Adaptive Precision in Homotopy Continuation

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The situation

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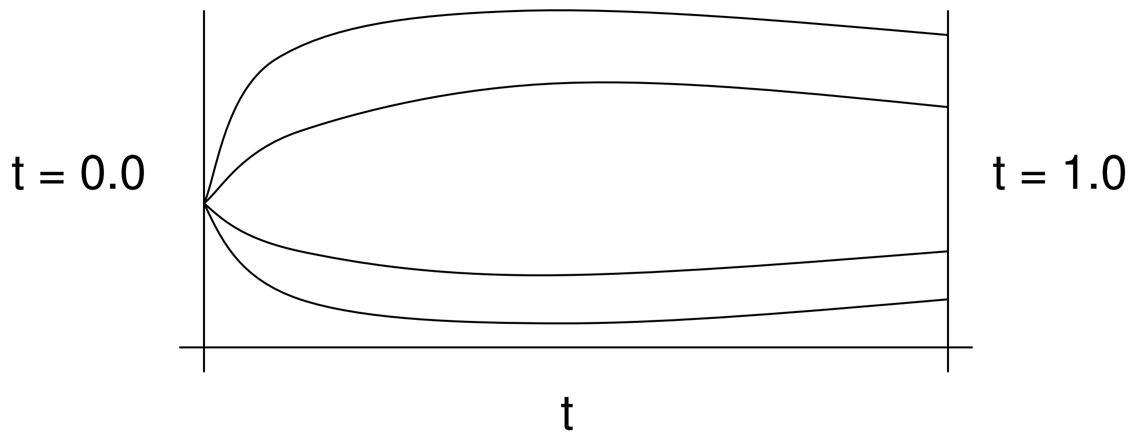
Examples

(1)Multiple endpoints



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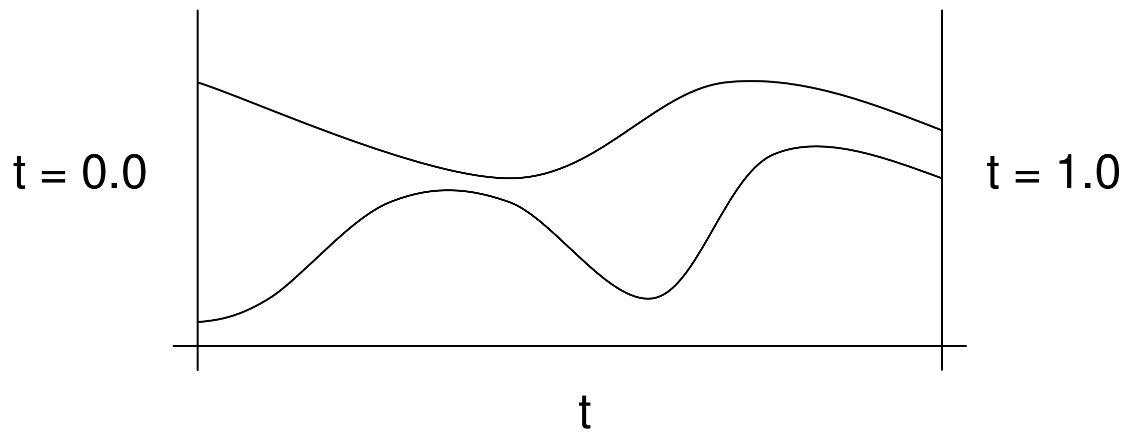
(2) Path-crossing



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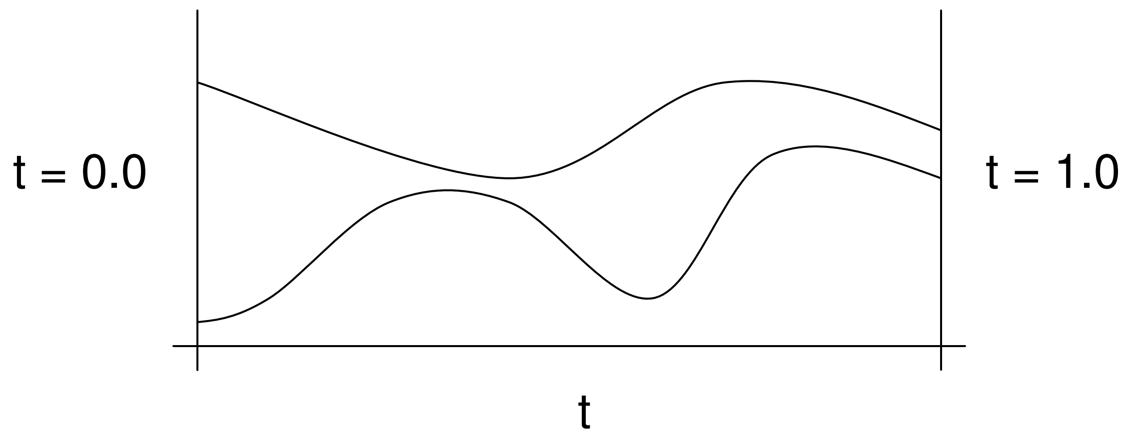

