Price Level Targeting and Stabilization Policy*

Aleksander Berentsen
University of Basel

Christopher Waller
University of Notre Dame
FRB St. Louis

July 18, 2008

Abstract

We construct a dynamic stochastic general equilibrium model to study optimal monetary stabilization policy. Prices are fully flexible and money is essential for trade. Our main result is that if the central bank pursues a price-level target, it can control inflation expectations and improve welfare by stabilizing short-run shocks to the economy. The optimal policy involves smoothing nominal interest rates which effectively smooths consumption across states.

*The paper has also benefitted from comments by participants at several seminar and conference presentations. Much of the paper was written while Berentsen was visiting the University of Pennsylvania. We also thank the Federal Reserve Bank of Cleveland and the CES in Munich for research support.

1 Introduction

A key objective of modern central banking is to keep inflation low and stable over some 'long' time horizon. However, central banks are also concerned with stabilizing the real economy in the 'short' run. Balancing these two objectives is a complex policy task for central bankers and thus there is an obvious need for economic models to guide central bankers in making informed policy choices.

After a long period of inactivity, the last decade has seen a tremendous resurgence of research focusing on how to conduct stabilization policy in the face of temporary shocks when there is a desire to keep inflation low and stable in the long-run. Nearly all of this work has come from the New Keynesian literature which, in the tradition of real business cycle models, constructs dynamic stochastic general equilibrium models to study optimal stabilization policy. What separates New Keynesian (NK) models from real business cycle models is their reliance on nominal rigidities, such as price or wage stickiness, that allows monetary policy to have real effects. A key policy recommendation coming out of NK models is that "good" monetary policy requires guiding inflation expectations in an appropriate manner.[1] In order to do so, it is often advocated that the central bank adopt some version of a price-level target or an inflation targeting. Which targeting approach is better is still an open question.

NK models typically are 'cashless' in the sense that there are no monetary trading frictions. Instead, the driving friction is some type of nominal rigidity but there has been considerable debate as to what nominal object should be rigid (output price, input price or nominal wage) as well as how the rigidity occurs (Calvo, Taylor, menu cost).

In this paper, we sidestep this debate on nominal rigidity and take the opposite approach – we assume all prices are flexible but there is a trading friction that money overcomes. Our objective is to study stabilization policy in this framework in order to compare it to policy prescriptions coming out of NK models. we construct a dynamic stochastic general

[1] See, for example, Woodford (2003) Chapters 1 and 7. Also see Clarida, Gali and Gertler (1999) p. 1663.

equilibrium model where money is essential for trade and prices are fully flexible.[2] There are shocks to preferences and technology. The existence of a credit sector generates a nominal interest rate that the monetary authority manipulates in its attempt to stabilize these shocks. The monetary authority maximizes the expected lifetime utility of the representative agent subject to the allocation being a competitive equilibrium and a given long-run inflation target.

We show that the monetary friction is enough to generate a welfare improving role for stabilization policy even if prices are fully flexible. We demonstrate that the critical element for effective stabilization policy is the central bank's control of long-run inflation expectations via price level targeting. By doing so monetary policy has real effects even though prices are fully flexible – a prescription similar to that of NK models. Whereas, controlling inflation expectations makes a central bank's stabilization response to aggregate shocks more effective in the NK models, our model makes a much stronger case for controlling expectations – failure to do so makes stabilization policy is completely neutral.

The characteristics of the optimal stabilization policy are the following. It involves smoothing nominal interest rates and *individual* consumption across states. When the marginal cost of production is roughly constant, optimal policy is procyclical. An interesting implication of this policy is that the central bank is essentially providing an elastic supply of currency – when demand for liquidity is high, it provides additional currency and withdraws it when the demand for liquidity is low. Furthermore, stabilization works through a liquidity effect. By injecting money the central bank lowers nominal interest rates, stimulating borrowing, which leads to higher consumption and production. On the other hand, if the central bank does not follow its targeted long-run price path, these injections simply raise inflation expectations and the nominal interest rate, as predicted by the Fisher equation.

The paper proceeds as follows. In Section 2 we describe the environment. In Section 3 agents' optimization problems are presented and in Section 4 we derive the first-best

[2]By essential we mean that the use of money expands the set of allocations (Kocherlakota (1998) and Wallace (2001)).

allocation. In Section 5 we present the central bank's stabilization problem and derive the policy response to shocks. Section 6 contains discussion of the results and Section 7 concludes.

2 The Environment

The basic environment is that of Berentsen, Camera, and Waller (2007) which builds on Lagos and Wright (2005). We use the Lagos-Wright framework because it provides a microfoundation for money demand and it allows us to introduce heterogenous preferences for consumption and production while keeping the distribution of money balances analytically tractable. Time is discrete and in each period there are three perfectly competitive markets that open sequentially.[3] Market 1 is a credit market while markets 2 and 3 are goods markets. There is a $[0,1]$ continuum of infinitely-lived agents and one perishable good produced and consumed by all agents.

At the beginning of the period agents receive a preference shock such that they either consume, produce or neither in the second market. With probability n an agent consumes, with probability s he produces and with probability $1-n-s$ he does neither. We refer to consumers as buyers and producers as sellers.

In the second market buyers get utility $\varepsilon u(q)$ from $q > 0$ consumption, where ε is a preference parameter and $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$ and $u'(\infty) = 0$. Furthermore, we impose that the elasticity of utility $e(q) = \frac{qu'(q)}{u(q)}$ is bounded. Producers incur utility cost $c(q)/\alpha$ from producing q units of output where α is a measure of productivity. We assume that $c'(q) > 0$, $c''(q) \geq 0$ and $c'(0) = 0$.

Following Lagos and Wright (2005) we assume that in the third market all agents consume and produce, getting utility $U(x)$ from x consumption, with $U'(x) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) \leq 0$.[4] Agents can produce one unit of the consumption good with

[3] Competitive pricing in the Lagos-Wright framework is a feature in Rocheteau and Wright (2005) and Berentsen, Camera, and Waller (2005).

[4] As in Lagos and Wright (2005), these assumptions allow us to get a degenerate distribution of money

one unit of labor which generates one unit of disutility. The discount factor across dates is $\beta \in (0, 1)$.

To motivate a role for fiat money, we assume that all goods trades are anonymous. In particular, trading histories of agents are private information, which rules out trade credit. Consequently, sellers require immediate compensation so buyers must pay with money. There is also no public communication of individual trading outcomes (public memory), which eliminates the use of social punishments to support gift-giving equilibria.

The first market is a credit market where agents can borrow or lend money at the nominal interest rate i. In contrast to the goods market, we assume the existence of a record-keeping technology so that financial trading histories are not private information. In all models with credit default is a serious issue. To focus on optimal stabilization, we simplify the analysis by assuming that some mechanism exists that ensures repayment of loans in the third market.[5] One can show that due to the quasi-linearity of preferences in market 3 there is no gain from multi-period contracts. Furthermore, since the aggregate states are revealed prior to contracting, the one-period nominal debt contracts that we consider are optimal.

