# ANOTHER LOOK AT STABLE FORKING IN 1-BASED SUPERSIMPLE THEORIES

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ABSTRACT. We give two alternative proofs that 1-based theories of finite SU-rank have stable forking, neither of which seems to require the full power of elimination of hyperimaginaries. We also show some miscellaneous results related to stable forking in simple theories.

## 1. Introduction

Recall that the class of simple theories is a proper extension of the class of stable theories. While the former can be characterized by, among other things, the symmetry of non-forking independence, forking in a stable theory (deviation from the definition of the type) is often rather easier to identify and smoother to work with than forking in a simple theory. For this reason, one might hope that forking in a simple unstable theory is always at least *witnessed* by a stable formula. Also, although there is a plethora of examples of simple unstable theories, many of these seem to be essentially stable up to some kind of "noise." One way to formalize this intuition is found in [5]:

**Definition 1.1.** *A simple theory $T$ has* stable forking *if whenever $q(x)$ is a complete type over a model $M$, $A \subset M$ and $q$ forks over $A$ then there is is a stable formula $\psi(x; b) \in q$ which forks over $A$.*

The Stable Forking Conjecture is simply that every simple theory has stable forking. The conjecture is unresolved—and not universally believed true—but it has been verified in several special classes including 1-based supersimple theories and stable theories expanded with a generic predicate.

Although the result is fairly well-known to model-theorists, demonstrations of the fact that 1-based supersimple theories have stable forking have all involved highly technical machinery spread over several publications. The original proof runs through [4], where Kim showed that 1-based simple theories with elimination of hyperimaginaries (EHI) have stable forking, and it terminates in a very technical paper, [1], where it is shown that all supersimple theories have EHI. In the standard textbook on simple theories, [8], it is proved as a sort of afterthought in an extended treatment of elimination of hyperimaginaries. In this note, we present some streamlined proofs of stable forking in 1-based theories of finite $SU$-rank. While still building on [1], we avoid a great deal of technical machinery.

The demonstrations in this article arose partially out of an attempt to obtain the results of [6] on stable forking for elements of "low $SU$-rank" in $\omega$-categorical simple theories while dropping the $\omega$-categoricity assumption. The original project was successful in that we proved (section 2) that in any simple theory with EHI, if a rank-2 element $a$ forks with $b$, and $b$ has finite SU-rank, then $tp(a/b)$ contains a stable forking formula. Our proof actually uses a condition, Dependence Witnessed by Imaginaries (DWIP), which seems to be somewhat weaker than full EHI (equivalent to weak elimination of hyperimaginaries in a simple theory), and building on this idea, we extend the method that worked for rank-2 elements against finite-rank elements to work for all finite-ranked elements in 1-based theory with DWIP (section 4). Finally, we present a second demonstration of stable forking in a 1-based theory (again, for finite-ranked elements) that does not require any mention of hyperimaginaries whatever.

In what follows we assume a familiarity with common conventions of model theory . We work in a suitably saturated model $\mathbb{M}$; we will use the terms forking, simple, supersimple, $SU$-rank, canonical base, and so on freely. One reference for this background material is [8]. All that said, we include a very brief reminder on the basics of hyperimaginaries because the word "hyperimaginary" makes at least one of the authors a bit queasy.

1.1. **Hyperimaginaries.** This subsection presents background on hyperimaginaries; there are no new results contained inside. For more details see [3], [1], and [8]. Readers familiar with hyperimaginaries may wish to skip to the next section.

Stability theory makes much use of imaginary elements, each of which is a name for a class of a definable equivalence relation. Imaginary elements can be "added" to a theory by adding sorts to represent each definable equivalence relation, and this "eq" construction preserves all of the relevant properties of the original first-order theory.

We can, then, do the same thing for type-definable equivalence relations, calling the name of such an equivalence class a *hyperimaginary*. If $E(\bar{x}, \bar{y})$ is a type definable equivalence relation with $\bar{x} = (x_i : i \in I)$

and $\bar{y} = (y_i : i \in I)$ for some (possibly infinite) linearly ordered index set $I$, and $\bar{a} = (a_i : i \in I)$ is some tuple of elements from $M^{\mathrm{eq}}$ then $\bar{a}_E$ is the hyperimaginary element representing the class of all tuples $E$-related to $\bar{a}$. It can be shown that any type-definable equivalence relation is equivalent to an infinite conjunction of equivalence relations defined by a countable types, so it suffices to add names only for the equivalence relations defined by countably-infinite conjunctions of formulas.

