Lecture No. 12

References: Roache, p. 64-67; papers by Leonard.

Problems with Computing Convection Dominated Transport

- There are significant numerical problems with central differencing schemes for convection dominated flows.
- We saw that due to phase propagation errors we left behind part of the solution (in particular the short wavelength components of the solution) and that for $\mathbf{P_e} > 2$ wiggles resulted (this was not related to stability). Only if we had a very substantial damping ratio (numerical to analytical) then the wiggles were eliminated but at the cost of damping the high wavenumber portion of the solution and sometimes the low wavenumber components as well.
- Since our numerical "problems" are associated with the central discretization of the convective terms, let's change the FD discretization for that term. We note that the central methods seem to work well for the diffusion terms.
- Let's examine the use of upwind differencing to improve our solutions to:

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$$

Upwind Differenced Explicit Scheme

$$\frac{u_{p,q+1} - u_{p,q}}{\Delta t} + V\frac{u_{p,q} - u_{p-1,q}}{\Delta x} = D\frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{\Delta x^2}$$

If $V > 0 \rightarrow$ use a backward difference operator for the convection term

If $V < 0 \rightarrow$ use a forward difference operator for the convection term

$$u_{p,q+1} = u_{p,q} - \mathbf{C}_{\#}(u_{p,q} - u_{p-1,q}) + \rho(u_{p+1,q} - 2u_{p,q} + u_{p-1,q})$$

Investigate Stability Using Fourier analysis

Let:

$$u_{p,q} = \xi'^{q}_{n} e^{\hat{i}\beta_n p\Delta x}$$

Substitute into the FD equation

$$\xi'^{q+1} e^{\hat{i}\beta_n p\Delta x} = \xi'^{q}_{n} e^{\hat{i}\beta_n p\Delta x} - \mathbf{C}_{\#}\left(\xi'^{q}_{n} e^{\hat{i}\beta_n p\Delta x} - \xi'^{q}_{n} e^{\hat{i}\beta_n (p-1)\Delta x}\right)$$

$$+\rho\left(\xi'^{q}_{n} e^{\hat{i}\beta_n (p+1)\Delta x} - 2\xi'^{q}_{n} e^{\hat{i}\beta_n p\Delta x} + \xi'^{q}_{n} e^{\hat{i}\beta_n (p-1)\Delta x}\right)$$

$$\Rightarrow$$

$$\xi'_{n} = 1 - \mathbf{C}_{\#}\left(1 - e^{-\hat{i}\beta_n\Delta x}\right) + \rho\left(e^{\hat{i}\beta_n\Delta x} + e^{-\hat{i}\beta_n\Delta x} - 2\right)$$

However:

$$e^{-\hat{i}\beta_n\Delta x} = \cos(\beta_n\Delta x) - \hat{i}\sin(\beta_n\Delta x)$$

$$e^{\hat{i}\beta_n\Delta x} + e^{-\hat{i}\beta_n\Delta x} = 2\cos\beta_n\Delta x$$

Thus:

$$\xi'_{n} = 1 - \mathbf{C}_{\#}\left(1 - \cos\beta_n\Delta x + \hat{i}\sin\beta_n\Delta x\right) + \rho\left(2\cos\beta_n\Delta x - 2\right)$$

The solution can be shown to be stable when (Roache, p.65):

$$|\xi'_n| \le 1$$

$$\Rightarrow$$

$$\Delta t \le \frac{1}{\frac{2D}{\Delta x^2} + \frac{|V|}{\Delta x}}$$

$$\Rightarrow$$

$$1 \le \frac{1}{2\rho + \mathbf{C}_{\#}}$$

$$\Rightarrow$$

$$1 \le \frac{1}{\mathbf{C}_{\#}\left(\frac{2}{\mathbf{P}_{\mathbf{e}}} + 1\right)}$$

- For $D = 0$ $(\rho = 0)$ stability requires that $\mathbf{C}_{\#} \le 1 \quad \Rightarrow \quad \Delta t \le \frac{\Delta x}{|V|}$. Recall that explicit central schemes were unconditionally unstable for $D = 0$ (static instability). However now we can ensure stability by making Δt small enough.
- Δt_s, the time step required for stability, gets smaller as D increases.

$$D = 0 \quad \Rightarrow \quad \mathbf{P}_{\mathbf{e}} = \infty \quad \Rightarrow \quad \mathbf{C}_{\#} \le 1 \text{ for stability}$$

$$\mathbf{P}_{\mathbf{e}} = 2 \quad \Rightarrow \quad \mathbf{C}_{\#} \le 0.5 \text{ for stability}$$

In general we have a scheme that we can make stable by decreasing Δt. Let's try it and see how it performs.

Numerical Experiments Using the Upwind Difference Explicit Scheme

See Figures L12.1 through L12.6

- Figure L12.1: $\mathbf{C}_{\#} = 0.1$ and $\mathbf{P}_{\mathbf{e}} = \infty$

  No wiggles but excessive damping of the entire solution.

- Figure L12.2: $\mathbf{C}_{\#} = 0.1$ and $\mathbf{P}_{\mathbf{e}} = 2.0$

  Excessive damping of the entire solution.

- Figure L12.3: $\mathbf{C}_{\#} = 1.0$ and $\mathbf{P}_{\mathbf{e}} = \infty$

  No wiggles and an exact numerical solution (i.e. the nodal values are exact).

- Figure L12.4: $\mathbf{C}_{\#} = 1.0$ and $\mathbf{P}_{\mathbf{e}} = 2$

  Unstable (as predicted by our stability constraint).

- Figure L12.5: $\mathbf{C}_{\#} = 1.1$ and $\mathbf{P}_{\mathbf{e}} = \infty$

  Unstable (as predicted by our stability constraint).

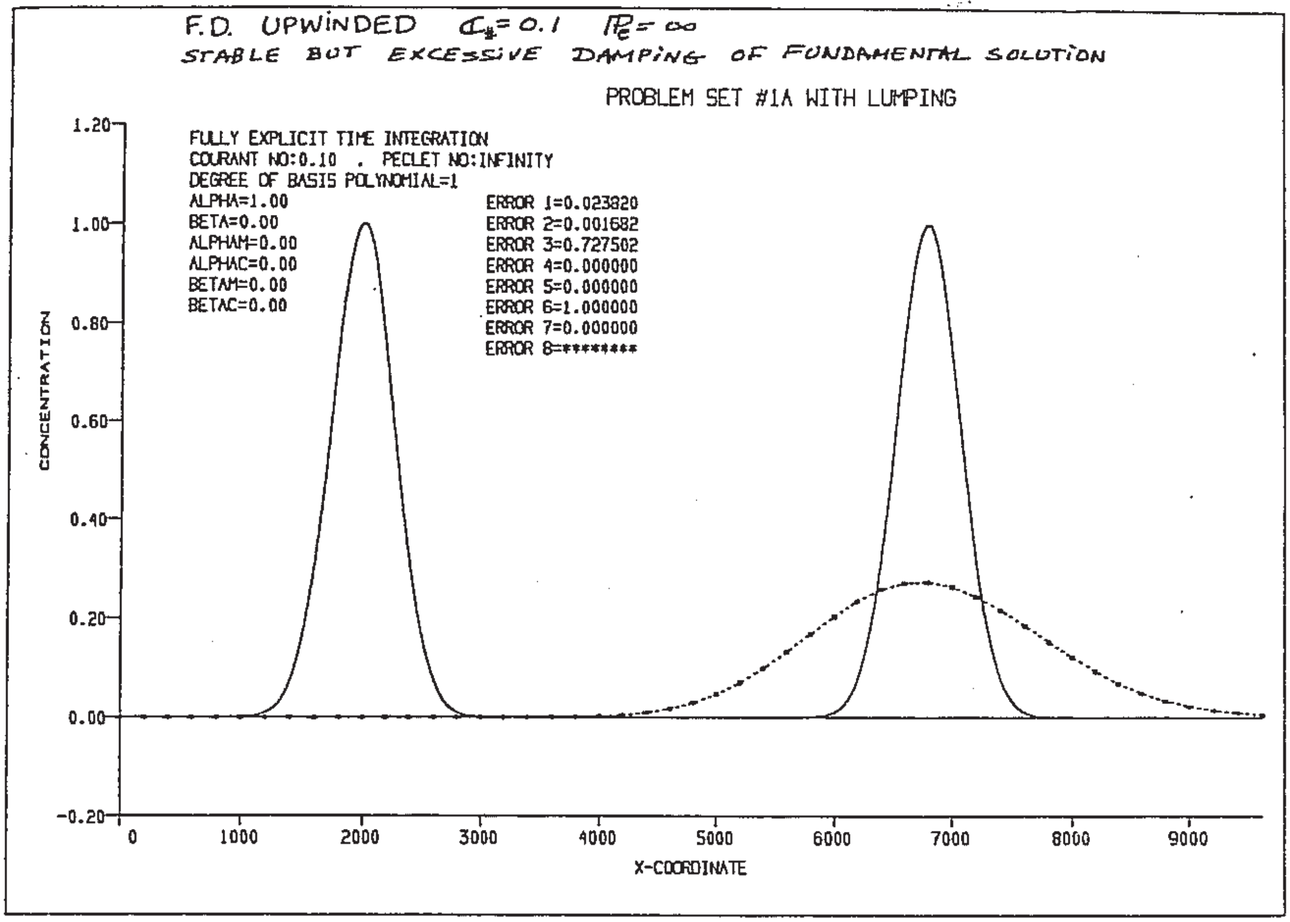
- Figure L12.6: $\mathbf{C}_{\#} = 1.1$ and $\mathbf{P}_{\mathbf{e}} = 2$

  Unstable (as predicted by our stability constraint).

