Solving Boundary Value Problems with Homotopy Continuation

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Motivation

We have:

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2) Homotopy continuation techniques, as well as an implementation.

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2) Homotopy continuation techniques, as well as an implementation.

Main idea: Use 2) to solve 1).

Outline

I. The Problem.
II. Sketch of the new method.
III. A few details of the method.
IV. Filters: A way to save some work.
V. A few numerical results.
VI. Potential Extensions.

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The general problem

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

The problem we wish to consider is a two-point boundary value problem on $[a, b] \subset \mathbb{R}$,

$$\begin{aligned} y'' &= f(x, y, y') \\ y(a) &= \alpha \\ y(b) &= \beta \end{aligned} \qquad (1)$$

In general, the number of solutions could be 0, 1, finitely many, or infinitely many.

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The standard discretization

I. The Problem.

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We may form the standard central difference approximation on a uniform mesh:

Let:

$h := \frac{b-a}{N+1}$ (N a positive integer),

$x_i := a + ih$ for $i = 0, \ldots, N+1$,

$y_0 = \alpha$, and

$y_{N+1} = \beta$.

The standard discretization

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Then we get the system $\mathcal{D}_N$:

$$\begin{array}{llll} y_0 & -2y_1 & +y_2 & = h^2 f(x_1, y_1, \frac{y_2 - y_0}{2h}) \\ \vdots & \vdots & \vdots & = \quad \vdots \\ y_{N-1} & -2y_N & +y_{N+1} & = h^2 f(x_N, y_N, \frac{y_{N+1} - y_{N-1}}{2h}) \end{array}$$

Solutions of the two-point boundary value problem (1) are approximated by vectors $\bar{y} \in \mathbb{R}^N$ that approximately satisfy the above system.

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Related ideas

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

If f is sufficiently smooth, we know that solutions of (1) will eventually be approximated with $\mathcal{O}(h^2)$ accuracy, as $h \to \infty$.

There are many numerical methods treating two-point BVPs, but these methods require good starting points, which our method may provide.

The basic algorithm

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

1. Find all solutions of $\mathcal{D}_N$ for some small N (details in a moment).

The basic algorithm

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VI. Extensions

1. Find all solutions of $\mathcal{D}_N$ for some small N.
2. Filter out solutions which are somehow "bad". Call the remaining solutions $\mathcal{V}_N$ (details in section IV).

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The basic algorithm

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VI. Extensions

1. Find all solutions of $\mathcal{D}_N$ for some small N.
2. Filter out solutions which are somehow "bad". Call the remaining solutions $\mathcal{V}_N$.
3. If we are satisfied with our current solution set (i.e., if the cardinality of $\mathcal{V}_N$ is stable and the mesh is adequately small), go to step 4. Otherwise, use continuation to generate solutions of $\mathcal{D}_{N+1}$ and go to 2.

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4. Refine solutions, as desired.

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4. Refine solutions, as desired.

Step 3 is the focus of the paper.

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The homotopy

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

Add $\mathcal{D}_N$,

$$\begin{array}{lllll}
\alpha & -2y_1 & + & y_2 & = h_1^2 f(x_1, y_1, \frac{y_2 - y_0}{2h}) \\
\vdots & \vdots & & \vdots & = \quad \vdots \\
y_{N-1} & -2y_N & + & \beta & = h_1^2 f(x_N, y_N, \frac{y_{N+1} - y_{N-1}}{2h})
\end{array}$$

times t to $(1-t)$ times $\mathcal{D}_{N+1}$,

$$\begin{array}{lllll}
\alpha & -2y_1 & + & y_2 & = h_2^2 f(x_1, y_1, \frac{y_2 - y_0}{2h}) \\
\vdots & \vdots & & \vdots & = \quad \vdots \\
y_N & -2y_{N+1} & + & \beta & = h_2^2 f(x_{N+1}, y_{N+1}, \frac{y_{N+2} - y_N}{2h})
\end{array}$$

to form the homotopy:

The homotopy

I. The Problem.

II. Sketch of method.

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VI. Extensions

$$H_{N+1}(y_1, \ldots, y_{N+1}, t) :=$$

$$\begin{bmatrix} y_0 - 2y_1 + y_2 & - & h(t)^2 f\left(x_1(t), y_1, \frac{y_2 - y_0}{2h(t)}\right) \\ & \vdots & \\ y_{N-2} - 2y_{N-1} + y_N & - & h(t)^2 f\left(x_{N-1}(t), y_{N-1}, \frac{y_N - y_{N-2}}{2h(t)}\right) \\ y_{N-1} - 2y_N + Y_{N+1}(t) & - & h(t)^2 f\left(x_N(t), y_N, \frac{Y_{N+1}(t) - y_{N-1}}{2h(t)}\right) \\ y_N - 2y_{N+1} + Y_{N+2}(t) & - & h(t)^2 f\left(x_{N+1}(t), y_{N+1}, \frac{Y_{N+2}(t) - y_N}{2h(t)}\right) \end{bmatrix}$$

with

$$\begin{aligned} y_0 &:= \alpha \\ h(t) &:= t\left(\frac{b-a}{N+1}\right) + (1-t)\left(\frac{b-a}{N+2}\right) \\ Y_{N+1}(t) &:= (1-t)y_{N+1} + \beta t \\ Y_{N+2}(t) &:= \beta(1-t) \\ x_i(t) &:= a + ih(t), \qquad i = 1, \ldots, N+1. \end{aligned}$$

This is the homotopy that we use, modulo a couple details which may be found in the paper.