A model $M$ enriched with these names is denoted $M^{\mathrm{heq}}$. It's not hard to see that $M \subset M^{\mathrm{eq}} \subset M^{\mathrm{heq}}$ up to some natural and obvious identifications. Automorphisms of $M$ lift to $M^{\mathrm{heq}}$, so if $e = \bar{a}_E$ is a hyperimaginary element we say an automorphism fixes $e$ iff it fixes the $E$ class of $\bar{a}$ as a set. This is an annoyance of dealing with hyperimaginaries: the formulas comprising the equivalence relation $E$ do not need to be $e$-invariant. The lifting of automorphisms does let us extend definable closure and algebraic closure to $M^{\mathrm{heq}}$; we will use dcl and acl to refer to these extensions for this paper.

Unfortunately, we cannot think of $M^{\mathrm{heq}}$ as a first-order structure. The main issue is that type-definable sets are closed but not clopen, so the negation of such is not, in general, type-definable. However, a fragment of logic can be developed. Specifically, we can extend the notions of *type* and *forking* to hyperimaginaries. If $a$ and $b$ are hyperimaginaries then there are equivalence relations $E(x, y)$ and $F(w, z)$ and tuples $\bar{a} = (a_i : i \in I)$, $\bar{b} = (b_i : i \in J)$ such that $a = \bar{a}_E$, $b = \bar{b}_F$. The type $\mathrm{tp}(a/b)$ can then be expressed as the union of the partial types over $\bar{b}$,

$$\exists wz[E(x, w) \wedge F(z, \bar{b}) \wedge \varphi(w', z')]$$

for all $\varphi$ such that the partial type is satisfied by $a$. In the equation above $w'$ and $z'$ represent some finite subtuples of $w$ and $z$. We could have chosen to base the type on any tuple $\bar{b}^*$ so long as $\bar{b}^*$ and $\bar{b}$ are $F$-equivalent. This definition is the right one since it preserves the idea of types representing orbits of elements over a base set.

**Fact 1.2.** [3, Proposition 1.4] *$a'$ realizes $\mathrm{tp}(a/b)$ if and only if there is an automorphism $f$ fixing $b$ such that $f(a) = a'$.*

From the notion of type one can derive the notion of an indiscernible sequence of hyperimaginaries. We then say a type $p(x, b) = \mathrm{tp}(a/b)$ divides over a set $C$ if there is a sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/C)$ indiscernible over $C$ such that $\bigcup_i p(x, b_i)$ is inconsistent. A type forks if it implies a finite disjunction of types which divide.

Just as it is terribly convenient to work in a theory which eliminates imaginaries, it is very nice to work with theories that eliminate hyperimaginaries. We say that a theory $T$ eliminates hyperimaginaries (has EHI) if for every hyperimaginary $e \in \mathbb{M}^{\mathrm{heq}}$, there is a set of imaginaries $\bar{b} \subset \mathbb{M}^{\mathrm{eq}}$ such that $\mathrm{dcl}(\bar{b}) = \mathrm{dcl}(e)$. Finally, we say that a theory $T$ has weak elimination of hyperimaginaries (has WEHI) if for every hyperimaginary $e \in \mathbb{M}^{\mathrm{heq}}$, there is a set of imaginaries $\bar{b} \subset \mathbb{M}^{\mathrm{eq}}$ such that $\mathrm{bdd}(\bar{b}) = \mathrm{bdd}(e)$.

## 2. Elements of Rank 2

In this section, we will show that whenever an element $a$ of $SU$ rank 2 forks with another tuple $b$ (of finite rank) over a set $C$, there is a witnessing stable forking formula. This extends a result of Peretz [6] by removing the assumption of $\omega$-catagoricity and allowing $b$ to have rank larger than 2. We include this argument mainly because of its connection to [6] and because it is slightly easier to follow than its more general cousin (see section 3). The proof builds the forking formula inductively using two observations: an element of $SU$-rank 2 which forks with an element of an indiscernible sequence, but is not algebraic over the sequence, is independent from the sequence over a single element, resembling a 1-based theory. Second, stable forking can be transfered "upward" through algebraic closure. We will prove the second observation en route to proving the theorem.

We will also need to use a result of Z. Shami in [7] that stable forking is symmetric. [7] uses the additional assumption that strong types and Lascar strong types coincide, but in private correspondence, Shami has shown that this assumption is unnecessary. Supersimplicity implies lstp = stp, but we don't know if DWIP/WEHI has the same power. Thus, we present Shami's proof of the more general result.