Thus we note that:

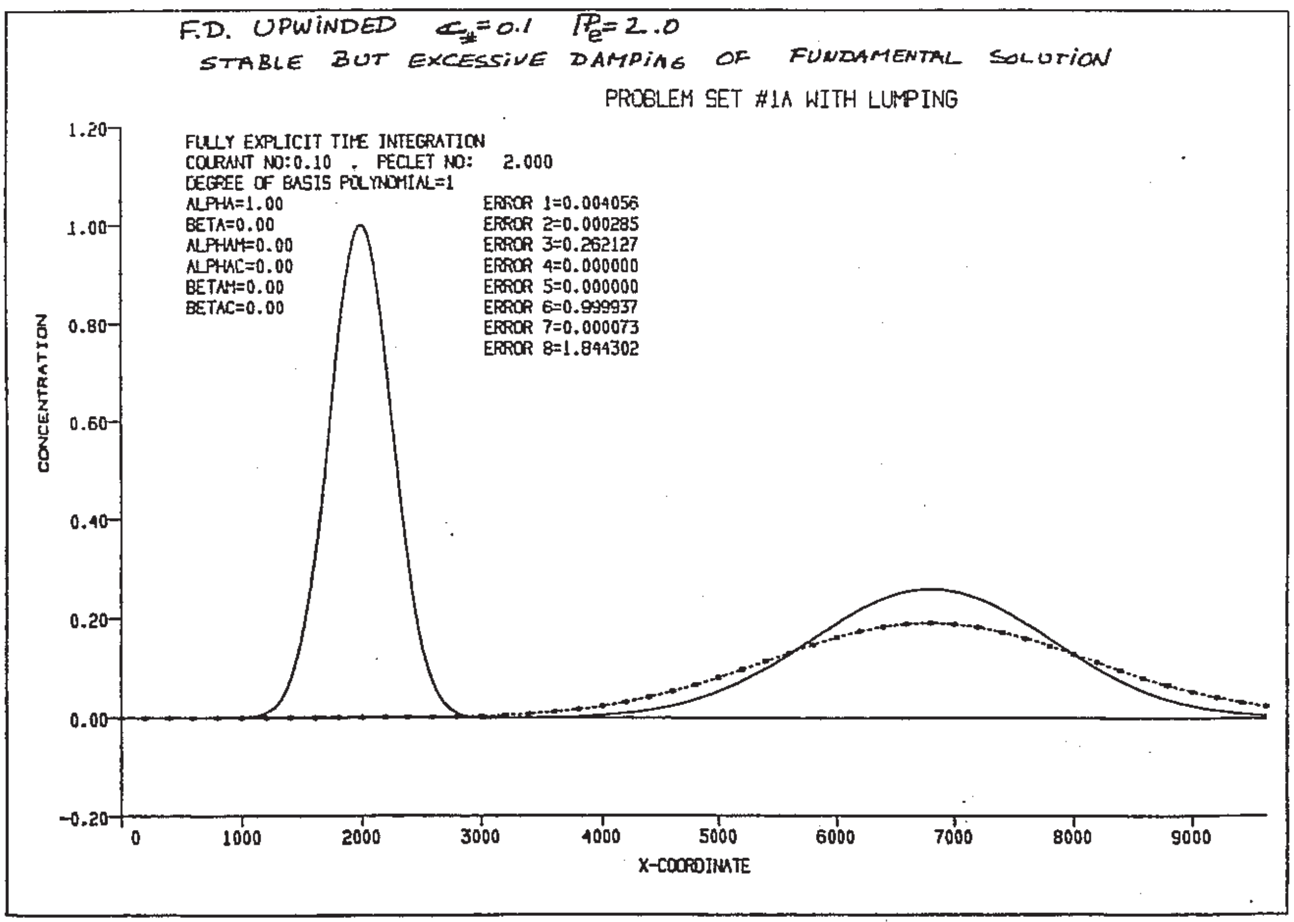
- The derived stability constraints are correct.
- For a very special case ($\mathbf{C}_{\#} = 1.0, \mathbf{P}_{\mathbf{e}} = \infty$) the scheme gives perfect results.
- The scheme always seems to get rid of wiggles.
- The scheme in general has tendency to be very overdamped.

L12-1



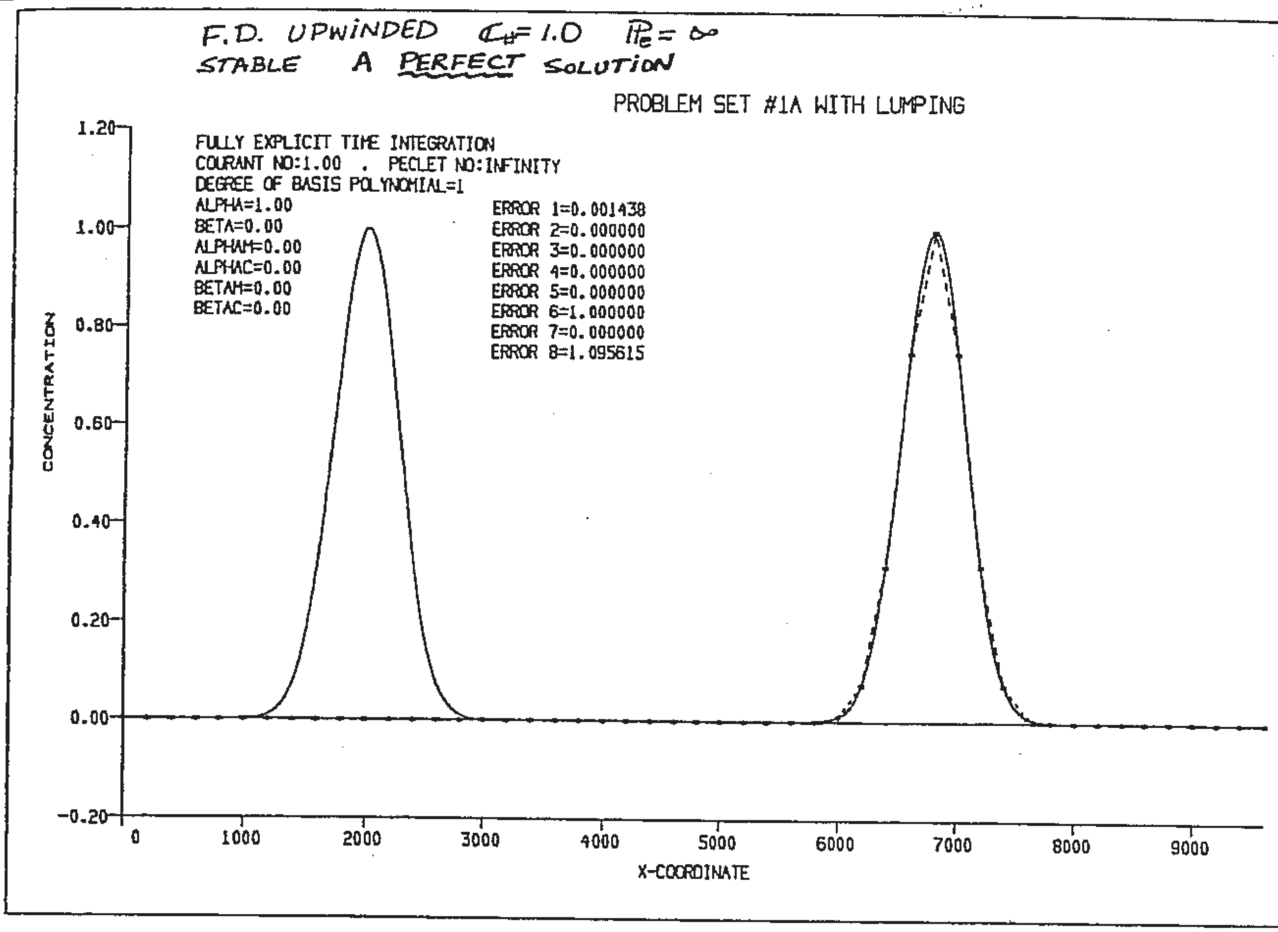
F.D. UPWINDED $C_{\#}=0.1$ $P_e=\infty$
STABLE BUT EXCESSIVE DAMPING OF FUNDAMENTAL SOLUTION
PROBLEM SET #1A WITH LUMPING
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:0.10 . PECLET NO:INFINITY
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=0.023820
ERROR 2=0.001682
ERROR 3=0.727502
ERROR 4=0.000000
ERROR 5=0.000000
ERROR 6=1.000000
ERROR 7=0.000000
ERROR 8=********
CONCENTRATION
X-COORDINATE
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