2.1 Shocks

To study the optimal response to shocks, we assume that n, s, α and ε are stochastic. The random variable n has support $[\underline{n}, \overline{n}] \in (0, 1/2]$, s has support $[\underline{s}, \overline{s}] \in (0, 1/2]$, α has support $[\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} < \overline{\alpha} < \infty$, and ε has support $[\underline{\varepsilon}, \overline{\varepsilon}]$, $0 < \underline{\varepsilon} < \overline{\varepsilon} < \infty$. Let $\omega = (n, s, \alpha, \varepsilon) \in \Omega$ be the aggregate state in market 1, where $\Omega = [\underline{n}, \overline{n}] \times [\underline{s}, \overline{s}] \times [\underline{\alpha}, \overline{\alpha}] \times [\underline{\varepsilon}, \overline{\varepsilon}]$ is a closed and compact subset on $\boldsymbol{R}^4_+$. The shocks are serially uncorrelated. Let $f(\omega)$ denote the density function of ω.

holdings at the beginning of a period. The different utility functions $U(.)$ and $u(.)$ allow us to impose technical conditions such that in equilibrium all agents produce and consume in the last market.

[5] One possibility would be that agents require a particular 'tool' to be able to consume in market 2. This tool can then be used as collateral against loans in market 1 so that for sufficiently high discount factors repayment occurs with probability one.

In Berentsen et al. (2007) we derive the equilibrium when the only punishment for strategic default is exclusion from the financial system in all future periods.

Shocks to n and ε are thought of as aggregate demand shocks, while shocks to s and α are aggregate supply shocks. We call shocks to ε and α intensive margin shocks since they change the desired consumption of each buyer and the productivity of each seller, respectively, without affecting the number of buyers or sellers. In contrast, shocks to n and s affect the number of buyers and sellers. Although we call these aggregate shocks, there are actually sectoral shocks since they do not affect demand or productivity in the third market. Nevertheless, as we see below, output in market 3 is constant so any volatility in total output per period is driven by shocks in market 2.

2.2 Monetary Policy

Monetary policy has a long and short-run component. The long-run component focuses on the trend inflation rate. The short-run component is concerned with the stabilization response to aggregate shocks.

We assume a central bank exists that controls the supply of fiat currency. We denote the gross growth rate of the money supply by $\gamma = M_t/M_{t-1}$ where M_t denotes the per capita money stock in market 3 in period t. The central bank implements its long-term inflation goal by providing deterministic lump-sum injections of money, τM_{t-1}, at the beginning of the period. These transfers are given to the private agents. The net change in the aggregate money stock is given by $\tau M_{t-1} = (\gamma - 1)M_{t-1}$. If $\gamma > 1$, agents receive lump-sum transfers of money. For $\gamma < 1$, the central bank must be able to extract money via lump-sum taxes from the economy. For notational ease variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1. There is an initial money stock $M_0 > 0$.

Throughout this paper, we will assume that γ determines the long-run desired inflation rate and that, for unspecified reasons, $\gamma > \beta$, i.e., the central bank does not run the Friedman rule. The inability to run the Friedman rule may occur in environments with limited enforcement. In such environments all trades must be voluntary and so lump-sum taxes of money are impossible because the central bank cannot impose any penalties on the agents (see Kocherlakota 2001).[6] On the other hand, the central bank might not choose to run the Friedman rule because it is not the optimal policy. For example, the Friedman rule can be suboptimal in models that display matching externalities (see Berentsen, Rocheteau and Shi (2007), Rocheteau and Wright (2005)). Another reason the central bank might be constrained from implementing the Friedman rule is that there are seigniorage needs implying $\gamma > 1$. Since our focus is stabilization policy we have not explicitly modeled reasons that give rise to deviations from the Friedman rule. However, we think doing so is an interesting research question to pursue.

The central bank implements its short-term stabilization policy through state contingent changes in the stock of money. Let $\tau_1(\omega) M_{-1}$ and $\tau_3(\omega) M_{-1}$ denote the state contingent cash injections in markets 1 and 3 received by private agents. Note that total injections at the beginning of the period are $T = [\tau + \tau_1(\omega)] M_{-1}$. We assume that $\tau_1(\omega) + \tau_3(\omega) = 0$. In short, any injections in market 1 are undone in market 3. This effectively means that the long-term inflation rate is still deterministic since τM_{-1} is not state dependent. Consequently, changes in $\tau_1(\omega)$ affect the money stock in market 2 without affecting the long-term inflation rate in market 3.[7] With $\tau_1(\omega) + \tau_3(\omega) = 0$ we are implicitly assuming the central bank chooses a path for the money stock in market 3. As we show later, this means the central bank is engaged in price level targeting (in terms of market 3 prices) which allows the central bank to control price expectations in market 3, which is critical for successful stabilization policy. An interesting implication of the optimal policy is that the central bank is essentially providing an elastic supply of currency – when demand for liquidity is high, it provides additional currency and withdraws it when the demand for liquidity is low.

[6] There is a difference between lump-sum taxation and loan repayment. Voluntary loan repayment can be supported with reputational strategies (see for example Berentsen, Camera, and Waller 2007). The reason is that default results in exclusion from financial markets and the loss of future benefits. In contrast, taxes typically finance public goods for which exclusion is not possible thus taxes must necessarily be forced on individual agents by society.

[7] Lucas (1990) employs a similar process for the money supply so that changes in nominal interest rates result purely from liquidity effects and not changes in expected inflation.

The state contingent injections of cash should be viewed as a type of repurchase agreement – the central bank 'sells' money in market 1 under the agreement that it is being repurchased in market 3. Alternatively, $\tau_1(\omega) M_{-1}$ can be thought of as a zero interest discount loan to households that is repaid in the night market. If $\tau_1(\omega) < 0$, agents would be required to lend to the central bank at zero interest. Since they can earn interest by lending in the credit market it is obvious that agents would never lend money to the central bank. Thus, $\tau_1(\omega) < 0$ is not feasible and so $\tau_1(\omega) \geq 0$ in all states.[8] Finally, to ensure repayment of loans we assume the central bank has the same recordkeeping and enforcement technologies as in the credit market. Thus, the only difference between the central bank and the credit market is the ability of the central bank to print fiat currency.

The precise sequence of action after the shocks are observed is as follows. First, the monetary injection τM_{-1} occurs and the central bank offers up to $\tau_1(\omega) M_{-1}$ units of cash per capita to agents at no cost. Then, agents move to the credit market where non-buyers lend their idle cash and buyers borrow money. Agents then move on to market 2 and trade goods. In the third market agents trade goods once again, all financial claims are settled and the central bank takes out $\tau_3(\omega) M_{-1} = -\tau_1(\omega) M_{-1}$ units of money.

3 First-best allocation

In a stationary equilibrium the expected lifetime utility of the representative agent at the beginning of period t is given by

$$(1-\beta)\mathcal{W} = U(x) - x + \int_{\Omega} \{n\varepsilon u[q(\omega)] - (s/\alpha) c[(n/s) q(\omega)]\} f(\omega) d\omega.$$

[8] Woodford (2003) p. 75, footnote 9, makes a similar argument.

The first-best allocation satisfies

$$U'(x^*) = 1 \text{ and} \tag{1}$$

$$\alpha\varepsilon u'[q^*(\omega)] = c'[(n/s)\,q^*(\omega)] \text{ for all } \omega. \tag{2}$$

These are the quantities chosen by a social planner who could force agents to produce and consume.

4 Monetary allocation

In period t, let P denote the nominal price of goods in market 3. It then follows that $\phi = 1/P$ is the real price of money. We study equilibria where end-of-period real money balances are time and state invariant

$$\phi M = \phi_{-1}M_{-1} = \phi_0 M_0 \equiv z, \qquad \omega \in \Omega. \tag{3}$$

We refer to it as a stationary equilibrium. This implies that ϕ is not state dependent and so $\phi_{-1}/\phi = P/P_{-1} = M/M_{-1} = \gamma$. This effectively means that the central bank chooses a price path $P = \gamma P_{-1}$ in market 3. Since the only state variable other than the shocks is M and ϕ is a jump variable, we can start the economy in steady state.