2.1. **Symmetry of Stable Forking.** This short section presents a proof that stable forking is symmetric in all simple theories. It builds on a foundation laid in [7], where stable forking symmetry is proved for theories where lstp = stp. We will recall the bare essentials of that paper, but only present a proof of the new result which we require. This proof and all the other material in this subsection are due to Shami.

Here is a rough outline of the original proof. First, one establishes that for stable formulas, generic satisfiability is unique for parameters that have the same Lascar strong type.

**Fact 2.1.** [7, Claim 6.5] *Let $T$ be simple, and let $\phi(x, y) \in L$ be stable. Assume $a \downarrow_A b$ and $a' \downarrow_A b$ and that $\mathrm{lstp}(a/A) = \mathrm{lstp}(a'/A)$. Then $\models \phi(a, b)$ if and only if $\models \phi(a', b)$.*

One uses this fact to identify an alternate definition of forking (more akin to forking in a stable theory) with the usual one for simple theories.

**Definition 2.2.** *Let $\phi(x, y)$ be a formula, and let $A \subseteq B \subset \mathbb{M}^{eq}$. Then* $p \in S_\phi(B)$ does not fork in the sense of LS *if, for some model $M$ containing $B$, some $p' \in S_\phi(M)$ extending $p$ is definable over* $\mathrm{acl}(A)$.

**Fact 2.3.** [7, Lemma 6.6] *Assume $T$ is a simple theory with* $\mathrm{lstp} = \mathrm{stp}$, *and let $\phi(x, y) \in L$ be a stable formula. Let $a \in \mathbb{M}^{eq}$ and $A \subseteq B \subset \mathbb{M}$. Then $\mathrm{tp}_\phi(a/B)$ does not fork over $A$ in the sense of LS if and only if $\mathrm{tp}_\phi(a/B)$ does not fork over $A$.*

Provided lstp = stp, it is a short step from this to the symmetry of "stable non-forking independence," from which the symmetry of stable forking is fairly obvious. Throughout Shami's argument, the assumption that lstp = stp appears only so that fact 2.1 may be used. Thus, to extend his argument to arbitrary simple theories, it suffices to prove an analog of fact 2.1 for strong types in place of Lascar strong types. Strangely, the proof of this extension actually uses its precursor.

**Lemma 2.4.** *Let $E(x, y)$ be a bounded, co-type definable equivalence relation. Then $E$ is a definable, finite equivalence relation.*

*Proof.* Suppose $E$ is as in the hypothesis. Let $r(x, y)$ be a partial type defining $\neg E$. If $E$ had an infinite number of equivalence classes, then we could build an indiscernible sequence where each 2-type realized $r$. The sequence would then imply the number of $E$ classes is unbounded, contradicting the hypothesis. Thus $E$ has only a finite number of classes. Let $a_1, \ldots, a_n$ for some $n < \omega$ consist of a single representative from each class. Then the type $\bigwedge_{i \leq n} r(x, a_i)$ is inconsistent, so there is a formula $\psi(x, y) \in r$ such that $\bigwedge_{i \leq n} \psi(x, a_i)$ is inconsistent by compactness. Observe that $\neg\exists x(\bigwedge_{i \leq n} \psi(x, a_i))$ is equivalent to $\forall x(\bigvee_{i \leq n} \neg\psi(x, a_i))$, so for every $x$ there is some $j$ such that $\neg\psi(x, a_j)$. Since $\psi \in r$, $\neg\psi(x, y) \models E(x, y)$. Let

$$\theta(x, y) \equiv \bigwedge_{i \leq n} (\neg\psi(x, a_i) \leftrightarrow \neg\psi(y, a_i)) .$$

We will show $\theta$ defines $E$. First suppose $\theta(b, c)$. For some $j$ we have $\neg\psi(b, a_j)$, and $\theta$ entails $\neg\psi(c, a_j)$. Thus $E(b, a_j)$ and $E(c, a_j)$ so by transitivity $E(b, c)$.