L12-2

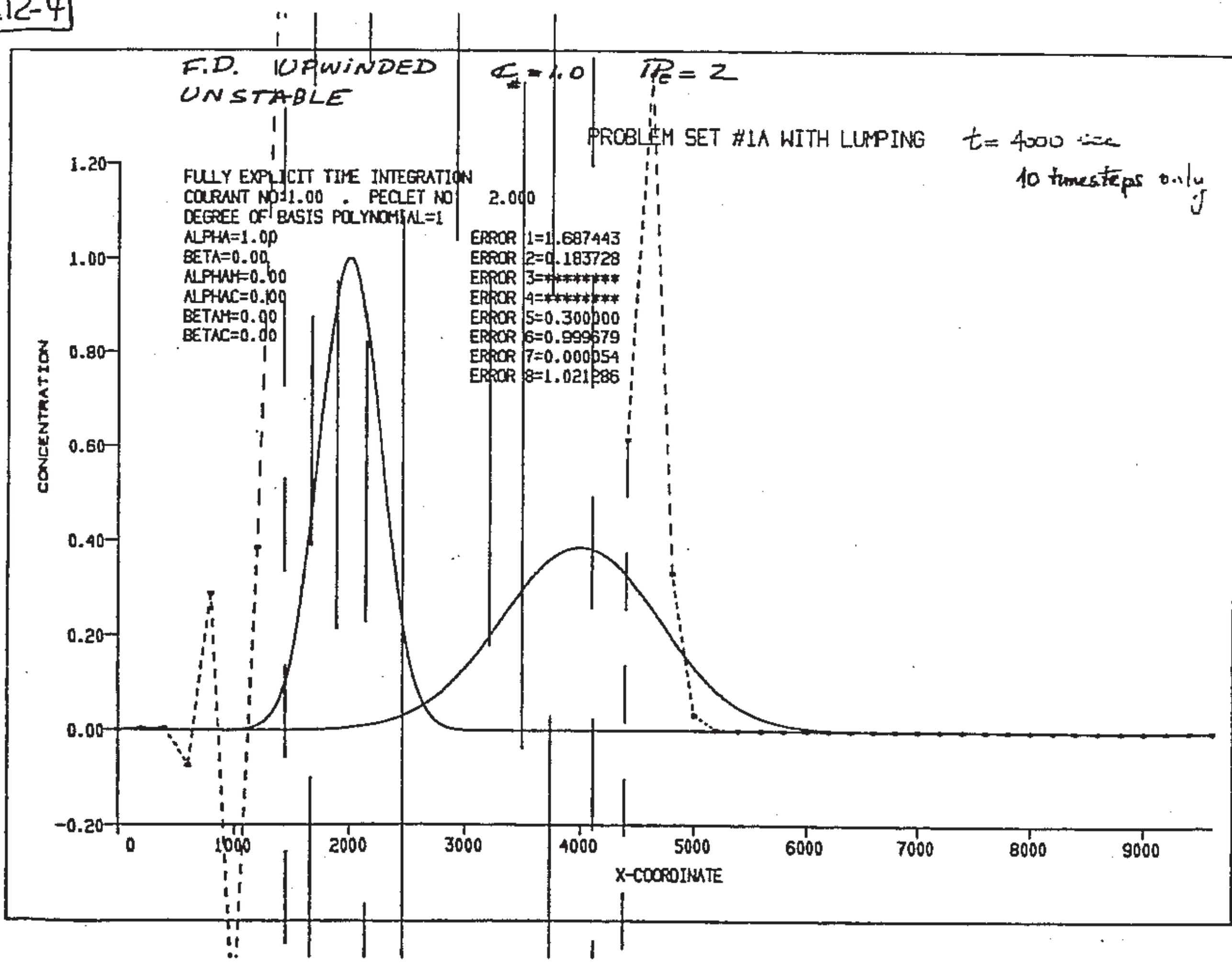


F.D. UPWINDED $C_{\#}=0.1$ $P_e=2.0$
STABLE BUT EXCESSIVE DAMPING OF FUNDAMENTAL SOLUTION
PROBLEM SET #1A WITH LUMPING
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:0.10 . PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=0.004056
ERROR 2=0.000285
ERROR 3=0.262127
ERROR 4=0.000000
ERROR 5=0.000000
ERROR 6=0.999937
ERROR 7=0.000073
ERROR 8=1.844302
CONCENTRATION
X-COORDINATE
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

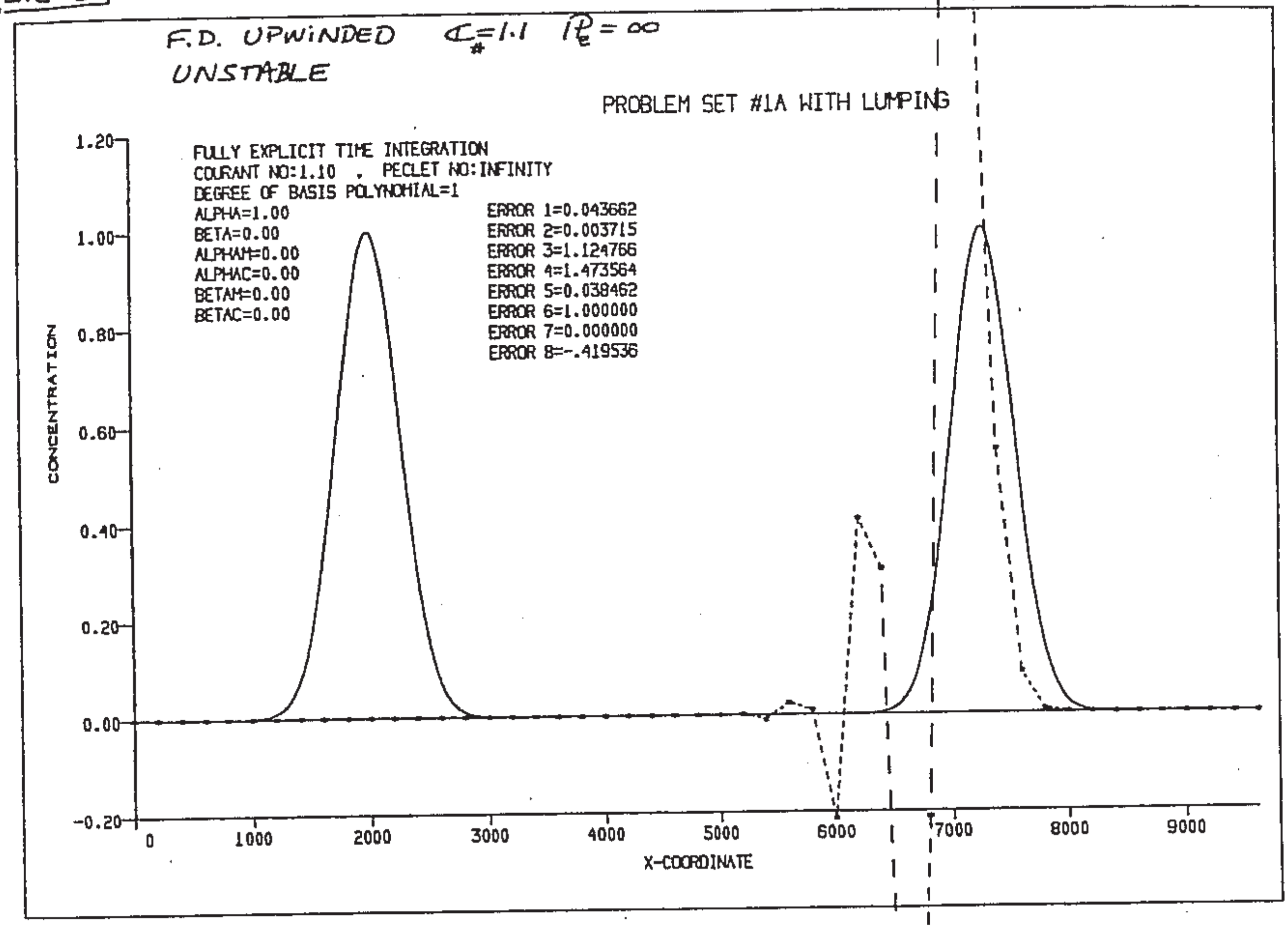
L12-3


F.D. UPWINDED $C_{\#}$=1.0 $\mathbb{P}_e$=∞
STABLE A PERFECT SOLUTION
PROBLEM SET #1A WITH LUMPING
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:1.00 . PECLET NO:INFINITY
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=0.001438
ERROR 2=0.000000
ERROR 3=0.000000
ERROR 4=0.000000
ERROR 5=0.000000
ERROR 6=1.000000
ERROR 7=0.000000
ERROR 8=1.095615
CONCENTRATION
X-COORDINATE

