has two fixed points and G is countable, so is Q. The set $S \setminus Q$ is the disjoint union of all orbits P_x of G:

$$P_x = \{\alpha \cdot x : \alpha \in G\}.$$

(If $\alpha \cdot x \in Q$ for some α then $\sigma\alpha \cdot x = \alpha \cdot x$ for some nonidentity σ. Hence $\alpha^{-1}\sigma\alpha \cdot x = x$, and since $\alpha^{-1}\sigma\alpha \neq 1$, we have $x \in Q$. But $x \in S \setminus Q$. contradiction)

Let $F = \{P_x : x \in S \setminus Q\}$ be the family of all orbits. By the axiom of choice, there is a set M which contains exactly one element in each P_x. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the decomposition of G given by the lemma. Put

$$\begin{aligned} A &= \{\rho \cdot x : \rho \in \mathcal{A}, x \in M\} \\ B &= \{\rho \cdot x : \rho \in \mathcal{B}, x \in M\} \\ C &= \{\rho \cdot x : \rho \in \mathcal{C}, x \in M\}. \end{aligned}$$

From the two lemmas, A, B, and C are all disjoint and congruent to each other and to $B \cup C$ since

$$\begin{aligned} \phi A &= \{\phi\rho \cdot x : \rho \in \mathcal{A}, x \in M\} = \{\tau \cdot x : \tau \in \mathcal{B} \cup \mathcal{C}, x \in M\} = B \cup C \\ \psi A &= \{\psi\rho \cdot x : \rho \in \mathcal{A}, x \in M\} = \{\tau \cdot x : \tau \in \mathcal{B}, x \in M\} = B \\ \psi^2 A &= \{\phi\rho \cdot x : \rho \in \mathcal{A}, x \in M\} = \{\tau \cdot x : \tau \in \mathcal{C}, x \in M\} = C \end{aligned}$$

□

To prove theorem 1 some properties of the $\approx$ relation are needed. This proof is similar to the Schroder-Bernstein theorem from set theory that two sets which can be embedded into each other have the same cardinality.

Lemma 3. *Let $\approx$ be as defined above. Then,*

1. *$\approx$ is an equivalence relation*

2. *If X and Y are disjoint unions of X_1, X_2 and Y_1, Y_2 respectively, and $X_i \approx Y_i$ for $i = 1, 2$ then $X \approx Y$.*

3. *If $X_1 \subseteq Y \subseteq X$ and $X \approx X_1$ then $X \approx Y$.*

Proof. The first two are easy. To prove the last one let $X = X^1 \cup \cdots \cup X^n$ and $X_1 = X_1^1 \cup \cdots \cup X_1^n$ such that each X^i is congruent to X_1^i for each $i = 1, \ldots, n$. Choose a congruence $f^i : X^i \to X_1^i$ for each $i = 1, \ldots, n$, and let $f : X \to X_1$ be the one-to-one mapping agreeing with f^i on X^i for each $i = 1, \ldots, n$. Define the sequences $X_0, X_1, X_2, \ldots$ and $Y_0, Y_1, \ldots$ by

$$X_0 = X, \qquad Y_0 = Y$$
$$X_{s+1} = f(X_s), \qquad Y_{s+1} = f(Y_s).$$

Put $Z = \bigcup_{s=0}^{\infty}(X_s \setminus Y_s)$.

Observe that if $x \in f(X_n \setminus Y_n)$ then $x \notin f(Y_n)$ since otherwise because f is one-to-one there must be an element $y \in Y_n$ with $x = f(y)$. But since $x \in f(X_n \setminus Y_n)$ we have a contradiction. Thus $f(X_n \setminus Y_n) = f(X_n) \setminus f(Y_n)$.

Then $f(Z) \subseteq Z$ since

$$\begin{aligned} f(Z) &= f(\bigcup_{s=0}^{\infty}(X_n \setminus Y_n)) \\ &= \bigcup_{s=0}^{\infty} f(X_n \setminus Y_n) \\ &= \bigcup_{s=0}^{\infty} f(X_n) \setminus f(Y_n) \\ &= \bigcup_{s=0}^{\infty} X_{n+1} \setminus Y_{n+1} \\ &\subseteq Z \end{aligned}$$

so $f(Z)$ and $X \setminus Z$ are disjoint, $Z \approx f(Z)$, and

$$X = Z \cup (X \setminus Z), \qquad Y = f(Z) \cup (X \setminus Z)$$

hence $X \approx Y$ by (2). $\square$

Proof of theorem 1. Let U be a closed ball and let $S = A \cup B \cup C \cup Q$ be the decomposition of its surface from theorem 2. We have

$$U = \bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{Q} \cup \{c\}$$

where c is the center of the ball, and for each $X \subset S$, $\bar{X}$ is the set of all points $x \in U$ such that its projection onto the surface is in X. Clearly $\bar{A} \approx \bar{B} \approx \bar{C} \approx \bar{B} \cup \bar{C}$. Let

$$X = \bar{A} \cup \bar{Q} \cup \{c\}, \qquad Y = U \setminus X$$

From the lemma and the above relations we have $\bar{A} \approx \bar{B} \cup \bar{C} \approx \bar{A} \cup \bar{C} \approx \bar{A} \cup \bar{B} \cup \bar{C}$. Thus $X \approx U$.

Find some rotation α (not in G) such that Q and $\alpha \cdot Q$ are disjoint. Using $\bar{C} \approx \bar{A} \cup \bar{B} \cup \bar{C}$, there exists $T \subset C$ such that $\bar{T} \approx \bar{Q}$. Pick an arbitrary $p \in \bar{T} \setminus \bar{C}$. Obviously,

$$\bar{A} \cup \bar{Q} \cup \{c\} \approx \bar{B} \cup \bar{T} \cup \{p\}.$$

Since

$$\bar{B} \cup \bar{T} \cup \{p\} \subseteq Y \subseteq U,$$

we have

$$\bar{B} \cup \bar{T} \cup \{p\} \approx X \approx U$$

and so by the lemma

$$Y \approx U.$$

□

References

[1] Felix Hausdorff. *Grundzüge der Mengenlehre*. Verlag, 1914.

[2] Thomas J. Jech. *The Axiom of Choice*. North Holland, 1973.

[3] Karl Stromberg. The Banach-Tarski paradox. *Amer. Math. Monthly*, 86:151–161, 1979.