Conversely, suppose $E(b, c)$. If $\neg\theta(b, c)$ then there is some index $i$ such that (without loss of generality) $\neg\psi(b, a_i) \wedge \psi(c, a_i)$ holds. There is another index $j$ such that $\neg\psi(c, a_j)$. Then we have $E(b, a_i)$, $E(b, c)$ and $E(c, a_j)$. By transitivity $E(a_i, a_j)$. But we chose $a_i$ and $a_j$ to be in different classes. The contradiction proves the claim and the lemma. □

**Lemma 2.5.** *Let $T$ be simple. Let $\phi(x, y) \in L$ be stable. Assume $a \downarrow_A b$ and $a' \downarrow_A b$ and that $\mathrm{stp}(a/A) = \mathrm{stp}(a'/A)$. Then $\models \phi(a, b)$ if and only if $\models \phi(a', b)$.*

*Proof.* Given a complete type $q(x)$ define the equivalence relation $E_q$ by

$$E_q^\phi(a, a') \iff \text{for every } b \models q \text{ with } b \downarrow aa', \models \phi(a, b) \leftrightarrow \phi(a', b).$$

The complement of $E_q^\phi$ is defined by a partial type over $A$. By fact 2.1, if $\mathrm{lstp}(a) = \mathrm{lstp}(a')$, then $E_q^\phi(a, a')$ holds; this implies $E_q^\phi$ is refined by the equality of Lascar strong type, so it is bounded. A bounded, co-type-definable equivalence relation is a finite, definable equivalence relation by lemma 2.4, so $E_q^\phi$ is an $A$-definable finite equivalence relation.

Now, suppose $\mathrm{stp}(a/A) = \mathrm{stp}(a'/A)$, $b \downarrow_A a$ and $b \downarrow_A a'$. Put $q = tp(b/A)$. By definition, $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$, so $a, a'$ must be in the same $E_q^\phi$-class. Let $a''$ realize $\mathrm{stp}(a/A)$ such that $a'' \downarrow_A aa'b$. By transitivity, $aa'' \downarrow_A b$ and $a'a'' \downarrow_A b$. Then, since $E_q^\phi(a, a'')$ and $E_q^\phi(a', a'')$ we have $\phi(a, b) \leftrightarrow \phi(a'', b) \leftrightarrow \phi(a', b)$, as desired. □

The remainder of Shami's argument (Lemma 6.6 to the end of section 6 of [7]) goes through essentially unchanged except for substituting our lemma 2.5 for fact 2.1.

**Theorem 2.6.** *Let $T$ be simple. If $a \not\downarrow_C b$ and there is a stable formula $\psi(x; y)$ such that $\psi(x; b)$ forks over $C$, then there is a stable formula $\varphi(x; y)$ such that $\varphi(x; a)$ forks over $C$.*

2.2. **Dependence-witnessed-by-imaginaries.** We now introduce a useful notion which, for wont of a better name, we call *dependence witnessed by imaginaries property*, and we show that it is equivalent to weak elimination of imaginaries. Every supersimple theory eliminates hyperimaginaries, so as we will see, every supersimple theory must have DWIP. The applications of DWIP in the our arguments for stable forking do not require that SU-rank is always defined—only that it is defined for the relevant elements—so it is possible that DWIP could be useful outside of the supersimple context.

**Definition 2.7.** *We say a simple theory has the* Dependence Witnessed by Imaginaries Property (DWIP) *if whenever $a \not\downarrow_C b$ for hyperimaginary elements $a$ and $b$ and set $C$ then there is an imaginary element $d \in \mathrm{acl}(bC)$ such that $a \not\downarrow_C d$.*

**Proposition 2.8.** *Suppose $T$ is simple and has DWIP. Then for every hyperimaginary $e$, there is a set of imaginary elements $D$ such that $D \in \mathrm{dcl}(e)$ and $e \in \mathrm{bdd}(D)$. In particular, $T$ has WEHI.*

*Proof.* Fix any hyperimaginary $e$. Let $C = \mathrm{acl}(e) \cap \mathbb{M}^{\mathrm{eq}}$. For each element $a \in C$, let $a'$ be an imaginary element representing the set of all $e$ automorphic images of $a$. Let $D$ be the set of all tuples $a'$ for $a \in C$.

**Claim.** $C = \mathrm{acl}(D)$.

*Proof of claim.* It is clear $D \subset C$ so $\mathrm{acl}(D) \subset \mathrm{acl}(C) = C$. For the other direction suppose $c \in C$. Then there is by construction a corresponding name $c' \in D$. There is an algebraic formula which says $c$ is in the set named by $c'$. Hence $c \in \mathrm{acl}(D)$. Then $D \in \mathrm{dcl}(e)$. □