One solution to basic numerical problems

Increase precision. Currently done by re-running part or all of the path at higher precision.

The situation

Difficulty with this solution

- Needs to be made automatic (not difficult).
- Can be very expensive (high precision is expensive!), especially if many levels of precision are used.
- Leads to failure if precision higher than pre-chosen threshold is needed.

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To resolve some of the issues above, we have developed a method for changing precision as needed.

<u>NOTE</u>: Adaptive precision tracking does NOT always solve the path-crossing problem! More on that later….

Outline

I. Homotopy continuation.

II. Facts about numerical linear algebra.

III. Criteria for determining when more precision is needed.

IV. Adaptive precision path-tracking.

V. Conclusions.

Homotopy Continuation

Given a polynomial system F: $\mathbb{C}^N \rightarrow \mathbb{C}^N$.

- Cast F(z) in a family of systems, one of which is easy to solve, yielding a homotopy H(z, t): $\mathbb{C}^N \times \mathbb{C} \rightarrow \mathbb{C}^N$.

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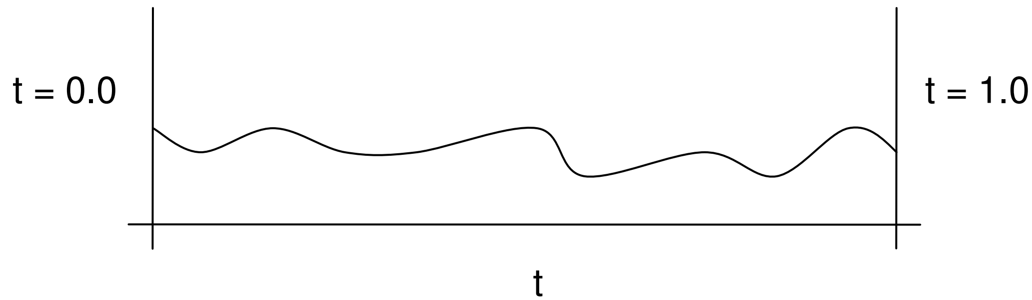

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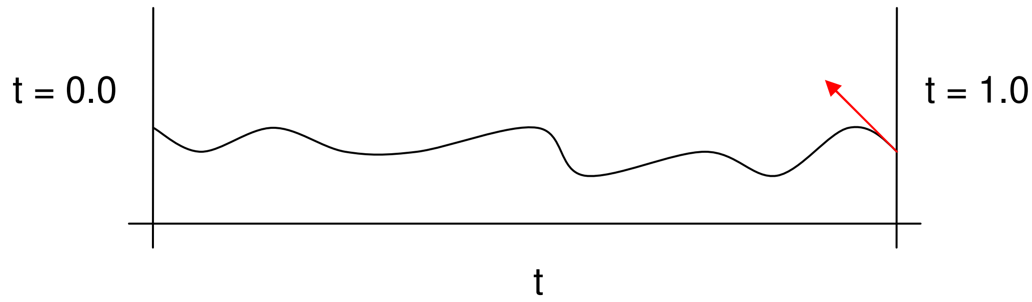

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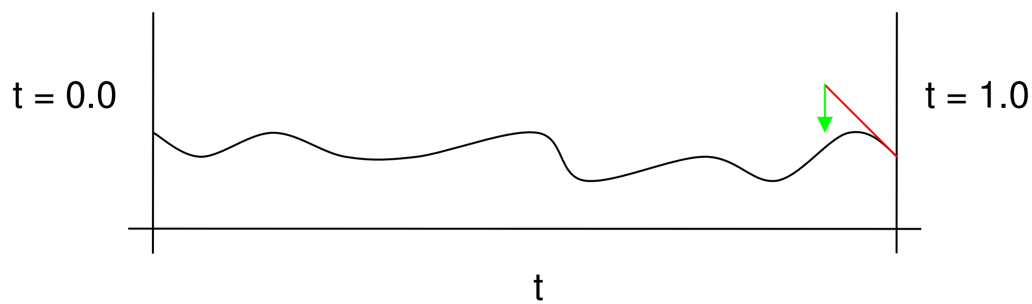