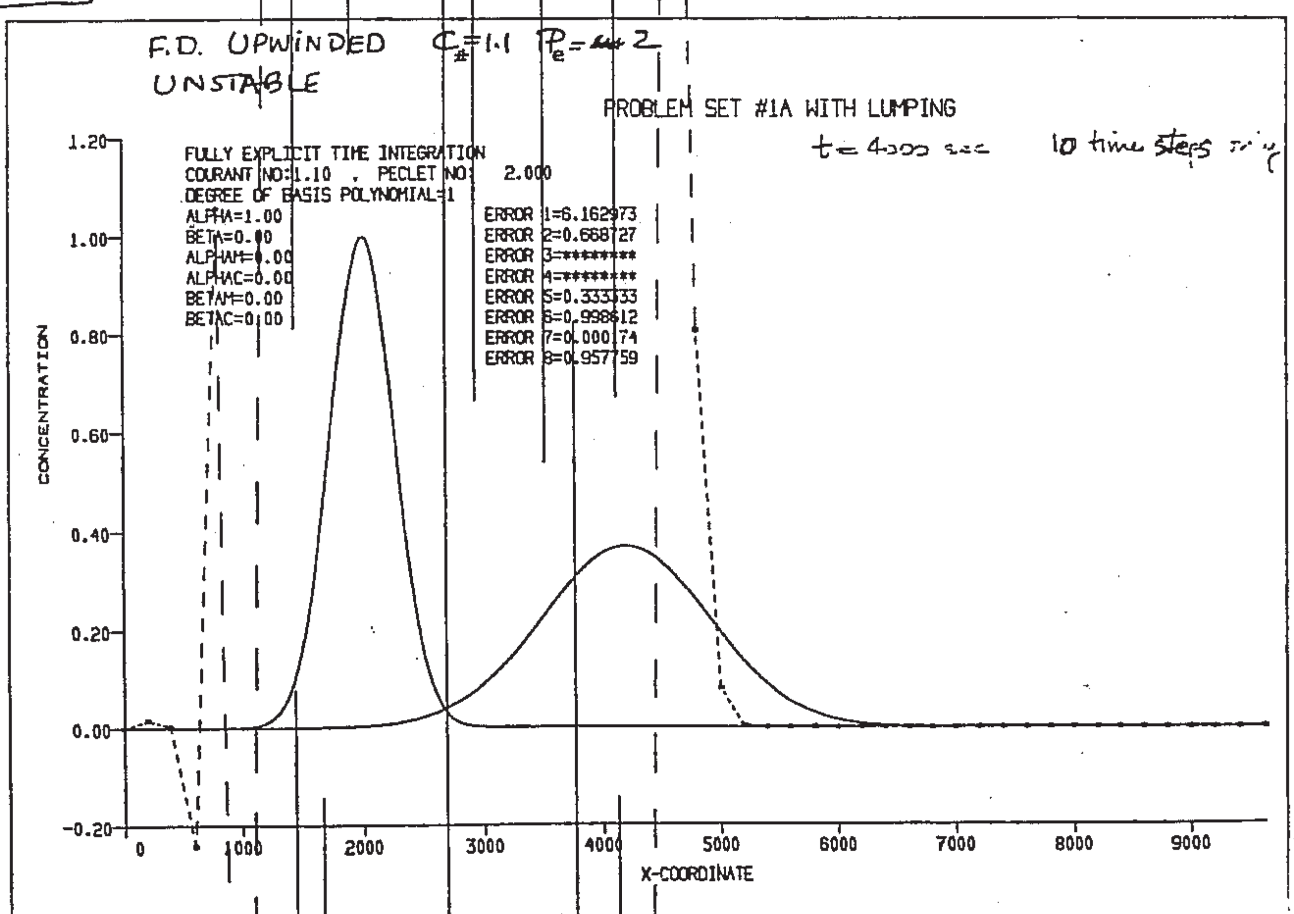
L12-4


F.D. UPWINDED $C_{\#}$=1.0 $\mathbb{P}_e$=2
UNSTABLE
PROBLEM SET #1A WITH LUMPING
t= 4000 sec
10 timesteps only
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:1.00 . PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=1.687443
ERROR 2=0.183728
ERROR 3=********
ERROR 4=********
ERROR 5=0.300000
ERROR 6=0.999679
ERROR 7=0.000054
ERROR 8=1.021286
CONCENTRATION
X-COORDINATE

L12-5


F.D. UPWINDED C#=1.1 Pe=∞
UNSTABLE
PROBLEM SET #1A WITH LUMPING
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:1.10 , PECLET NO:INFINITY
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=0.043662
ERROR 2=0.003715
ERROR 3=1.124766
ERROR 4=1.473564
ERROR 5=0.038462
ERROR 6=1.000000
ERROR 7=0.000000
ERROR 8=-.419536
CONCENTRATION
X-COORDINATE
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

L12-6


F.D. UPWINDED C#=1.1 Pe=2
UNSTABLE
PROBLEM SET #1A WITH LUMPING
t=4000 sec
10 time steps
FULLY EXPLICIT TIME INTEGRATION
COURANT NO:1.10 , PECLET NO: 2.000
DEGREE OF BASIS POLYNOMIAL=1
ALPHA=1.00
BETA=0.00
ALPHAM=0.00
ALPHAC=0.00
BETAM=0.00
BETAC=0.00
ERROR 1=6.162973
ERROR 2=0.668727
ERROR 3=********
ERROR 4=********
ERROR 5=0.333333
ERROR 6=0.998612
ERROR 7=0.000174
ERROR 8=0.957759
CONCENTRATION
X-COORDINATE
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
0
1000
2000
3000
4000
5000
6000
7000
8000
9000

- Even the implicit central scheme was not overdamped at $\mathbf{C}_{\#} = 0.1$, $\mathbf{P}_{\mathbf{e}} = 2.0$. However the upwind scheme introduces substantial damping for this relatively easy case.

Truncation Error Analysis of the Upwind Differenced Explicit Scheme

Let's investigate the accuracy of this scheme by examining the pure convection equation only:

$$\frac{\partial U}{\partial t} + V\frac{\partial U}{\partial x} = 0$$

This equation is discretized as (using the upwind scheme):

$$u_{p,\,q+1} = u_{p,\,q} - \mathbf{C}_{\#}\,(u_{p,\,q} - u_{p-1,\,q})$$

The truncation error is defined as:

$$\tau_{p,\,q} = U_{p,\,q+1} - U_{p,\,q} + \mathbf{C}_{\#}\,(U_{p,\,q} - U_{p-1,\,q})$$

Using Taylor series:

$$U_{p,\,q} = U_{p,\,q}$$

$$U_{p,\,q+1} = U_{p,\,q} + \Delta t\frac{\partial U_{p,\,q}}{\partial t} + \frac{\Delta t^2}{2!}\frac{\partial^2 U_{p,\,q}}{\partial t^2} + 0\,(\Delta t)^3$$

$$U_{p-1,\,q} = U_{p,\,q} - \Delta x\frac{\partial U_{p,\,q}}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 U_{p,\,q}}{\partial x^2} + 0\,(\Delta x)^3$$

$$U_{p+1,\,q} = U_{p,\,q} + \Delta x\frac{\partial U_{p,\,q}}{\partial x} + \frac{\Delta x^2}{2!} + \frac{\partial^2 U_{p,\,q}}{\partial x^2} + 0\,(\Delta x)^3$$

Substitute into the expression for the truncation error:

$$\tau_{p,q} = U_{p,q} + \Delta t \left.\frac{\partial U}{\partial t}\right|_{p,q} + \frac{\Delta t^2}{2!} \left.\frac{\partial^2 U}{\partial t^2}\right|_{p,q} + 0\,(\Delta t)^3$$

$$-U_{p,q} + \frac{V\Delta t}{\Delta x}\left[U_{p,q} - \left(U_{p,q} - \Delta x \left.\frac{\partial U}{\partial x}\right|_{p,q} + \frac{\Delta x^2}{2!} \left.\frac{\partial^2 U}{\partial x^2}\right|_{p,q} + O\,(\Delta x)^3\right)\right]$$

Divide by Δt and re-arrange:

$$\tau_{p,q} = \frac{\partial U_{p,q}}{\partial t} + V\frac{\partial U_{p,q}}{\partial x} - \frac{V\Delta x}{2}\frac{\partial^2 U_{p,q}}{\partial x^2} + \frac{\Delta t}{2}\frac{\partial^2 U_{p,q}}{\partial t^2} + 0\,(\Delta t)^2 + 0\,(\Delta x)^2$$

In general this scheme is only $0\,(\Delta t)$, $0\,(\Delta x)$ accurate.