**Claim.** *$A \downarrow_D e$ for any set $A$..*

*Proof of claim.* Suppose not. Then there is some set $A$ such that $A \not\downarrow_D e$. By DWIP there is an imaginary element $d \in \mathrm{acl}(D) \cap \mathbb{M}^{\mathrm{eq}}$ such that $A \not\downarrow_D d$. But $d \in C = \mathrm{acl}(D)$. Hence $A \downarrow_D d$. Contradiction proves the claim. □

Thus we may consider the case where $A = e$. Then $e \downarrow_D e$, which implies $e \in \mathrm{bdd}(D)$. □

**Proposition 2.9.** *Weak elimination of hyperimaginaries implies DWIP.*

*Proof.* Suppose $T$ weakly eliminates hyperimaginaries and $a$,$b$ are hyperimaginaries such that $a \not\downarrow_C b$. We need to find a real element $d \in \mathrm{acl}(bC)$ such that $a \not\downarrow_C d$. WEHI gives a set of real elements $B$ such that $\mathrm{bdd}(B) = \mathrm{bdd}(b)$. Hence $a \not\downarrow_C \mathrm{bdd}(B)$, implying $a \not\downarrow_C B$. Then by the finite character of forking, there is a finite tuple $d \in B$ such that $a \not\downarrow_C d$. Since $d \in B \subseteq \mathrm{acl}(b)$ we are done. □

**Corollary 2.10.** *A simple theory $T$ has DWIP if any only if it has weak elimination of hyperimaginaries.*

2.3. **Stable Forking with Rank 2 Elements.** At last, we come to the proof that forking between a finite-rank element and a rank-2 is always witness by a stable formula. The first result along this line comes from the observation that algebraic formulas are always stable.

**Proposition 2.11.** *Let $C \subset \mathbb{M}$, and let $a, b \in \mathbb{M}$ be finite tuples. If $SU(b/C) = 1$ and $a \not\downarrow_C b$ then there are stable forking formulas in both $tp(a/bC)$ and $tp(b/aC)$.*

*Proof.* From $a \not\mathrel{\smile\!\!\!\!|}_C b$, we have $SU(b/aC) < SU(b/C) = 1$, so $b \in \operatorname{acl}(aC) \setminus \operatorname{acl}(C)$. Let $\theta(y; ac) \in tp(b/aC)$ be an algebraic formula. Then $\theta(y; ac)$ forks over $C$ because $b \notin \operatorname{acl}(C)$, and it is stable because it is algebraic. The stable forking formula inside $\operatorname{tp}(b/aC)$ then follows from theorem 2.6. □

We next show stable forking can be passed "upward" through algebraic closure. It is similar to the result of Kim [4] that if $E(y; z)$ is a finite equivalence relation and $\varphi(x; y)$ is any formula then $\exists y[\varphi(x; y) \wedge E(y; z)]$ is stable. For this one claim we do not need to assume $T$ is simple.

**Lemma 2.12.** *Let $T$ be an arbitrary theory. Suppose $\zeta(x; y)$ is a stable formula and $\theta(y; zw)$ is algebraic in $y$. Let $\psi(x; zw)$ be the formula $\exists y[\zeta(x; y) \wedge \theta(y; zw)]$. Then,*

1. *$\psi(x; zw)$ is stable;*
2. *if there are elements $a, b, c, d$ and a set $D$ containing $d$ such that $a \models \zeta(x; c)$, $\zeta(x; c)$ forks over $D$, and $\theta(x; bd)$ isolates $\operatorname{tp}(c/bD)$ (so $c \in \operatorname{acl}(bD)$) then $\psi(x; b)$ forks over $D$.*

*Proof of (1).* Towards a contradiction, suppose $\psi(x; zw)$ is unstable. Let $(a_i b_i c_i : i < \omega + \omega)$ be an indiscernible sequence witnessing the order property – i.e. $\models \psi(a_i; b_j c_j)$ iff $i \leq j$. For $i < \omega$, let $d_i$ be the element witnessed by the existential in $\psi(a_i; b_\omega c_\omega)$. Since $\theta(y; b_\omega c_\omega)$ is algebraic, there are only finitely many possible $d_i$, so by the pigeonhole principle at least one, $d'$, is repeated infinitely often. As $\zeta$ is stable and $(a_i)_{i<\omega+\omega}$ is indiscernible, the set $\{i < \omega+\omega : \models \zeta(a_i; d')\}$ is either finite or cofinite, and by our choice of $d'$, it must be cofinite. Consequently, there are indices $k > \omega$ such that $\models \zeta(a_k; d')$, and this entails $\models \psi(a_k; b_\omega c_\omega)$, a contradiction . □