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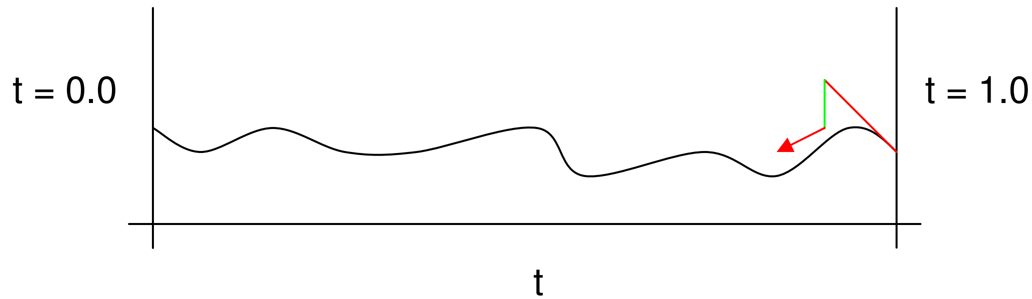

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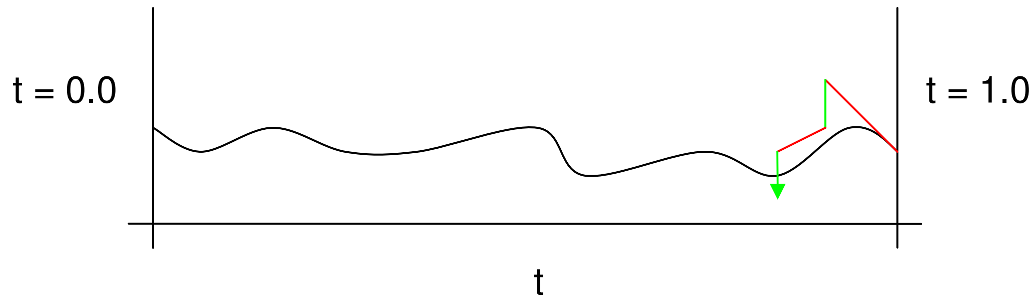

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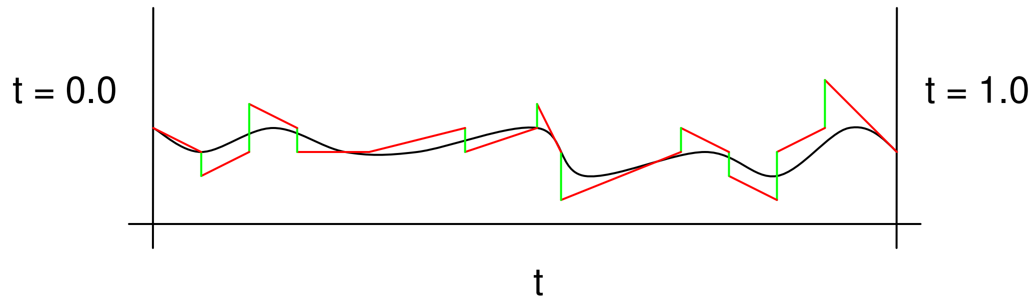

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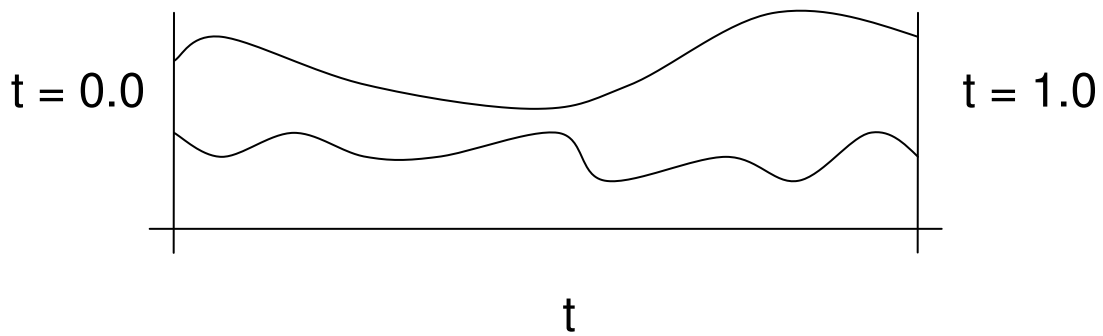