In what follows, we look at a representative period t and work backwards from the third to the first market to examine the agents' choices.

4.1 The third market

In the third market agents consume x, produce h, and adjust their money balances taking into account cash payments or receipts from the credit market. If an agent has borrowed l units of money, then he repays $(1+i)\,l$ units of money.

Consider a stationary equilibrium. Let $V_1(m,t)$ denote the expected lifetime utility at the beginning of market 1 with m money balances prior to the realization of the aggregate state ω. Let $V_3(m, l, \omega, t)$ denote the expected lifetime utility from entering market 3 with m units of money and net borrowing l when the aggregate state is ω in period t. For notational simplicity we suppress the dependence of the value functions on the aggregate state and time.

The representative agent's program is

$$V_3(m, l) = \max_{x,h,m_{+1}} [U(x) - h + \beta V_{1,+1}(m_{+1})] \quad (4)$$
$$\text{s.t.} \quad x + \phi m_{+1} = h + \phi(m + \tau_3 M_{-1}) - \phi(1 + i)l,$$

where m_{+1} is the money taken into period $t + 1$. Rewriting the budget constraint in terms of h and substituting into (4) yields

$$\begin{aligned} V_3(m, l) &= \phi[m + \tau_3 M_{-1} - (1 + i)l] \\ &\quad + \max_{x,m_{+1}} [U(x) - x - \phi m_{+1} + \beta V_1(m_{+1})]. \end{aligned}$$

The first-order conditions are $U'(x) = 1$ and

$$-\phi_{-1} + \beta V_1^m = 0, \quad (5)$$

where the superscript denotes the partial derivative with respect to the argument m. Note that the first-order condition for money has been lagged one period. Thus, V_1^m is the marginal value of taking an additional unit of money into the first market in period t. Since the marginal disutility of working is one, $-\phi_{-1}$ is the utility cost of acquiring one unit of money in the third market of period $t - 1$.

The envelope conditions are

$$V_3^m = \phi; V_3^l = -\phi(1 + i). \quad (6)$$

As in Lagos and Wright (2005) the value function is linear in wealth. The implication is that all agents enter the following period with the same amount of money.

4.2 The second market

At the beginning of the second market there are three trading types: buyers (b), sellers (s) and others (o). Accordingly, let $V_{2j}(m,l)$ denote the expected lifetime utility of an agent of trading type $j=b,s,o$. Let q_b and q_s, respectively, denote the quantities consumed by a buyer and produced by a seller and let p be the nominal price of goods. Since $j=o$ agents are inactive in this market we have $V_{2o}(m,l)=V_3(m,l)$.

A seller who holds m money and l loans at the opening of the second market has expected lifetime utility

$$V_{2s}(m,l) = -c\left(q_s\right)/\alpha + V_3(m+pq_s,l),$$

where $q_s = \arg\max_{q_s}\left[-c\left(q_s\right)/\alpha + V_3(m+pq_s,l)\right]$. Using (6), the first-order condition reduces to

$$c'\left(q_s\right) = \alpha p\phi, \qquad \omega\in\Omega. \tag{7}$$

A buyer who has m money and l loans at the opening of the second market has expected lifetime utility

$$V_{2b}(m,l) = \varepsilon u(q_b) + V_3(m-pq_b,l),$$

where $q_b = \arg\max_{q_b}\ \varepsilon u(q_b) + V_3(m-pq_b,l)$ s.t. $pq_b\leq m$. Using (6) and (7) the buyer's first-order condition can be written as

$$\lambda_q = \phi\left[\alpha\varepsilon u'\left(q_b\right)/c'\left(q_s\right)-1\right], \qquad \omega\in\Omega, \tag{8}$$

where $\lambda_q=\lambda_q\left(\omega\right)$ is the multiplier on the buyer's budget constraint in state ω. If the budget constraint is not binding, then $\alpha\varepsilon u'(q_b)=c'\left(q_s\right)$, which means trades are efficient. If it is binding, then $\alpha\varepsilon u'(q_b)>c'\left(q_s\right)$ which means trades are inefficient. In this case the buyer spends all of his money, i.e. $pq_b = m$.

The marginal value of a loan at the beginning of the second market is the same for all agents and so

$$V_{2j}^l = -(1+i)\,\phi, \tag{9}$$

for $j = b, s, o$. Using the envelope theorem and equations (6) and (8), the marginal values of money for $j = b$ and $j = s, o$ in the second market are

$$\begin{aligned} V_{2b}^m &= \phi\alpha\varepsilon u'(q_b)/c'(q_s) & (10)\\ V_{2s}^m &= V_{2o}^m = \phi. & (11) \end{aligned}$$

4.3 The first market

An agent who has m money at the opening of the first market has expected lifetime utility

$$V_1(m) = \int_\Omega [nV_{2b}(m_b, l_b) + sV_{2s}(m_s, l_s) + (1-n-s)V_{2o}(m_o, l_o)]\, f(\omega)\, d\omega, \tag{12}$$

where $m_j = m+T+l_j$, $j = b, s, o$. Once trading types are realized, an agent of type $j = b, s, o$ solves

$$\max_{l_j} V_{2j}(m_j, l_j) \text{ s.t. } 0 \le m_j.$$

The constraint means that money holdings cannot be negative. The first-order condition is

$$V_{2j}^m + V_{2j}^l + \lambda_j = 0, \qquad \omega \in \Omega,$$

where $\lambda_j = \lambda_j(\omega)$ is the multiplier on the agent's non-negativity constraint in state ω. It is straightforward to show that buyers will become net borrowers while the others become net lenders. Consequently, we have $\lambda_b = 0$ and $\lambda_s = \lambda_o > 0$.

Using (9)-(11), the first-order conditions for $j = b$ and for $j = s, o$ can be written as

$$\alpha\varepsilon u'(q_b) = c'(q_s)(1+i), \qquad \omega \in \Omega, \tag{13}$$

$$\lambda_s = \lambda_o = i\phi, \qquad \omega \in \Omega. \tag{14}$$

Note that if $i = 0$, trades are efficient and if $i > 0$, they are inefficient.

Using the envelope theorem and equations (8), (13), and (14), the marginal value of money satisfies

$$V_1^m = \int_\Omega \left[\phi\alpha\varepsilon u'(q_b)/c'(q_s)\right] f(\omega)\, d\omega. \tag{15}$$

Differentiating (15) shows that the value function is concave in m.

4.4 Stationary Equilibrium

We now derive the symmetric stationary monetary equilibrium. In a symmetric equilibrium all agents of a given type behave equally. Then, market clearing in market 2 implies

$$q(\omega) \equiv q_b(\omega) = (s/n)\, q_s(\omega), \qquad \omega \in \Omega, \tag{16}$$

while in the credit market it implies that all buyers receive a loan of size

$$l_b(\omega) = \frac{(1-n)\left[1+\tau+\tau_1(\omega)\right] M_{-1}}{n}, \qquad \omega \in \Omega. \tag{17}$$

In any monetary equilibrium the buyer's budget constraint must hold with equality in at least one state. In these states we have

$$(n/\alpha)\, q(\omega)\, c'\left[(n/s)\, q(\omega)\right] = \upsilon(\omega)\, z. \tag{18}$$

where $z = \phi M$ is the real stock of money and $\upsilon(\omega) = \left[1+\tau+\tau_1(\omega)\right]/(1+\tau)$. It follows from (18) that in binding states $q(\omega, z) < q^*(\omega)$ where $q(\omega, z)$ is an increasing function of

z. In non-binding states we have $q(\omega, z) = q^*(\omega)$ where $q^*(\omega)$ solves (2).