*Proof of (2).* Let $(c_i)_{i<n}$ enumerate all $n$ elements satisfying $\theta(y; bd)$. We may assume all the $c_i$ have the same type over $bD$ as $c$ since $\theta$ isolates $\operatorname{tp}(c/bD)$. Since $\zeta(x; c)$ forks over $D$, so does $\bigvee_{i<n} \zeta(x; c_i)$. As $\psi(x; bd) \vdash \bigvee_{i<n} \zeta(x; c_i)$, $\psi(x; bd)$ forks. □

We restate the previous lemma to have a more useful form. This corollary is what we mean by stable forking passing "upward" in algebraic closure. The stable forking of $a$ with $d$ is passed to $a$ and $b$.

**Corollary 2.13.** *Let $T$ be an arbitrary theory. Suppose $a$ and $b$ are tuples where $tp(a/Cb)$ forks. Moreover, suppose there is an element $d \in acl(Cb)$ such that $tp(a/Cd)$ forks via a stable formula. Then $tp(a/Cb)$ also contains a stable forking formula.*

**Theorem 2.14.** *Let $T$ be a simple theory with DWIP, and let $C \subset \mathbb{M}^{eq}$. Further, let $a, b \in \mathbb{M}^{eq}$, and assume that $SU(a/C) = 2$ and $SU(b/C) < \omega$. Then, if $a \not\downarrow_C b$ then there is a stable formula in $tp(a/bC)$ which forks over $C$.*

*Proof.* Suppose $a \not\downarrow_C b$. The proof is by induction on the $SU$ rank of $b$ over $C$. If $SU(b/C) = 1$ then proposition 2.11 produces a stable forking formula in $tp(a/bC)$.

Now suppose $SU(b/C) = r < \omega$ and the proposition holds for all elements having smaller rank. Let $\varphi(x; bc)$ with $c \in C$ be an forking formula in $tp(a/bC)$. Choose a sequence $I = (b_i c)_{i \in \omega}$ containing $b$ which is indiscernible over $aC$ and for which $a \notin \operatorname{acl}(IC)$. One way to find such a sequence is to take non-forking extensions of $\operatorname{tp}(b/aC)$. The dependence $a \not\downarrow_C b$ implies $a \not\downarrow_C I$ which gives the rank inequalities $1 \leq SU(a/IC) \leq SU(a/bC) < SU(a/C) = 2$. Hence, $SU(a/IC) = SU(a/bC)$, and so $a \downarrow_{bC} I$. Put $e = \operatorname{Cb}(a/IC)$. Since $a$ and $I$ are independent over $bC$, $e \in \operatorname{bdd}(bC)$. In fact, we could have done the above rank argument with any $b' \in I$ to give $e \in \operatorname{bdd}(b'C)$. However, $b \notin \operatorname{acl}(e)$, since $b \in \operatorname{acl}(e)$ would imply $b \in \operatorname{acl}(b'C)$ for any $b' \in I$, which is impossible since $b$ and $b'$ are both in the same indiscernible sequence $I$.

Hence $SU(b/e) > 0$ and $SU(e/b) = 0$. The Lascar inequalities then show $SU(e) < SU(b)$:

$$SU(e) < SU(b/e) + SU(e) \leq SU(eb) \leq SU(e/b) \oplus SU(b) = SU(b)$$

Both $a \not\downarrow_C I$ and $a \downarrow_e I$ imply $a \not\downarrow_C e$ by transitivity. DWIP gives a finite, real tuple $d \in \operatorname{acl}(e)$ with $a \not\downarrow_C d$. The induction hypothesis gives a stable forking formula inside $tp(a/dC)$ since $SU(dC) \leq SU(eC) < SU(bC)$. Then because $d \in \operatorname{acl}(bC)$ corollary 2.13 gives a stable forking formula in $tp(a/bC)$. □

This proof relies on the fact that $a$ is essentially 1-based with respect to the indiscernible sequence $I$. The main obstacle to extending the proof to elements $a$ with SU-rank larger than 2 is that it is, then, no longer possible to force $\operatorname{Cb}(a/CI)$ to lie inside the bounded closure of a single element of the indiscernible sequence $I$; in general the canonical base just would be inside the bounded closure of $I$.

The observation that 1-basedness is what really makes the induction step work suggested looking at how well this proof generalizes to, well, 1-based theories. Naturally, the argument requires the presence of SU ranks for the induction. In trying to remove other assumptions we found that DWIP, at least, seems essential, but it is sufficient for stable forking of finite-rank elements in a 1-based theory.