Main aspects of the homotopy

- At t=0, we have the discretization $\mathcal{D}_{N+1}$.

I. The Problem.

II. Sketch of method.

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VI. Extensions

Main aspects of the homotopy

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

- At t=0, we have the discretization $\mathcal{D}_{N+1}$.
- At t=1, we have the discretization $\mathcal{D}_N$, PLUS the final equation

$$y_N - 2y_{N+1} = h(t)^2 f\left(b, y_{N+1}, \frac{-y_N}{2h}\right)$$

which must be solved for y_{N+1} in order to expand the solutions of $\mathcal{D}_N$ in $\mathcal{V}_N$ to start points (of N+1 coordinates) for the homotopy.

Main aspects of the homotopy

I. The Problem.

II. Sketch of method.

III. Details.

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VI. Extensions

- At t=0, we have the discretization $\mathcal{D}_{N+1}$.

- At t=1, we have the discretization $\mathcal{D}_N$, PLUS the final equation

$$y_N - 2y_{N+1} = h(t)^2 f\left(b, y_{N+1}, \frac{-y_N}{2h}\right)$$

which must be solved for y_{N+1} in order to expand the solutions of $\mathcal{D}_N$ in $\mathcal{V}_N$ to start points (of N+1 coordinates) for the homotopy.

- For this final reason, we from now on restrict our attention to the case in which $f(x, y, y')$ is a real polynomial $p(y)$, since the final equation is easily solved in that case.

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A potential problem….

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

Let $d = \deg p(y)$.

By Bézout's theorem, if all tracking goes smoothly, we would expect d^N solutions in the set $\mathcal{V}_N$.

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Let $d = \deg p(y)$.

By Bézout's theorem, if all tracking goes smoothly, we would expect d^N solutions in the set $\mathcal{V}_N$.

<u>Example</u>:

$$y'' = -\lambda\left(1 + y^2\right),\ \lambda > 0$$
$$y(0) = 0$$
$$y(1) = 0$$

For $N = 17$, it took nearly an hour to track all 2^{17} paths and would take more than a year to track all of the paths for $N = 30$!

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A potential solution.

If we know something about the solutions (symmetry, positivity, oscillatory nature, etc.), we may filter out "bad" solutions that do not exhibit that behavior.

A potential solution.

I. The Problem.

II. Sketch of method.

III. Details.

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VI. Extensions

If we know something about the solutions (symmetry, positivity, oscillatory nature, etc.), we may filter out "bad" solutions that do not exhibit that behavior.

BEWARE: Some potential solutions which eventually converge to actual solutions do not resemble the final solution until N >> 0. More research is needed!

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Bertini

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

Bertini is a new polynomial system solver under development by Bates, Sommese, and Wampler, with some early work by Chris Monico (Texas Tech University).

Bertini will be able to compute the numerical irreducible decomposition of polynomial systems, using adaptive multiprecision. It should be released later this year.

For this project, we used the homotopy continuation module of Bertini with some data management in Maple.

A few examples.

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

1. $$y'' = -\lambda\left(1+y^2\right),\ \lambda > 0$$
$$y(0) = 0$$
$$y(1) = 0$$

Theoretically, we have:

- 2 real solutions for λ > 4.
- 1 real solution for λ = 4.
- 0 real solutions for λ < 4.

We were able to confirm these results for λ = 2 and λ = 6 to N=17. We made it to N=100 using a filter based on symmetry about the midpoint.

A few examples, cont'd.

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

2. $$y'' = -\lambda y^3, \ \lambda > 0$$
$$y(0) = 0$$
$$y(1) = 0$$

From theory, there are infinitely many real solutions.

We found:

# mesh pts	2	3	4	5	6	7	8
# solutions	3	3	7	11	23	47	91

A few examples, cont'd.

3. $$y'' = -\lambda e^{y},\ \lambda > 0$$
$$y(0) = 0$$
$$y(1) = 0$$

(the Bratu problem)

For polynomial nonlinearity, we use a truncated power series:

$$y'' = -\lambda\left(1 + y + \frac{y^2}{2}\right)$$

From theory, we expect 2 real solutions if $\lambda \approx 0$ and none if $\lambda >> 0$. We confirmed this for the values $\lambda = 0.5$ and $\lambda = 10$.

Ideas for extensions

I. The Problem.

II. Sketch of method.

III. Details.

IV. Filters.

V. Numerical results.

VI. Extensions

1. Filters need to be studied.

Ideas for extensions

1. Filters need to be studied.
2. Add multiple mesh points at each step, at various points in the interval.

Ideas for extensions

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VI. Extensions

1. Filters need to be studied.
2. Add multiple mesh points at each step, at various points in the interval.
3. Use non-uniform grids.

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Ideas for extensions

I. The Problem.

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VI. Extensions

1. Filters need to be studied.
2. Add multiple mesh points at each step, at various points in the interval.
3. Use non-uniform grids.
4. Extend this method to work for systems of ODEs.

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