Finally, use (5) to eliminate V_1^m and (16) to eliminate q_s from (15). Then, multiply the resulting expression by M_{-1} to get

$$\frac{\gamma - \beta}{\beta} = \int_\Omega \left\{ \frac{\alpha \varepsilon u' [q(\omega, z)]}{c' [(n/s) q(\omega, z)]} - 1 \right\} f(\omega) \, d\omega. \tag{19}$$

We can now define the equilibrium as the value of z that solves (19). The reason is that once the equilibrium stock of money is determined all other endogenous variables can be derived.

Definition 1 *A symmetric monetary stationary equilibrium is a z that satisfies (19).*

Before moving on to stabilization policy it is important to note that there are two nominal interest rates in our model. First there is the interest rate paid on a riskless, one-period nominal bond issued in market 3 and redeemed in the following market 3 (quasi-linearity means the agents are risk neutral). Although such a bond is never traded we can price it and its interest rate is given by $1 + i_3 = \gamma/\beta$. Thus the right-hand side of (19) corresponds to this nominal interest rate. The second nominal interest rate in the model is the state contingent rate occuring in the market 1 financial market, $i(\omega)$. This is the nominal interest rate controlled by the central bank to stabilize the shocks. Thus, (19) can be rewritten as

$$1 + i_3 = \int_\Omega [1 + i(\omega)] f(\omega) \, d\omega$$

which is just an arbitrage condition between market 3 bonds and money. In short, by holding a unit of money an agent gives up the 'long-term' interest rate i_3 but earns the expected nominal interest rate $i(\omega)$ in state ω in market 1 (either by depositing the unit of money if its not needed or by avoiding having to borrow a unit of money in market 1). Thus, (19) equates the nominal return of a market 3 bond to the expected nominal rate on a market 1 bond.

5 Stabilization Policy

The central bank's objective is to maximize the welfare of the representative agent. It does so by choosing the quantities consumed and produced in each state subject to the constraint that the chosen quantities satisfy the conditions of a competitive equilibrium. The policy is implemented by choosing state contingent injections $\tau_1(\omega)$ and $\tau_3(\omega)$ accordingly.

The Ramsey problem facing the central bank is

$$\underset{q(\omega),x}{M\alpha x}\; U(x) - x + \int_{\Omega} \{n\varepsilon u[q(\omega)] - (s/\alpha)\, c[(n/s)\, q(\omega)]\} f(\omega)\, d\omega \tag{20}$$

$$\text{s.t. } (19),$$

where the constraint facing the central bank is that the quantities chosen must be compatible with a competitive equilibrium. It is obvious that $x = x^*$ so all that remains is to choose $q(\omega)$.

Proposition 1 *If $\gamma = \beta$, the optimal policy is $i(\omega) = 0$ with $q(\omega) = q^*(\omega)$ for all states.*

According to Proposition 1, if $\gamma = \beta$ is feasible, the central bank should implement the Friedman rule $i(\omega) = 0$ for all states. The reason is that the only friction in our model is the cost of holding money across periods and the Friedman rule eliminates it. So agents can perfectly self-insure against all consumption risk. Consequently, there are no welfare gains from stabilization policies.[9]

Now consider the case in which $\gamma > \beta$. For this case we have the following result.

Proposition 2 *If $\gamma > \beta$, the optimal policy is $i(\omega) > 0$ with $q(\omega) < q^*(\omega)$ for all states.*

Surprisingly, in this case the central bank never chooses $i(\omega) = 0$ for any state. The reason is that the central bank wants to smooth consumption across states. Intuitively,

[9]Ireland (1996) derives a similar result in a model with nominal price stickiness. He finds that at the Friedman rule there is no gain from stabilizing aggregate demand shocks.

consider two states $\omega, \omega' \in \Omega$ with $i(\omega) = 0$ implying $q(\omega) = q^*(\omega)$ and $i(\omega') > 0$ implying $q(\omega') < q^*(\omega')$. Then, the first-order loss from decreasing $q(\omega)$ is zero while there is a first-order gain from increasing $q(\omega')$. This gain can be accomplished by increasing $i(\omega)$ and lowering $i(\omega')$. Thus, the central bank's optimal policy is to smooth interest rates across states.

According to Propositions 1 and 2, unless $i(\omega) = 0$ can be done *for all* states, it is optimal to *never* set $i(\omega) = 0$. Hence, zero nominal interest rates should be an all-or-nothing policy.

For the remainder of the paper, we will study the behavior of stabilization policy under the condition that $\gamma > \beta$ in order to understand how the central bank responds to the individual shocks.

An example To illustrate how the optimal policy is implemented, consider a simple example in which the only shock is the intensive margin demand shock ε. Let ε be uniformly distributed and assume $\alpha\underline{\varepsilon} > 1$. Preferences are $u(q) = 1 - \exp^{-q}$ and $c(q) = q$. With these functions the first-order conditions for the central bank (26) yields[10]

$$\alpha\varepsilon \exp^{-q} = \frac{n}{n - \alpha\lambda}, \tag{21}$$

where λ is multiplier on the constraint. Substituting this expression in the central bank's constraint we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \frac{\alpha\lambda}{n - \alpha\lambda} f(\varepsilon)\, d\varepsilon = \frac{\alpha\lambda}{n - \alpha\lambda}.$$

Solving for λ and substituting back into (21) yields

$$q(\varepsilon) = \ln \alpha\varepsilon - \ln(\gamma/\beta) = q^*(\varepsilon) - \ln(\gamma/\beta), \tag{22}$$

[10] With these utility and cost functions, the central bank's second-order condition is satisfied.

where $q(\varepsilon)$ is increasing in ε.[11] Furthermore,

$$i(\omega) = \frac{\gamma - \beta}{\beta}.$$

Note that this example generates perfect interest rate smoothing by the central bank. When demand for goods (and loans) increases, the central bank accomodates this higher demand by injecting funds into the market 1 financial market thereby keeping interest rates constant. While smoothing interest rates is a general property of our model, perfect smoothing is a special case resulting from the functional forms used.

From the buyer's budget constraint we have

$$q(\varepsilon) = \frac{\alpha\left[1 + \tau + \tau_1(\varepsilon)\right] z}{n(1 + \tau)}. \tag{23}$$

Since z is not state dependent, taking the ratios of (23) for all $q(\varepsilon)$ relative to $q(\underline{\varepsilon})$ gives

$$q(\varepsilon) = \frac{1 + \tau + \tau_1(\varepsilon)}{1 + \tau + \tau_1(\underline{\varepsilon})} q(\underline{\varepsilon}). \tag{24}$$

Since the transfers are nominal objects, there is one degree of freedom in $\tau_1(\varepsilon)$ so let $\tau_1(\underline{\varepsilon}) = 0$. Thus

$$z = (n/\alpha)\left[\ln \alpha\underline{\varepsilon} - \ln(\gamma/\beta)\right]$$

and using (22) and (24) gives us the sequence of transfers that implements this desired allocation

$$1 + \frac{\tau_1(\varepsilon)}{1 + \tau} = \left[\frac{\ln \alpha\varepsilon - \ln(\gamma/\beta)}{\ln \alpha\underline{\varepsilon} - \ln(\gamma/\beta)}\right] > 1 \text{ for all } \varepsilon > \underline{\varepsilon},$$

so $\tau_1(\varepsilon) > 0$ for all $\varepsilon > \underline{\varepsilon}$ and increasing in ε.