We present two approaches, the first one keeps the same argument as in the rank-2 case, and we use the 1-basedness of the theory to eliminate the rank-2 assumption.

The second approach uses a well known coordinatization result for 1-based theories to stand as a replacement for DWIP. The trade-off is that we must assume the entire theory (and not just the elements $a$ and $b$) is finite-ranked.

## 3. A broader application of DWIP

In this section, we will show that the proof for theorem 2.14 has much broader applicability than it would initially seem. More precisely, in that argument, we used an indiscernible sequence $I = (b_i : i < \omega)$ with the following properties:

(1) $I$ is indiscernible over $aC$
(2) $a \downarrow_{b_iC} I$ for all $i < \omega$

There we relied on that fact of $SU(a/C) = 2$ to make such a sequence, but as we noted earlier, this was akin to saying that "$a$ is 1-based with respect to $I$." Here, we will see that this statement is precise; that is, assuming $T$ is 1-based is sufficient to extend the argument to all elements of finite SU-rank.

**Lemma 3.1.** *Suppose $T$ is a simple, 1-based theory. If $a$,$b$ are elements such that $a \not\downarrow_C b$ for some set $C$ then there is sequence $I = (b_i : i < \omega)$ indiscernible over $aC$ such that $b_0 = b$ and satisfying $a \downarrow_{b_iC} I$ for all $i < \omega$.*

*Proof.* Let $a$,$b$,$C$ be as in the statement of the lemma. We will consider "negative" ordinals for indexing purposes, and let $J = (b_i : -|T|^+ \leq i \leq |T|^+)$ be a Morley sequence in $\mathrm{tp}(b/aC)$. By using an automorphism, we may assume $b_0 = b$. Let $L = (b_i : 1 \leq i \leq |T|^+)$ and $K = (b_i : -|T|^+ \leq i \leq -1)$. For the moment we will focus on $L$. By 1-basedness $L$ is Morley over $b_0C$. For some subset $D \subset L$ such that $|D| \leq |T|$ we have $a \downarrow_{b_0CD} L$. Let $L^* = L \setminus D$. Since $L^*$ is Morley over $b_0C$, $D \downarrow_{b_0C} L^*$. Apply transitivity to get $a \downarrow_{b_0C} L^*$. In the same way we find a subset $K^* \subset K$ such that $a \downarrow_{b_0C} K^*$. There is a subset $D \subset K^*$ such that $a \downarrow_{b_0CL^*D} K^*$, by replacing $K^*$ with $K^* \setminus D$ we may assume $a \downarrow_{b_0C} K^*b_0L^*$ (note that in either case we have $|K^*| = |T|^+$). Let $I' = K^*{}^\frown b_0{}^\frown L^*$. $I'$ is indiscernible over $aC$, which implies for each $b_i \in I$, $a \downarrow_{b_iC} I'$. Taking $I = (b_i : 0 \leq i < \omega) \subset I'$ then works for the conclusion of the lemma. □

Now we have the theorem.

**Theorem 3.2.** *Let $T$ be a 1-based supersimple theory. Let $C$ be a set. Suppose $a$ is an imaginary element and $b$ is an arbitrary imaginary element with $SU(b/C) < \omega$. If $a \not\downarrow_C b$ then there is a stable formula in $tp(a/bC)$ which forks over $C$.*

*Proof.* This proof closely follows the proof of theorem 2.14. Suppose $a \not\downarrow_C b$. We proceed by induction on $SU(b/C)$. If $SU(b/C) = 1$ then proposition 2.11 produces a stable forking formula in $tp(a/bC)$. Now suppose $SU(b/C) = \alpha$ and the proposition holds for all elements having smaller rank (over $C$). Choose a forking formula $\varphi(x; bc) \in \mathrm{tp}(a/bC)$ (for some $c \in C$), and a sequence $I = (b_i c)_{i\in\omega}$ containing $b$ which is indiscernible over $aC$ and for which $a \notin \mathrm{acl}(IC)$ and $a \downarrow_{bC} I$; such a sequence exists by lemma 3.1. Let $e = \mathrm{Cb}(a/IC)$. $e \in \mathrm{bdd}(bC)$ since $a \downarrow_{bC} IC$. In fact, the above rank argument could be repeated with any $b' \in I$ to give $e \in \mathrm{bdd}(b'C)$. However, $b \notin \mathrm{acl}(e)$, since $b \in \mathrm{acl}(e)$ would imply $b \in \mathrm{acl}(b'C)$ for any $b' \in I$, which is impossible since $b$ and $b'$ are both in the same indiscernible sequence $I$. Hence $SU(b/e) > 0$ and $SU(e/b) = 0$. The Lascar inequalities then show $SU(e) < SU(b)$. $a \not\downarrow_C I$, and $a \downarrow_e I$ imply $a \not\downarrow_C e$. DWIP produces a finite, real tuple $d \subset e$ with $a \not\downarrow_C d$. Note that $SU(d) \leq SU(e)$ and $d \in \mathrm{acl}(bC)$. Since $SU(dC) < SU(bC)$, apply the induction hypothesis to get a stable forking formula inside $tp(a/\hat{e}C)$. Then corollary 2.13 gives a stable forking formula in $tp(a/bC)$. □