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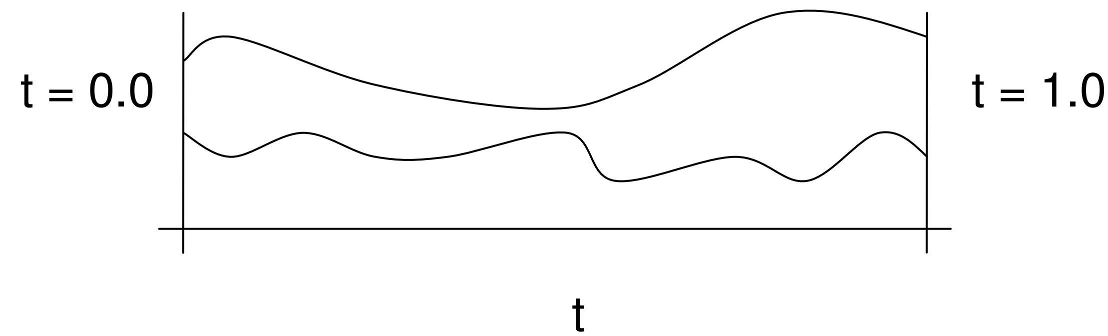

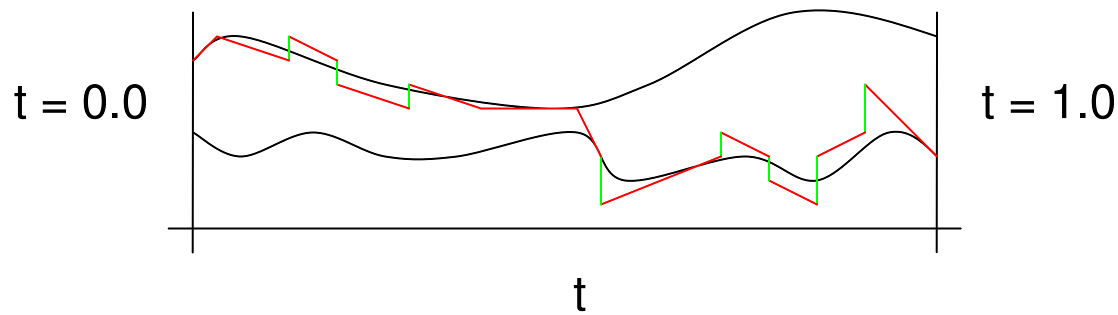

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$$H(z+\Delta z, t+\Delta t) \approx H(z, t) + J(z, t)\Delta z + H_t(z, t)\Delta t$$

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Adaptive steplength:

-Increase Δt after N consecutive successes.

-Decrease Δt after one failure.

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SUCCESS

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FAILURE

- Detect that path is going to ∞.
- Δt drops below some threshold.
- "Too many" steps have been taken.

Wilkinson's result (see [BF])

When solving $Ax = b$ for x, we have that

$$\frac{||x - x'||}{||x||} \leq K(A)\frac{||r||}{||b||}.$$

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The Bottom Line:

accurate digits ≈ # digits – CN, where CN is the exponent of the condition number.

Criterion I

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Safety digits

CN is computed approximately, so there is some uncertainty about CN. Also, never want to get close to PREC = CN, so we have:

(A) PREC – CN > SD_1

Criterion II

Another obvious one:

(B) PREC – TOL > SD_2

This means that we must have enough working digits to represent the data to the desired accuracy.

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Need enough accurate digits and a good enough predictor step so that there is hope to attain the desired final tolerance after some fixed number of corrector steps.

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x = _ . _ _ _ _ _ █ … █ (CR=5 good digits)

Δx = 0 . 0 0 0 0 _ _ _ █ … █ (PREC-CN = 3)

Δx = 0 . 0 0 0 0 0 0 0 _ _ _ █ … █ (same)

So, after a prediction and 2 corrections, we have at most 11 correct digits.