[11] Since the Inada condition does not hold for this utility function $q(\underline{\varepsilon}) = 0$ when $\gamma = \beta\alpha\underline{\varepsilon}$. Thus for all $\beta \leq \gamma < \beta\alpha\underline{\varepsilon}$ an equilibrium exists. For $\gamma \geq \beta\alpha\underline{\varepsilon}$ no monetary equilibrium exists.

6 Discussion

In this section we discuss why stabilization policy requires control of price expectations and we explore the gains from optimal stabilization by comparing the allocations under the optimal policy with the one when the central bank is passive. Finally we discuss a benchmark with "sticky" prices.

6.1 Liquidity and inflation expectation effects

The optimal stabilization policy in our model works through a liquidity effect. For this effect to operate, the central bank must control inflation expectations by choosing a price path in market 3. Without it, injections in the first market simply change price expectations and the nominal interest rate as predicted by the Fisher equation.

To see this note from (13) that the interest rate associated with the optimal policy is

$$i(\omega) = \frac{\alpha \varepsilon u'[q(\omega)]}{c'[(n/s)\, q(\omega)]} - 1 > 0, \qquad \omega \in \Omega. \tag{25}$$

Assume for simplicity that the marginal cost is constant and equal to 1. Then rewrite the buyer's budget constraint (18) and (25) to get

$$\begin{aligned} q(\omega) &= (\alpha/n)\left[1 + \tau + \tau_1(\omega)\right] \phi M_{-1} \text{ and} \\ i(\omega) &= \alpha \varepsilon u'[q(\omega)] - 1 > 0, \qquad \omega \in \Omega. \end{aligned}$$

Since the central bank has committed to a price path for ϕ, changes in $\tau_1(\omega)$ do not affect ϕ in the first equation. Hence, ϕM_{-1} is constant. It then follows that increasing $\tau_1(\omega)$ raises real balances for all agents. This decreases the real demand for loans and increases the real supply of loans. As a result, the nominal interest rate falls lowering the cost of borrowing and so $q(\omega)$ increases. Consequently, state contingent injections are not neutral as long as changes in $\tau_1(\omega)$ do not affect ϕ.

What happens if the central bank never undoes the state contingent injections of market 1? In this case $\tau_3(\omega) = 0$ for all t and $\omega \in \Omega$. We can then state the following

Proposition 3 *Assume that $\tau_3(\omega) = 0$ for all $\omega \in \Omega$. Then, changes in $\tau_1(\omega)$ have no real effects and any stabilization policy is ineffective.*

If the central bank does not reverse the state contingent injections of the first market, the price of goods in market 1 changes proportionately with changes in $\tau_1(\omega)$. Consequently, the real money holdings of the buyers are unaffected and so consumption in market 2 does not react to changes in $\tau_1(\omega)$. Such a policy only affects the expected nominal interest rate. To see this note that the gross growth rate of the money supply is $\gamma_t = \tau_1(\omega) + \tau + 1$. Then substitute this and (25) into the constraint of the central bank problem to get

$$\frac{\tau_1(\omega) + \tau + 1 - \beta}{\beta} = \int_{\Omega} i(\omega)\, dF(\omega).$$

An increase in $\tau_1(\omega)$ increases the expected nominal interest rate. This is simply the inflation expectation effect from the Fisher equation.

6.2 The inefficiency of a passive policy

What are the inefficiencies arising from a passive policy? In order to study this question we now derive the allocation when the central bank follows a policy where the injections are not state dependent, i.e., $\tau_1(\omega) = \tau_3(\omega) = 0$, and compare it to the central bank's optimal allocation. We do so under the assumption that the central bank cannot use lump-sum taxes meaning $\gamma \geq 1$. We also analyze each shock separately to understand their individual effects on the equilibrium allocation.

Extensive margin demand shocks For the analysis of shocks to n, we assume that α, ε and s are constant. Note that the first-best quantity $q^*(n)$ is non-increasing in n.

Proposition 4 *For $\gamma \geq 1$, a unique monetary equilibrium exists with $q = q^*(n)$ if $n \leq \tilde{n}$ and $q < q^*(n)$ if $n > \tilde{n}$, where $\tilde{n} \in (0, \overline{n}]$. Moreover, $d\tilde{n}/d\gamma < 0$.*

With a passive policy buyers are constrained when there are many borrowers (high n) and are unconstrained when there are many creditors (low n). Since $d\tilde{n}/d\gamma < 0$, the higher is the inflation rate, the larger is the range of shocks where the quantity traded is inefficiently low. Note that for large γ we can have $\tilde{n} \leq \underline{n}$ which implies that $q < q^*(n)$ in all states.

How does this allocation differ from the one obtained by following an active policy? We illustrate the differences in Figure 1 for a linear cost function. The curve labelled "Passive q" represents equilibrium consumption under a passive policy and the curve labelled "Active q" consumption when the central bank behaves optimally.

As shown earlier, with an active policy buyers never consume q^*, and with linear cost the central bank wants q to be increasing in n.[12] This is just the opposite from what happens when the central bank is passive. With a passive policy, buyers consume $q = q^*$ in low n states and $q < q^*$ in high n states. Moreover, q is strictly decreasing in n for $n > \tilde{n}$. These differences are also reflected in the nominal interest rates. With an active policy the nominal interest rate is strictly positive in all states and decreasing in n. In contrast, with a passive policy the nominal interest rate is $i = 0$ for $n \leq \tilde{n}$ and $i = \varepsilon\alpha u'(q) - 1 \geq 0$ for $n > \tilde{n}$, and increasing in n.

[12] This can be shown by differentiating the central bank's first-order condition with respect to n to find $\partial q/\partial n$.

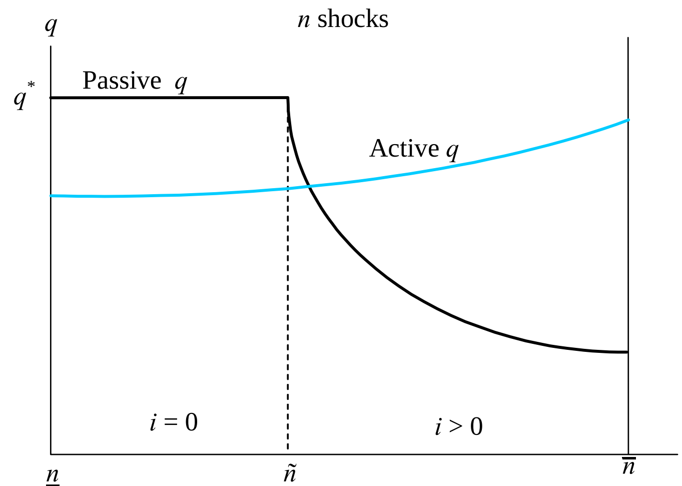

Figure 1: Shocks to the number of buyers.

What is the role of the credit market? With a linear cost function and no credit market, the quantities consumed are the same across all n-states since buyers can only spend the cash they bring into market 1, which is independent of the state that is realized. In contrast, with a credit market, idle cash is lent out to buyers. This makes individual consumption higher on average but also more volatile. The reason is that when n is high demand for loans is high and the supply of loans is low. This pushes up the nominal interest rate and decreases individual consumption. The opposite occurs when n is low.