## 4. Stable forking via coordinatization

In this section, we will prove 1-based theories of finite SU-rank have stable forking using a coordinatization lemma and avoiding any mention of hyperimaginaries. This coordinatization lemma is a fairly well-known fact, exposed in [2] and also to be found in [8], and it is the engine that drives many results on theories of finite $SU$-rank.

**Fact 4.1.** [2, Lemma 3.1] *Assume $T$ is supersimple of finite rank and 1-based. Let $a \in \mathbb{M}^{eq}$ and $A \subset \mathbb{M}^{eq}$ such that $SU(a/A) = \alpha + 1$. Then $SU(a/Ab) = 1$ for some $b \in acl^{eq}(a)$. (And by the Lascar inequalities, $SU(b/A) = \alpha$.)*

**Theorem 4.2.** *Assume $T$ is a 1-based simple theory of finite $SU$-rank. Let $a, b \in \mathbb{M}^{eq}$, $C \subset \mathbb{M}$, $SU(b/C) < \omega$ and suppose $a \not\downarrow_C b$. Then there is a formula $\psi(x; d) \in \mathrm{tp}(a/bC)$ such that $\psi(x; y)$ is stable and $\psi(x; d)$ forks over $C$.*

*Proof.* The proof is by induction on $SU(b/C)$.

If $C$ is an arbitrary base set and $b$ is an arbitrary tuple such that $SU(b/C) = 1$ then by proposition 2.11, there is such a stable forking formula inside $\mathrm{tp}(a/bC)$.

Now suppose the conclusion holds for all base sets $C$ and all elements $b$ with $SU(b/C) < \alpha$. Let $C$ be a set and $b$ a tuple such that $SU(b/C) = \alpha$ and $a \not\downarrow_C b$. Since $SU(b/C) > 1$ fact 4.1 finds an element $d \in \mathrm{acl}^{\mathrm{eq}}(b)$ such that $SU(b/dC) = 1$ and $SU(b/C) = SU(d/C) + 1$. There are now two possible cases: either $a$ and $d$ are dependent over $C$, or they are independent. If $a \not\downarrow_C d$ then since $SU(d/C) < SU(b/C)$ by induction there is a stable forking formula inside $\mathrm{tp}(a/Cd)$. Since $d \in \mathrm{acl}(bC)$, there is a stable forking formula inside $\mathrm{tp}(a/bC)$ and we are done.

For the other case $a \downarrow_C d$. By transitivity of forking, we must have $a \not\downarrow_{Cd} b$, and $SU(b/Cd) \leq 1 < \alpha$. By induction there is a stable formula $\psi(x; y)$ such that some instance $\psi(x; bdc)$ is in $\mathrm{tp}(a/bdC)$ and $\psi(x; bdc)$ forks over $Cd$. Note that $\psi(x; bdc)$ still forks over $C$, and is in $\mathrm{tp}(x; bdC)$. Since $d \in \mathrm{acl}(bC)$ there is some algebraic formula $\theta(w; yz)$ and tuple $c' \subset C$ such that $d \models \theta(w; bc')$. In what follows we may assume $c = c'$. Let $\varphi(x; yz)$ be the formula $\exists w[\psi(x; ywz) \wedge \theta(w; yz)]$. It's routine to verify that $\varphi(x; yz)$ is stable, so we need only show that $\varphi(x, bc)$ forks over $C$. To show this, suppose $\{d_1, ..., d_n\}$ is a maximal set of realizations of $\theta(w; bc)$. Now, $\varphi(x; bc) \models \bigvee_i \psi(x; bd_ic)$, and the disjunction forks over $C$; hence, $\varphi(x; bc)$ forks over $C$, as required. $\square$

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