Criterion III (continued)

So the rule is:

(C) CR+ItsLeft*(CN-PREC) > TOL+SD_3

Here, CR is the corrector residual and ItsLeft is the number of Newton steps remaining. For more safety, set ItsLeft to 1.

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There are several other possible criteria, but we have found that each reduces to a combination of (A), (B), and (C).

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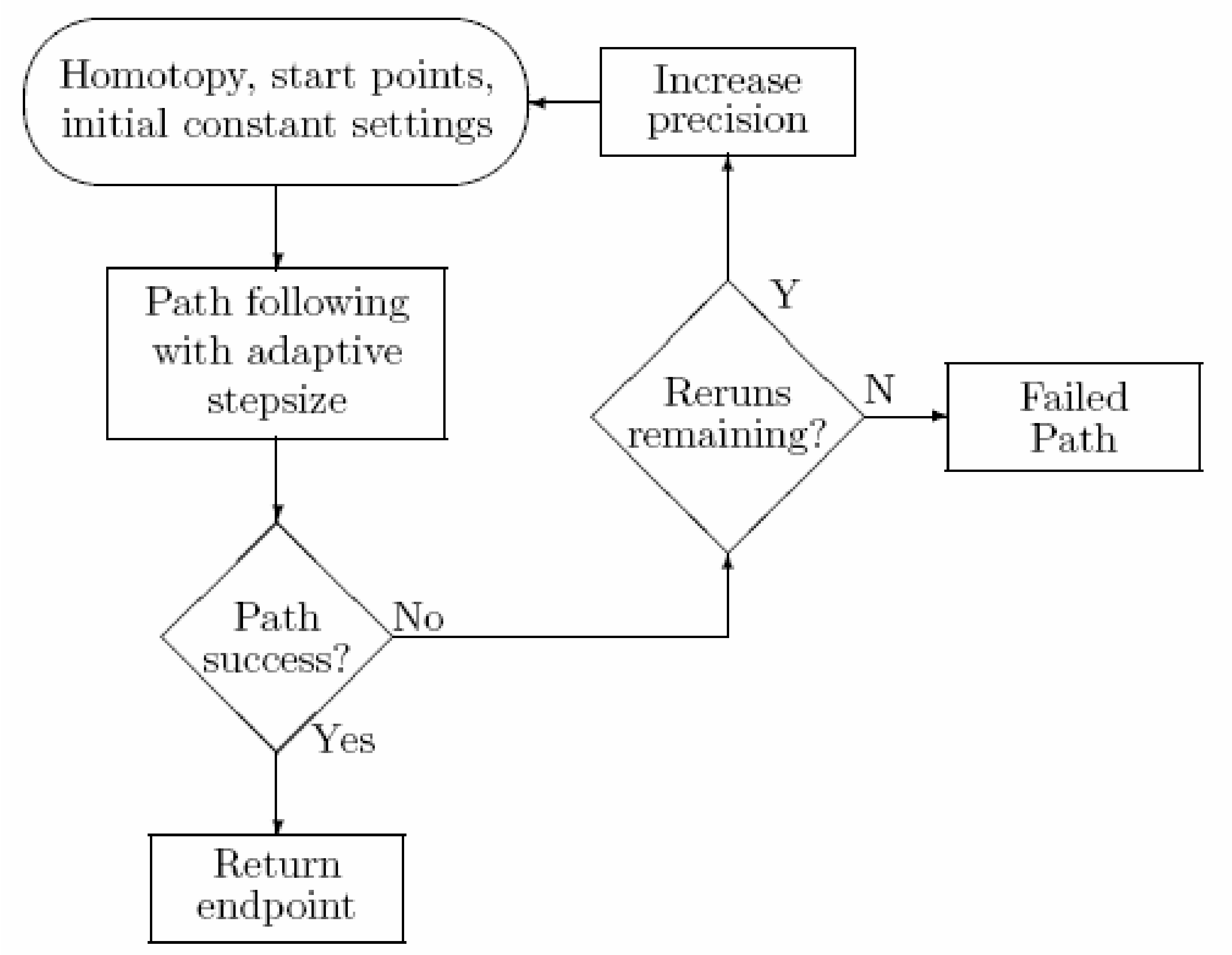