Intensive margin demand shocks To study ε shocks we assume that α, n and s are constant. It then follows that $\omega = \varepsilon$. Note that the first-best quantity $q^*(\varepsilon)$ is strictly increasing in ε.

Proposition 5 *For $\gamma \geq 1$, a unique monetary equilibrium exists with $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$ and $q = q^*(\varepsilon)$ for $\varepsilon < \tilde{\varepsilon}$, where $\tilde{\varepsilon} \in [0, \bar{\varepsilon}]$. Moreover, $d\tilde{\varepsilon}/d\gamma < 0$.*

With a passive policy, buyers are constrained in high marginal utility states but not in low states. If γ is sufficiently high, buyers are constrained in all states. Note that with a passive policy $dq/d\varepsilon > 0$ for $\varepsilon \leq \tilde{\varepsilon}$ and $dq/d\varepsilon = 0$ for $\varepsilon > \tilde{\varepsilon}$. For $\varepsilon \leq \tilde{\varepsilon}$, buyers have more than enough real balances to buy the efficient quantity. So when ε increases, they simply spend more of their money balances. For $\varepsilon > \tilde{\varepsilon}$, buyers are constrained. So when ε increases, the demand for loans increases but the supply of loans is unchanged so no additional loans can be made. Thus, the interest rate simply increases to clear the credit market.

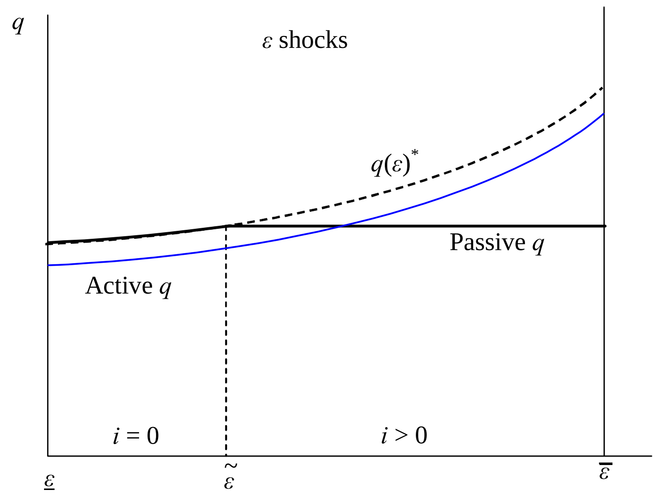

Figure 2: Marginal utility shocks.

Figure 2 illustrates how the allocation resulting from a passive policy differs from the one obtained under an active policy. The dashed curve represents the first-best quantities $q^* (\varepsilon)$. The curve labelled "Passive q" represents equilibrium consumption under a passive policy and the curve labelled "Active q" consumption when the central bank behaves optimally. The central bank's optimal choice is strictly increasing in ε for any cost function.

Finally, we have also derived the equilibrium under a passive policy for the extensive, s, and the intensive, α, supply shocks. The results and figures are qualitatively the same and we therefore do not present them here. They typically involve a cutoff value such that the nominal interest rate is zero either above or below this value. These derivations are available by request.

7 Conclusion

In this paper we have constructed a dynamic stochastic general equilibrium model where money is essential for trade and prices are fully flexible. Our main result is that if the central bank engages in price-level targeting, it can successfully stabilize short-run aggregate shocks to the economy and improve welfare. The optimal policy works through a liquidity effect and involves smoothing nominal interest rates, thereby smoothing consumption across states. If it does not adhere to the price path, stabilization attempts are ineffective. Monetary injections simply raise price expectations and the nominal interest rate as predicted by the Fisher equation.

There are many extensions of this model that would be interesting to pursue. For example, how would the optimal policy be affected if repayment of loans were endogenous? In particular, does the risk of default alter stabilization? Furthermore, we have assumed that the shocks are known to the central bank. An interesting question is what is the optimal policy if the central bank has imperfect information about the nature of the aggregate shocks? A further extension would be to incorporate capital into the model to generate an intertemporal trade-off for the optimal policy. Finally, how would the existence of inside money affect the equilibrium and optimal policy. For example, would inside money act as an automatic stabilizer, eliminating the need for the central bank to stabilize the economy? We leave this to future research.

Appendix

Proof of Proposition 1. From (20) the unconstrained optimum corresponds to $q = q^*(\omega)$ for all $\omega = \Omega$. From the constraint of the central bank problem, since $\gamma < \beta$ is not feasible, the only value that is consistent with the unconstrained optimum is $\gamma = \beta$. ■

Proof of Proposition 2. The first-order conditions for the central bank are

$$n\varepsilon u'[q(\omega)] - (n/\alpha)c'[(n/s)q(\omega)] + \lambda\Psi(\omega) = 0 \qquad \omega \in \Omega, \tag{26}$$

where

$$\Psi(\omega) = \alpha\varepsilon\left\{\frac{u''[q(\omega)]c'[(n/s)q(\omega)] - (n/s)c''[(n/s)q(\omega)]u'[q(\omega)]}{c'[(n/s)q(\omega)]^2}\right\} < 0.$$

Note that λ is independent of ω. Sufficient conditions for a maximum are

$$\alpha\varepsilon u''[q(\omega)] - (n/s)c''[(n/s)q(\omega)] - \{\alpha\varepsilon u'[q(\omega)] - c'[(n/s)q(\omega)]\}\Phi(\omega) < 0,$$

where

$$\begin{aligned}\Phi(\omega) &= \frac{u'''[q(\omega)]c'[(n/s)q(\omega)] - (n/s)^2 c'''[(n/s)q(\omega)]u'[q(\omega)]}{u''[q(\omega)]c'[(n/s)q(\omega)] - (n/s)c''[(n/s)q(\omega)]u'[q(\omega)]} \\ &\quad - \frac{2(n/s)c''[(n/s)q(\omega)]}{c'[(n/s)q(\omega)]}\end{aligned}$$

for all $\omega \in \Omega$. The rest of the proof immediately follows from inspecting the first-order conditions (26). ■

Proof of Proposition 3. In any equilibrium buyers' money holdings are

$$M_{-1}[1 + \tau + \tau_1(\omega)] + l_b(\omega) = \frac{M_{-1}[1 + \tau + \tau_1(\omega)]}{n} = \frac{M(\omega)}{n},$$

since the end-of-period nominal money stock is $M(\omega) = M_{-1}[1 + \tau + \tau_1(\omega)]$. Thus, in any equilibrium we must have

$$\phi(\omega)\, p(\omega)\, q(\omega) \leq \frac{\phi(\omega)\, M(\omega)}{n}, \qquad \omega \in \Omega.$$

The first-order conditions of the sellers (7) imply

$$\alpha^{-1} c' \left[(n/s)\, q(\omega)\right] q(\omega) \leq \frac{\phi(\omega)\, M(\omega)}{n}, \qquad \omega \in \Omega.$$

In a steady-state equilibrium $\phi(\omega)\, M(\omega) = \phi_{-1}(\omega)\, M_{-1}(\omega) = z(\omega)$ for all $\omega \in \Omega$. Hence,

$$\alpha^{-1} c' \left[(n/s)\, q(\omega)\right] q(\omega) \leq \frac{z(\omega)}{n}, \qquad \omega \in \Omega. \tag{27}$$

We now show that in any stationary equilibrium $z(\omega) = z$ is a constant. Use (5) to eliminate V_1^m and (16) to eliminate q_s from (15) to get

$$\phi_{-1}(\omega_{-1})/\beta = \int_{\Omega} \left\{\phi(\omega)\, \alpha\varepsilon u' \left[q(\omega)\right] / c' \left[(n/s)\, q(\omega)\right]\right\} f(\omega)\, d\omega.$$