Since:

$$\frac{\partial U}{\partial t} = -V\frac{\partial U}{\partial x}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right) = \frac{\partial}{\partial t}\left(-V\frac{\partial U}{\partial x}\right)$$

$$\Rightarrow$$

$$\frac{\partial^2 U}{\partial t^2} = -V\frac{\partial}{\partial x}\left(-V\frac{\partial U}{\partial x}\right)$$

$$\Rightarrow$$

$$\frac{\partial^2 U}{\partial t^2} = V^2\frac{\partial^2 U}{\partial x^2}$$

Substituting for the second time derivative in our expression for $\tau_{p,q}$:

$$\tau_{p,q} = \frac{\partial U_{p,q}}{\partial t} + V\frac{\partial U_{p,q}}{\partial x} - \frac{V\Delta x}{2}\frac{\partial^2 U_{p,q}}{\partial x^2} + \frac{\Delta t}{2}V^2\frac{\partial^2 U_{p,q}}{\partial x^2} + 0\,(\Delta x)^2 + 0\,(\Delta t)^2 + H.O.T.$$

Since $\frac{\partial U}{\partial t} = -V\frac{\partial U}{\partial x}$ for the exact solution, the truncation error equals:

$$\tau_{p,q} = \left(-\frac{V\Delta x}{2} + \frac{\Delta t}{2}V^2\right)\frac{\partial^2 U_{p,q}}{\partial x^2} + H.O.T.$$

$$\Rightarrow$$

$$\tau_{p,q} = \frac{V\Delta x}{2}\left(-1 + \frac{\Delta t V}{\Delta x}\right)\frac{\partial^2 U_{p,q}}{\partial x^2} + H.O.T.$$

$$\Rightarrow$$

$$\tau_{p,q} = \frac{V\Delta x}{2}(\mathbf{C}_{\#} - 1)\frac{\partial^2 U_{p,q}}{\partial x^2} + H.O.T.$$

This truncation term looks exactly like a diffusion term with:

$$D_{num} = \frac{V\Delta x}{2}(\mathbf{C}_{\#} - 1)$$

- This is known as Numerical Diffusion, Artificial Diffusion, Artificial Damping.
- In general using a fully upwinded scheme we are really solving:

$$\frac{\partial U}{\partial t} + V\frac{\partial U}{\partial x} = (D + D_{num})\frac{\partial^2 U}{\partial x^2}$$

- ***Thus a set of first order truncation terms is introduced which with their second order associated spatial derivatives are of identical form to the physical diffusion term. This introduces artificial or numerical diffusion which often competes with the physical diffusion term. This explains the generally over-damped nature of some of the solutions we examined in the numerical experiments.***
- Often $D_{num} > D$, introducing excessive artificial diffusion and therefore leading to a particularly poor solution
- ***The upwinded solution is however consistent!***
- At $\mathbf{C}_{\#} = 1$ and $\mathbf{P}_{e} = \infty$, our truncation error analysis indicates that $D_{num} = 0$ and that the higher order terms in the truncation series (*H.O.T.*) are all equal to zero as well. Therefore all truncation error terms are eliminated and a perfect solution results (as we saw in the numerical experiments).

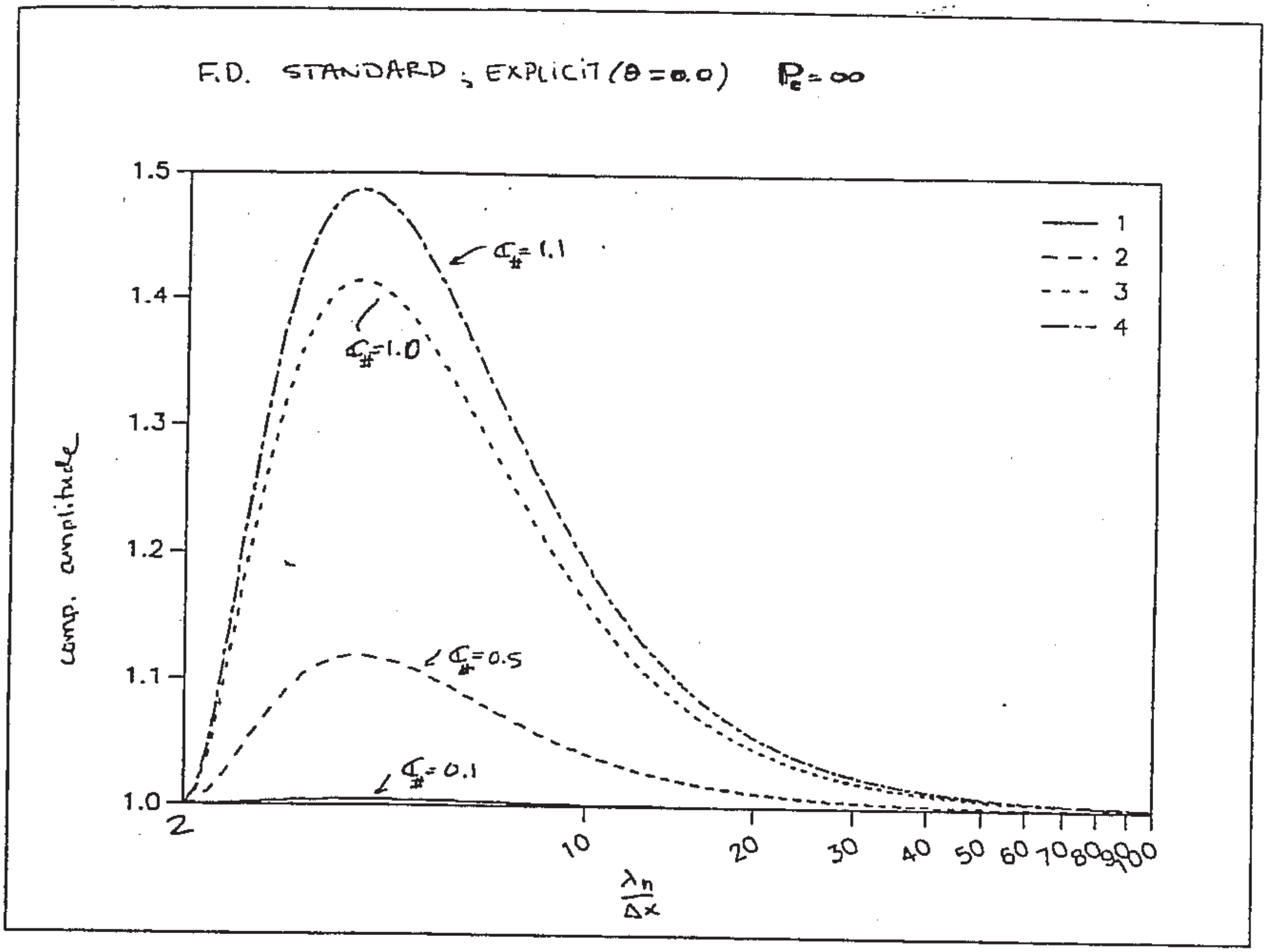
<u>Fourier Analysis of the Upwind Differenced Explicit Scheme</u>

Plot up

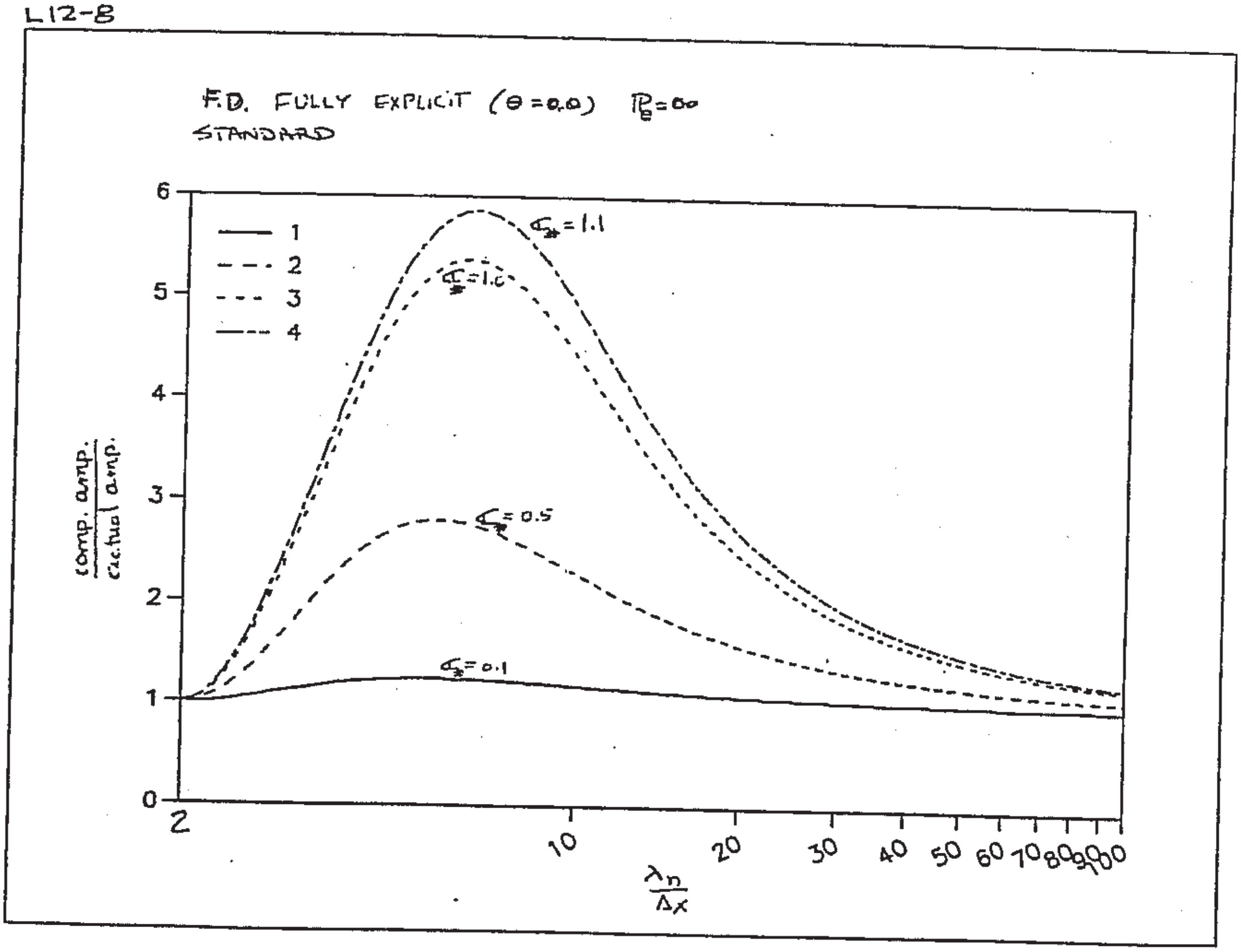
- $|\xi'_n|$ versus $\frac{\lambda_n}{\Delta x}$ demonstrates the stability for given values of $\mathbf{P}_{e}$ and $\mathbf{C}_{\#}$.
- $\left[\frac{|\xi'_n|}{|\xi_n|}\right]^{N_n}$ ratio versus $\frac{\lambda_n}{\Delta x}$ indicates whether the numerical solution is over-damped or not.