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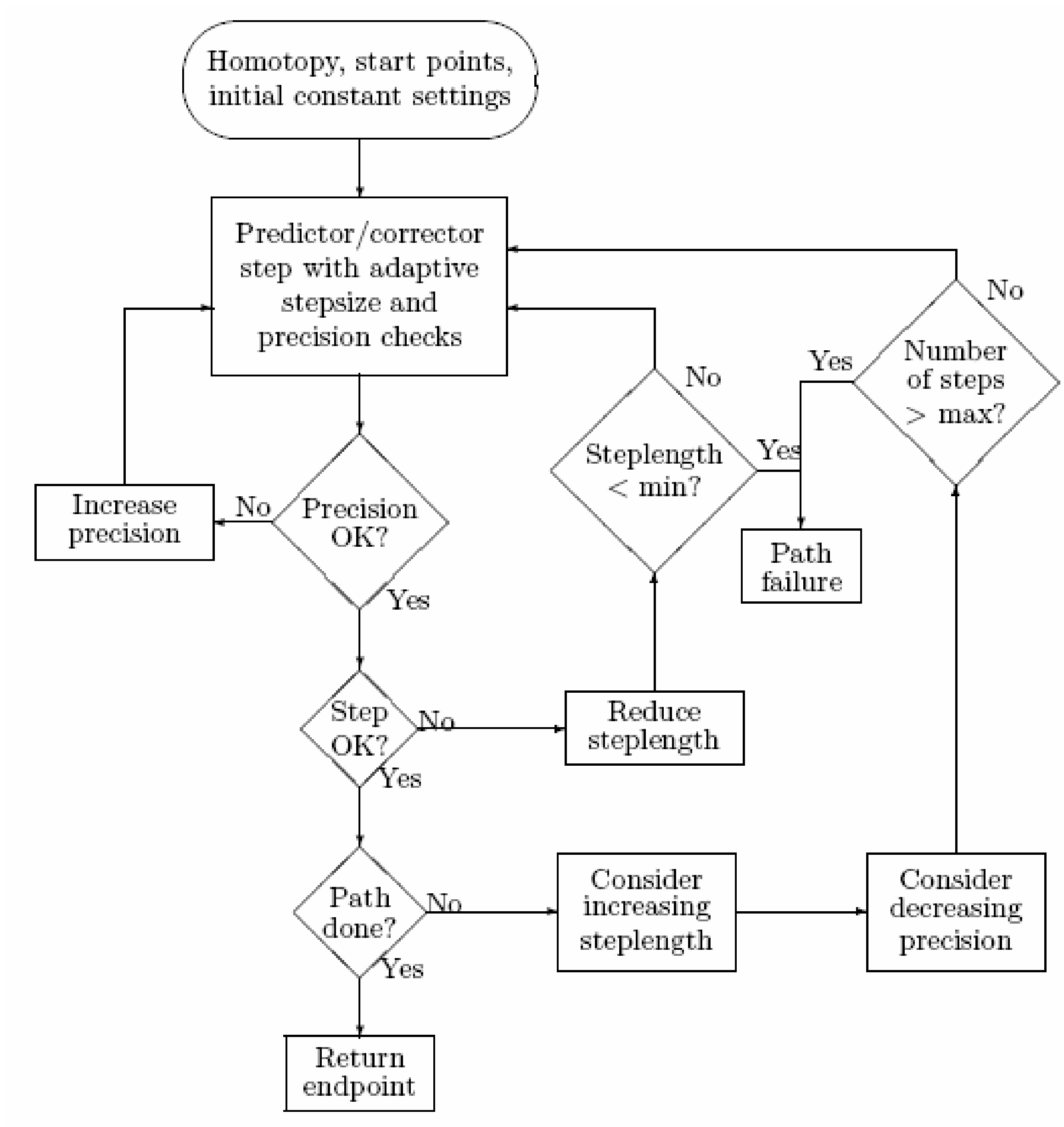

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- Adaptive precision, with endgames and deflation (see [LVZ]), makes it possible to track along singular components.

Works Cited

[BSW] D. Bates, A. Sommese, and C. Wampler. Adaptive precision path-tracking. Preprint. (available on my website, www.nd.edu/~dbates1/)

[B] T. Brown. flow (GNU software for flowcharts).

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[LVZ] A. Leykin, J. Verschelde, and A. Zhao. Newton's method with deflation for isolated singularities of polynomial systems. Preprint.

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Thank you to the organizers for having me and all of you for coming to the earliest talk!