Multiply this expression by $M_{-1}(\omega_{-1})$ to get

$$M_{-1}(\omega_{-1})\, \phi_{-1}(\omega_{-1})/\beta = \int_{\Omega} \left\{\frac{M(\omega)\, \phi(\omega)}{\gamma(\omega)} \frac{\alpha\varepsilon u' \left[q(\omega)\right]}{c' \left[(n/s)\, q(\omega)\right]}\right\} f(\omega)\, d\omega.$$

since $M(\omega) = \left[1 + \tau + \tau_1(\omega)\right] M_{-1}(\omega_{-1}) = \gamma(\omega)\, M_{-1}(\omega_{-1})$. Note that in any steady-state equilibrium the right-hand side is independent of ω_{-1} and therefore a constant. This immediately implies that $M_{-1}(\omega_{-1})\, \phi_{-1}(\omega_{-1}) = z_{-1}$ is constant for all $\omega_{-1} \in \Omega$. Since in a stationary equilibrium we have $z_{-1} = z$ we can rewrite this equation as follows

$$1/\beta = \int_{\Omega} \left\{\frac{\alpha\varepsilon u' \left[q(\omega)\right]}{\gamma(\omega)\, c' \left[(n/s)\, q(\omega)\right]}\right\} f(\omega)\, d\omega.$$

Finally from (27) we have

$$\alpha^{-1}c'\left[(n/s)\,q\,(\omega)\right]q\,(\omega) \leq \frac{z}{n}, \qquad \omega \in \Omega.$$

Since the right-hand side is independent of $\gamma\,(\omega)$, changes is $\gamma\,(\omega)$ are neutral. Hence, stabilization policy is ineffective. ■

We use Lemma 1 in the proofs of Propositions 4 and 5.

Lemma 1 *Under efficient trading, real aggregate spending* $n\phi p\,(\omega)\,q^*\,(\omega)$ *is increasing in* ε*. It is increasing in* n *and decreasing in* s *and* α *if*

$$\Phi = 1 + \frac{q^*u''\,(q^*)}{u'\,(q^*)} - \frac{q^*c''\left[(n/s)q^*\right](n/s)}{c'\left[(n/s)\,q^*\right]} < 0.$$

Proof of Lemma 1. In equilibrium buyer's real money holdings are $(\upsilon/n)\,z = z/n$ since $\upsilon = 1$ with a passive policy. Thus, in any equilibrium $n\phi pq \leq z$. The right-hand side is the aggregate real money stock in market 1 which is independent of ω. The left-hand side is real aggregate spending measured in market 3 prices which is a function of ω. For a given state ω, trades are efficient if $n\phi p\,(\omega)\,q^*\,(\omega) \leq z$ and inefficient if $n\phi p\,(\omega)\,q^*\,(\omega) > z$ where $p = p\,(\omega)$ is a function of ω but ϕ is not. We would like to know how real aggregate spending $g\,(\omega) = n\phi p\,(\omega)\,q^*\,(\omega)$ changes in ω when trades are efficient:

$$dg\,(\omega) = \phi p\,(\omega)\,q^*\,(\omega)\,dn + n\phi q^*\,(\omega)\,dp + n\phi p\,(\omega)\,dq^*.$$

The first term reflects the change in real liquidity that is intermediated in the economy. This effect only occurs if n changes. The second term reflects changes in the relative price ϕp of goods and the third term changes in the efficient quantity. Rewrite it as follows

$$dg\,(\omega) = n\phi pq^*\left[\frac{dn}{n} + \frac{dp}{p} + \frac{dq^*}{q^*}\right].$$

The term $\frac{dp}{p}$ can be derived from (7) as follows

$$\frac{dp}{p} = \frac{c''\left[(n/s)\,q^*\right]}{c'\left[(n/s)\,q^*\right]}\frac{q^* n}{s}\left[\frac{dn}{n} - \frac{ds}{s}\right] - \frac{d\alpha}{\alpha}$$

and the term $\frac{dq^*}{q^*}$ can be derived from $\varepsilon\alpha u'(q^*) = c'\left[(n/s)q^*\right]$ as follows

$$\begin{aligned}\frac{dq^*}{q^*} &= \frac{c''\left[(n/s)q^*\right](n/s)}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)}\left[\frac{dn}{n} - \frac{ds}{s}\right] \\ &\quad -\frac{\varepsilon\alpha u'(q^*)/q^*}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)}\left[\frac{d\alpha}{\alpha} + \frac{d\varepsilon}{\varepsilon}\right].\end{aligned}$$

Investigating each shock separately we get

$$\begin{aligned}\frac{\partial g(n)}{\partial n} &= c'\left[(n/s)q^*\right]q^*(n/\alpha s)\left\{1 + \frac{c''\left[(n/s)\,q^*\right]\Phi}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)}\right\} \geq 0 \\ \frac{\partial g(n)}{\partial s} &= -\frac{c'\left[(n/s)q^*\right]q^*(n/s)^2 c''\left[(n/s)\,q^*\right]\Phi}{\alpha\left\{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)\right\}} \leq 0 \\ \frac{\partial g(n)}{\partial \alpha} &= \frac{-c'\left[(n/s)q^*\right]n\varepsilon u'(q^*)\,\Phi}{\alpha\left\{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)\right\}} < 0 \\ \frac{\partial g(n)}{\partial \varepsilon} &= -\frac{c'\left[(n/s)q^*\right]nu'(q^*)}{\alpha\varepsilon u''(q^*) - c''\left[(n/s)q^*\right](n/s)} > 0.\end{aligned}$$

■

Proof of Proposition 4. Here $\omega = n$.

Critical value: From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{n}}^{\overline{n}}\left\{\frac{\alpha\varepsilon u'\left[q(n,z)\right]}{c'\left[(n/s)\,q(n,z)\right]} - 1\right\}f(n)\,dn. \tag{28}$$

Lemma 1 gives $\frac{\partial g(n)}{\partial n} \geq 0$. If $g(\underline{n}) > z$, then agents are constrained in all states. If $g(\overline{n}) < z$, then agents are never constrained. If $g(\overline{n}) \geq z \geq g(\underline{n})$, for a given value of z there is a unique critical value $\tilde{n}$ such that

$$g(\tilde{n}) = z. \tag{29}$$

This implies that $q = q^*(n)$ for $n \leq \tilde{n}$ and $q < q^*(n)$ for $n > \tilde{n}$. Note that $\frac{\partial \tilde{n}}{\partial z} \geq 0$.