See Figures L12.7 through L12.14.

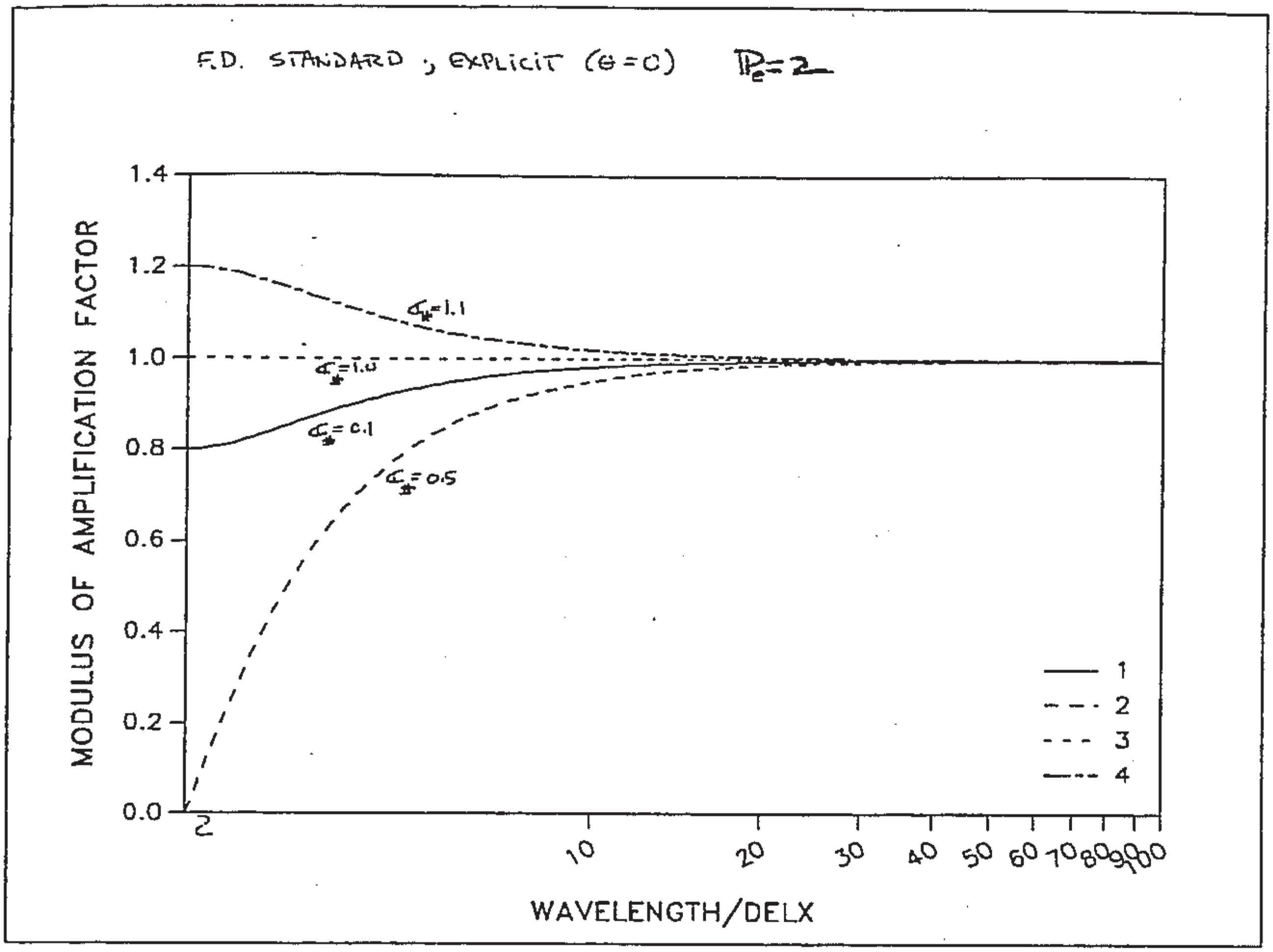
L12-7


F.D. STANDARD ; EXPLICIT (θ = 0.0) Pe = ∞
1.5
1.4
1.3
1.2
1.1
1.0
comp. amplitude
1
2
3
4
C# = 1.1
C# = 1.0
C# = 0.5
C# = 0.1
2
10
20
30
40
50
60
70
80
90
100
λn / Δx

L12-8

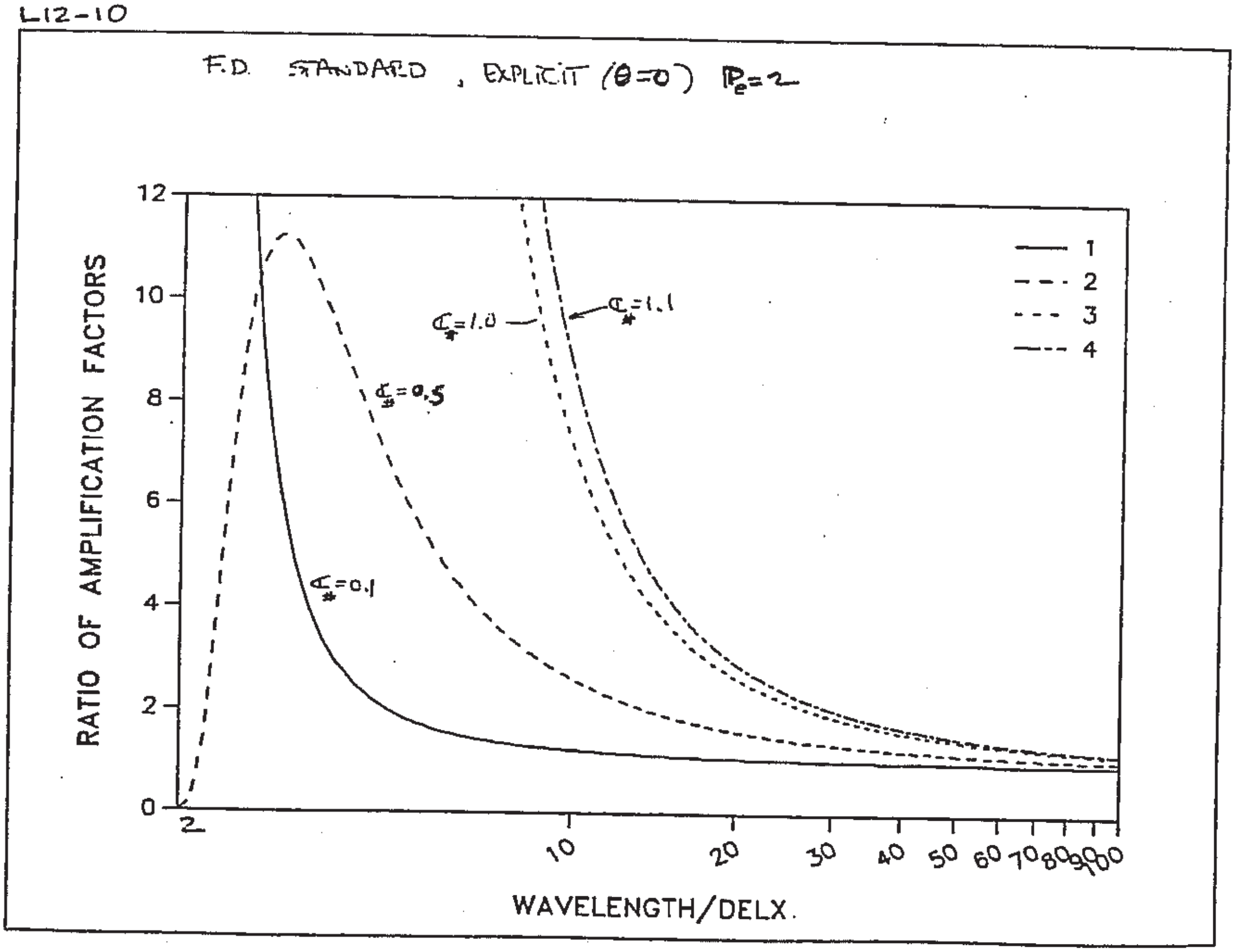

F.D. FULLY EXPLICIT (θ = 0.0) Pe = ∞
STANDARD
6
5
4
3
2
1
0
comp. amp. / actual amp.
1
2
3
4
C# = 1.1
C# = 1.0
C# = 0.5
C# = 0.1
2
10
20
30
40
50
60
70
80
90
100
λn / Δx

L12-9



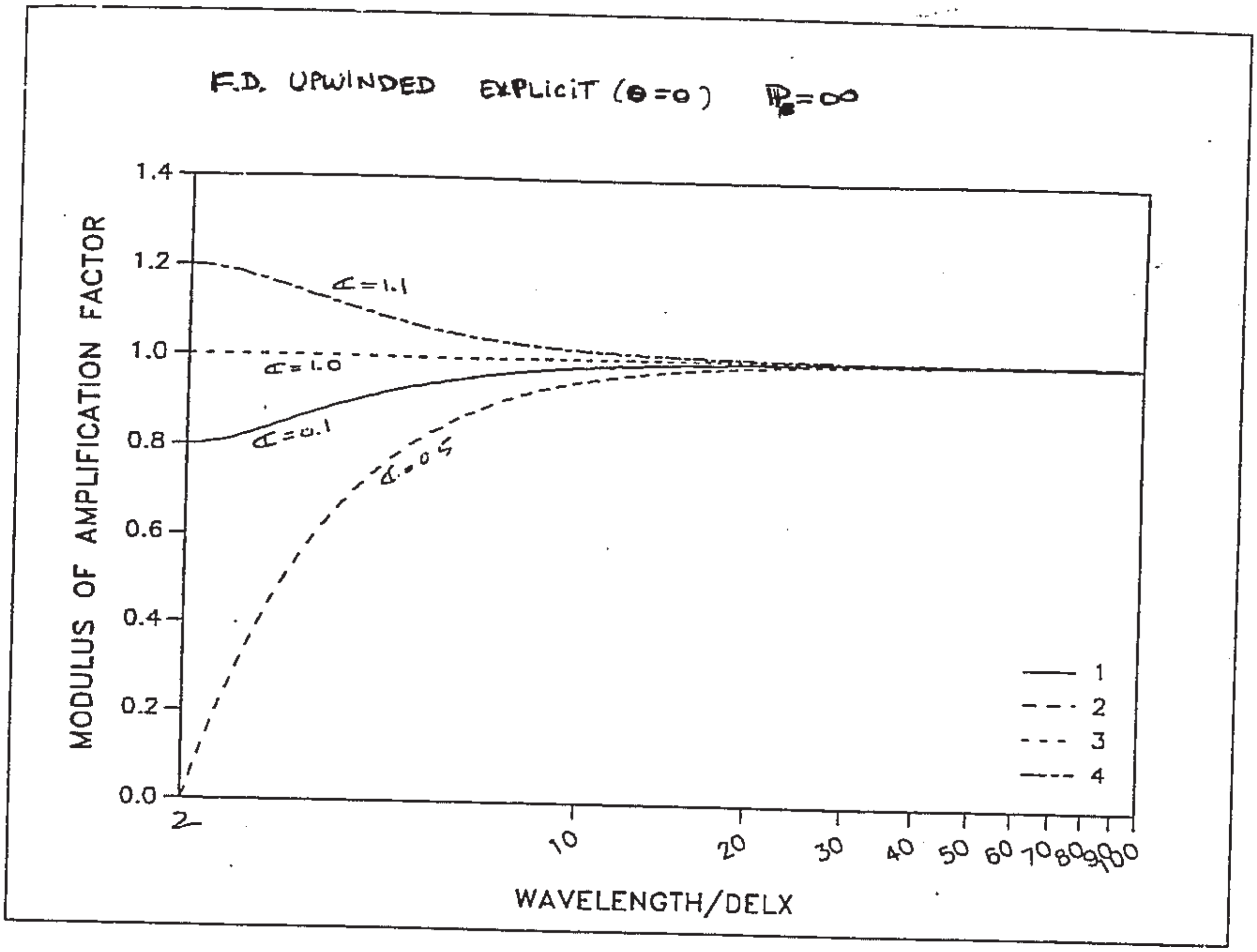
F.D. STANDARD, EXPLICIT (θ=0) Pe=2
MODULUS OF AMPLIFICATION FACTOR
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
C*=1.1
C*=1.0
C*=0.1
C*=0.5
1
2
3
4
2
10
20
30
40
50
60
70
80
90
100
WAVELENGTH/DELX

L12-10

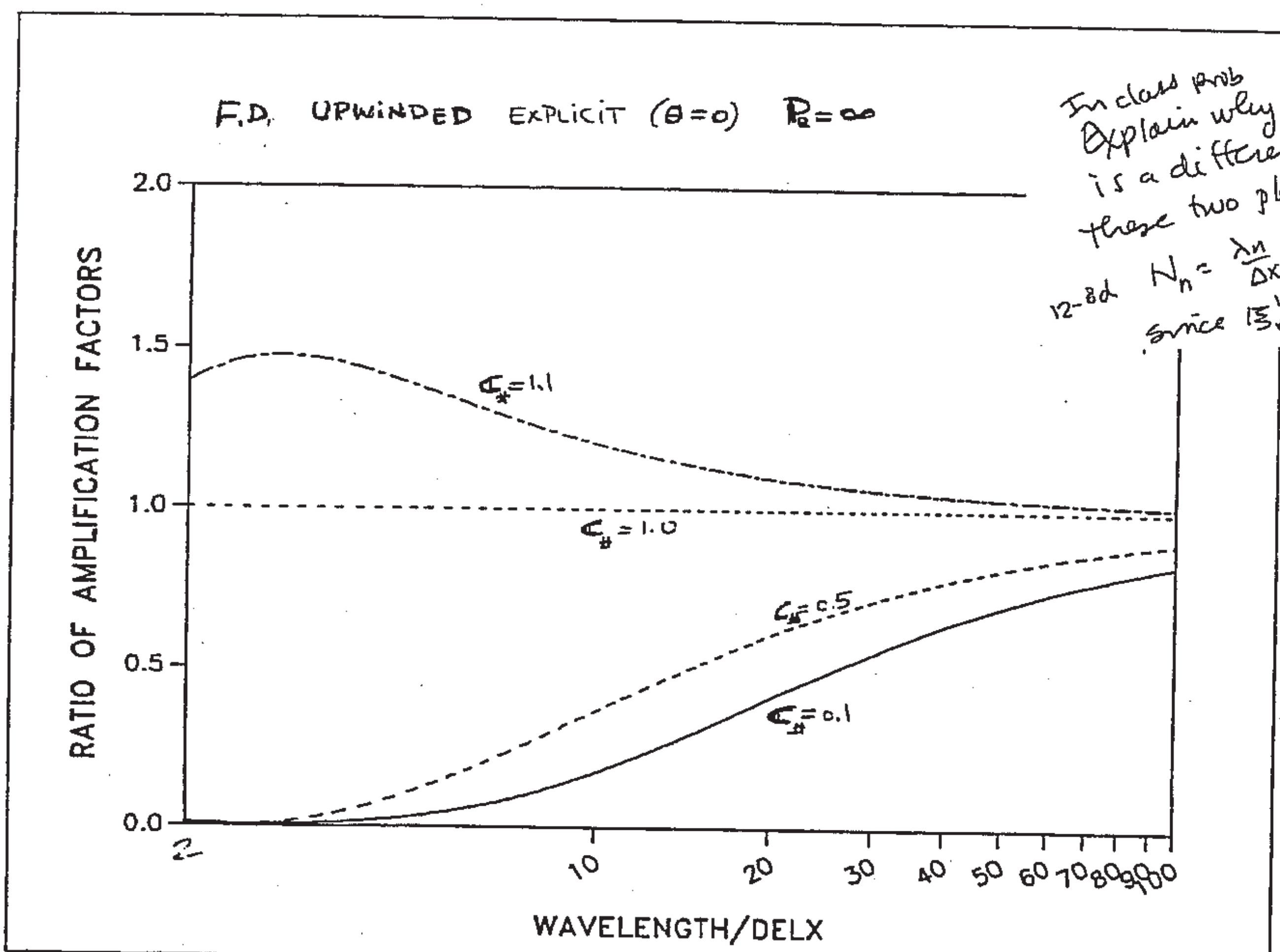


F.D. STANDARD, EXPLICIT (θ=0) Pe=2
RATIO OF AMPLIFICATION FACTORS
12
10
8
6
4
2
0
C*=1.0
C*=1.1
C*=0.5
C*=0.1
1
2
3
4
2
10
20
30
40
50
60
70
80
90
100
WAVELENGTH/DELX.