Existence: Using (29) we can write (28) as follows

$$\frac{\gamma-\beta}{\beta}=\int_{\check{n}}^{\overline{n}}\left\{\frac{\alpha\varepsilon u'\left[q\left(n,z\right)\right]}{c'\left[\left(n/s\right)q\left(n,z\right)\right]}-1\right\}f\left(n\right)dn\equiv RHS, \tag{30}$$

where $\check{n}=\max\left\{\tilde{n},\underline{n}\right\}$. Only the right-hand side is a function of z. Note that $\lim_{z\rightarrow 0} RHS=\infty$. For $\overline{z}=g\left(\overline{n}\right)$ we have $\tilde{n}=\overline{n}$ and therefore $RHS\left|_{z=\overline{z}}\right.=0\leq\frac{\gamma-\beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (30) is monotonically decreasing in z. To see this use Leibnitz's rule to get

$$\begin{aligned}\frac{\partial RHS}{\partial z} &= \int_{\check{n}}^{\overline{n}}\frac{\alpha\varepsilon\left[u''c'-\left(n/s\right)c''u'\right]}{\left(c'\right)^{2}}\frac{\partial q\left(n,z\right)}{\partial z}f\left(n\right)dn \\ &\quad -\left\{\frac{\alpha\varepsilon u'\left[q\left(\check{n},z\right)\right]}{c'\left[\left(\check{n}/s\right)q\left(\check{n},z\right)\right]}-1\right\}f\left(\check{n}\right)\frac{\partial\check{n}}{\partial z}.\end{aligned}$$

Since $q\left(\check{n},z\right)=q^{*}\left(\check{n}\right)$ by construction we have

$$\frac{\partial RHS}{\partial z}=\int_{\check{n}}^{\overline{n}}\frac{\alpha\varepsilon\left[u''c'-\left(n/s\right)c''u'\right]}{\left(c'\right)^{2}}\frac{\partial q\left(n,z\right)}{\partial z}f\left(n\right)dn<0.$$

Since the right-hand side is decreasing in z, we have a unique z that solves (30). Consequently, we have

$$q=q^{*}\left(n\right)\ \text{ if } n\leq\check{n} \text{ and } q<q^{*}\left(n\right)\ \text{ otherwise.}$$

Hours worked: Finally, if buyers have been constrained in market 1 money holdings at the opening of the third market are $m_3=pq_s$ for sellers $m_3=0$ for non-sellers. Solving for equilibrium consumption and production in the third market, with $x^{*}=U'^{-1}\left(1\right)$, gives

$$\begin{aligned}h_b &= x^{*}+ne_c\left(q\right)c\left[\left(n/s\right)q\right]+\left(1-n\right)e_u\left(q\right)u\left(q\right)\\ h_s &= x^{*}-ne_c\left(q\right)c\left[\left(n/s\right)q\right]\left(1-s\right)s^{-1}-ne_u\left(q\right)u\left(q\right)\\ h_o &= x^{*}+ne_c\left(q\right)c\left[\left(n/s\right)q\right]-ne_u\left(q\right)u\left(q\right).\end{aligned}$$

Notice that $nh_b + sh_s + (1-n-s)\, h_o = x^*$. Moreover, we have $h_b \geq h_o \geq h_s$. For existence we need that all agents work a positive amount in the third market. This, it is sufficient to show that $h_s > 0$.

Given $\underline{s} > 0$, n/s is bounded and since the elasticities $e_c(q)$ and $e_u(q)$ are bounded, we can scale $U(x)$ such that there is a value $x^* = U'^{-1}(1)$ greater than the last term for all $q \in [0, q^*]$. Hence, h_s is positive for for all $q \in [0, q^*]$ ensuring that the equilibrium exists. Note that the states where the buyers are constrained are the ones where the sellers have all the money after trading. Therefore, if h_s is positive in constrained states it is positive in all unconstrained states. ■

Proof of Proposition 5. Here $\omega = \varepsilon$.

Critical value: From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left\{ \frac{\alpha \varepsilon u' \left[q\left(\varepsilon, z\right) \right]}{c' \left[\left(n/s \right) q\left(\varepsilon, z\right) \right]} - 1 \right\} f\left(\varepsilon\right) d\varepsilon. \tag{31}$$

Lemma 1 gives $\frac{\partial g(\varepsilon)}{\partial \varepsilon} \geq 0$. If $g(\underline{\varepsilon}) > z$, then agents are constrained in all states. If $g(\overline{\varepsilon}) < z$, then agents are never constrained. If $g(\overline{\varepsilon}) \geq z \geq g(\underline{\varepsilon})$, for a given value of z there is a unique critical value $\tilde{\varepsilon}$ such that

$$g\left(\tilde{\varepsilon}\right) = z. \tag{32}$$

This implies that $q = q^*(\varepsilon)$ for $\varepsilon \leq \tilde{\varepsilon}$ and $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$. Note that $\frac{\partial \tilde{\varepsilon}}{\partial z} \geq 0$.

Existence: Using (32) we can write (31) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\breve{\varepsilon}}^{\overline{\varepsilon}} \left\{ \frac{\alpha \varepsilon u' \left[q\left(\varepsilon, z\right) \right]}{c' \left[\left(n/s \right) q\left(\varepsilon, z\right) \right]} - 1 \right\} f\left(\varepsilon\right) d\varepsilon \equiv RHS, \tag{33}$$

where $\breve{\varepsilon} = \max\{\tilde{\varepsilon}, \underline{\varepsilon}\}$. Only the right-hand side is a function of z. Note that $\lim_{z \to 0} RHS = \infty$. For $\overline{z} = g(\overline{\varepsilon})$ we have $\tilde{\varepsilon} = \overline{\varepsilon}$ and therefore $RHS\,|_{z=\overline{z}} = 0 \leq \frac{\gamma-\beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (33) is monotonically decreasing in z. To see this use Leibnitz's rule and note that by construction $q(\breve{\varepsilon}, z) = q^*(\breve{\varepsilon})$ to get

$$\frac{\partial RHS}{\partial z} = \int_{\breve{\varepsilon}}^{\overline{\varepsilon}} \left\{ \frac{\alpha\varepsilon \left[u''c' - (n/s)\, c''u'\right]}{(c')^2} \frac{\partial q(\varepsilon, z)}{\partial z} \right\} f(\varepsilon)\, d\varepsilon < 0.$$

Since the right-hand side is strictly decreasing in z, we have a unique z that solves (33). Consequently, we have

$$q = q^*(\varepsilon) \text{ if } \varepsilon \leq \breve{\varepsilon} \text{ and } q < q^*(\varepsilon) \text{ otherwise.}$$

Finally, it is straightforward to show that the hours worked in market 3 are bounded away from zero. ■

References

Berentsen, A., G. Camera and C. Waller (2007), "Money, Credit and Banking," *Journal of Economic Theory,*.135, 171-195.

Berentsen, A., G. Camera and C. Waller (2005). "The Distribution of Money Balances and the Non-Neutrality of Money," *International Economic Review*, 46, 465-487.

Berentsen, A., G. Rocheteau and S. Shi (2007), "Friedman Meets Hosios: Efficiency in Search Models of Money," *Economic Journal*, 117, 174–195.

Clarida, R., J. Gali and M. Gertler (1999). "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature* XXXVII, 1661-1707.

Ireland, P. (1996). "The Role of Countercyclical Monetary Policy," *Journal of Political Economy* 104 vol. 4, 704-723.

Lagos, R. and R. Wright (2005). "A Unified Framework for Monetary Theory and Policy Evaluation," *Journal of Political Economy* 113, 463-484.

Lucas, R. (1990). "Liquidity and Interest Rates," *Journal of Economic Theory* 50, 237-264.

Kocherlakota, N. (1998). "Money is Memory," *Journal of Economic Theory,* 81, 232–251.

Kocherlakota, N. (2001). "Money: What's the Question and Why We Care About the Answer," Staff paper, Federal Reserve Bank of Minneapolis.

Rocheteau, G. and R. Wright (2005). "Money in Search Equilibrium, in Competitive Equilibrium and in Competitive Search Equilibrium," *Econometrica*, 73, 175-202.

Wallace, N. (2001). "Whither Monetary Economics?" *International Economic Review*, 42, 847-869.

Woodford, M. (2003). "Interest and Prices: Foundations of a Theory of Monetary Policy," Princeton University Press.