L12-11

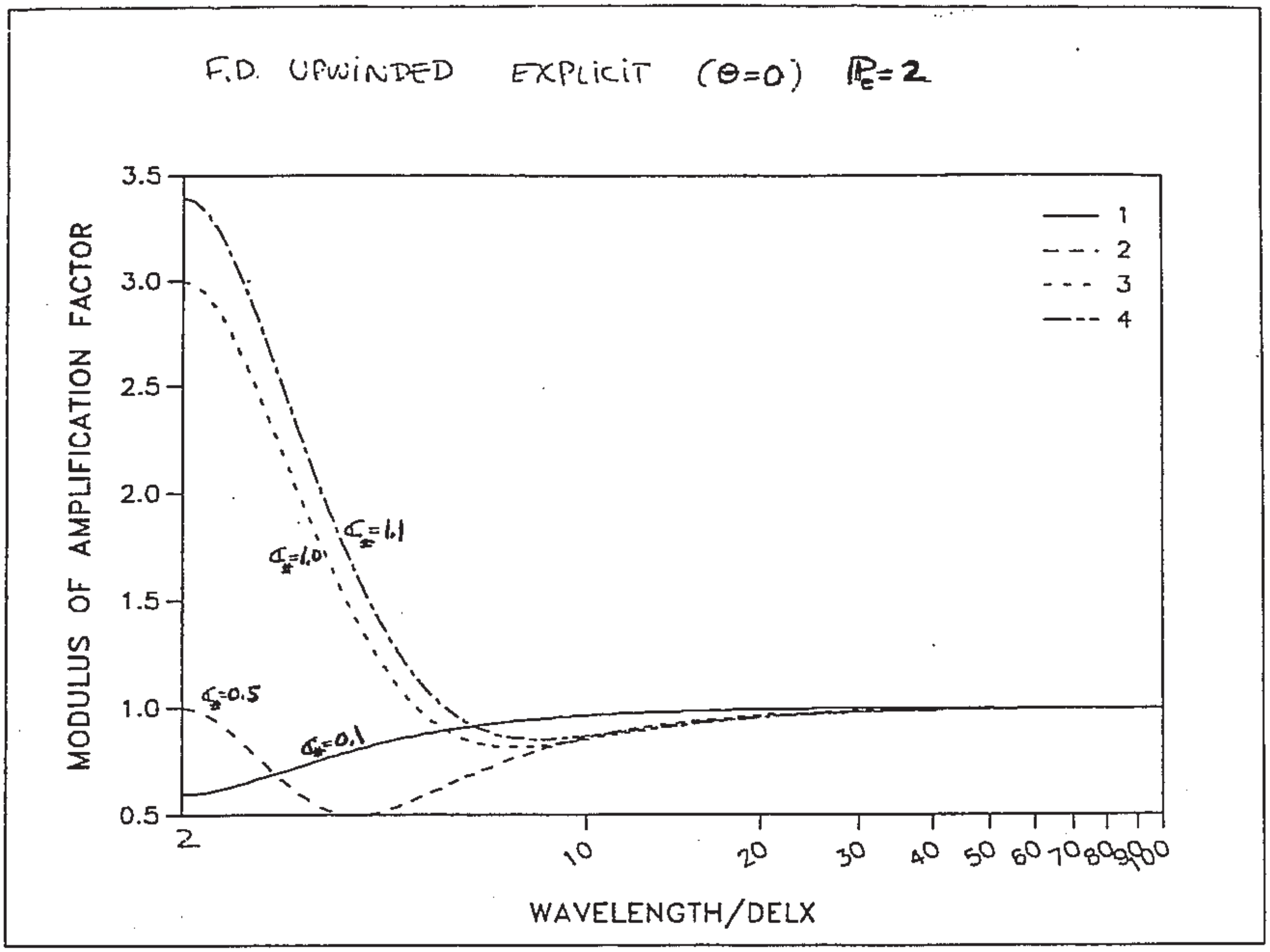


L12-12

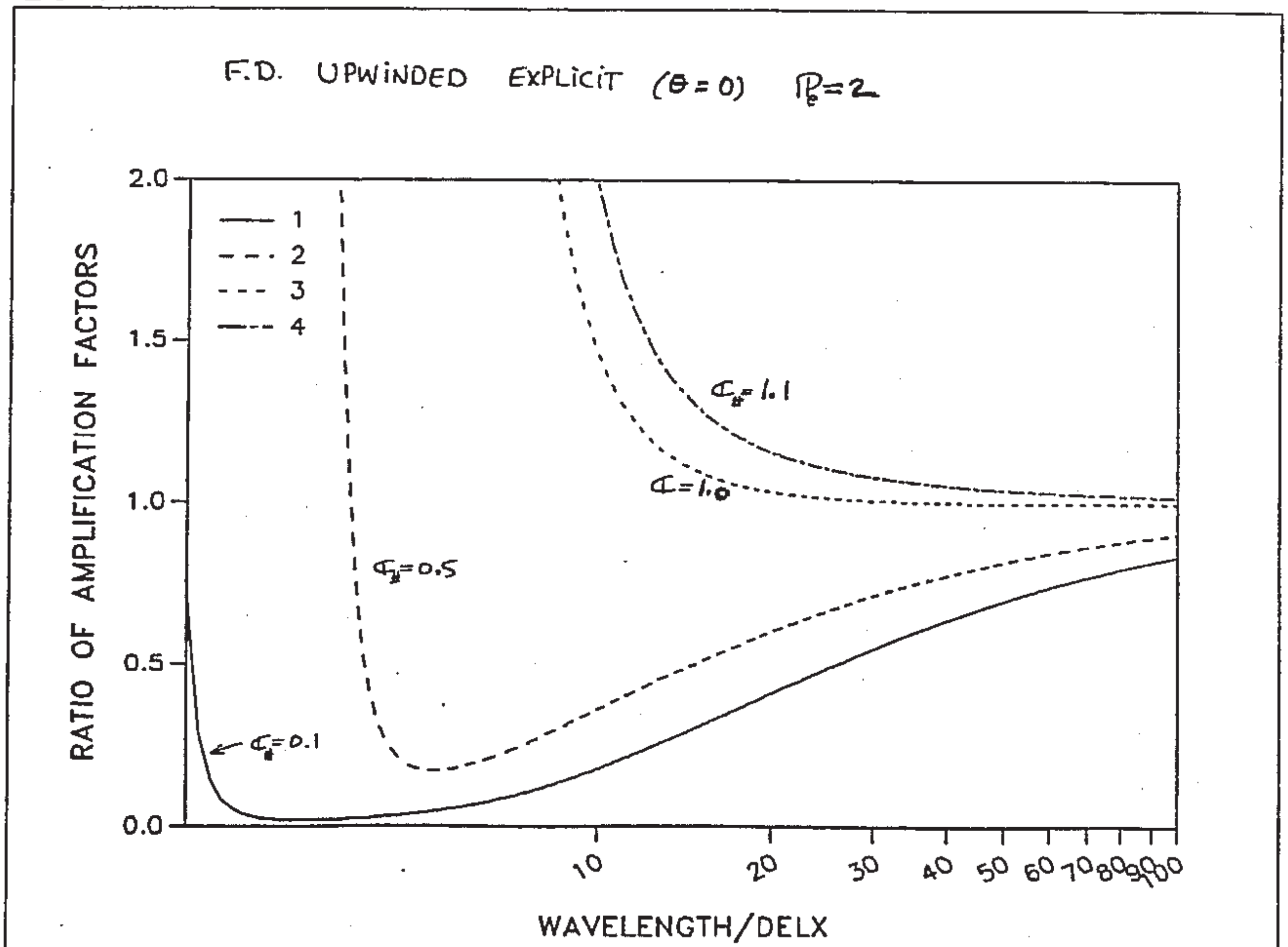


In class prob
Explain why there is a difference b/w these two plots!
12-8d $N_n = \frac{\lambda_n}{\Delta x} \cdot \frac{1}{C}$
since $|\xi_n'| = ? \geq 1$

L12-13


F.D. UPWINDED EXPLICIT (θ=0) Pe=2
MODULUS OF AMPLIFICATION FACTOR
WAVELENGTH/DELX
3.5
3.0
2.5
2.0
1.5
1.0
0.5
2
10
20
30
40
50
60
70
80
90
100
1
2
3
4
C#=1.1
C#=1.0
C#=0.5
C#=0.1

L12-14


F.D. UPWINDED EXPLICIT (θ=0) Pe=2
RATIO OF AMPLIFICATION FACTORS
WAVELENGTH/DELX
2.0
1.5
1.0
0.5
0.0
10
20
30
40
50
60
70
80
90
100
1
2
3
4
C#=1.1
C=1.0
C#=0.5
C#=0.1

- Figures L12.7/8: FD standard solution, fully explicit for $\mathbf{P_e} = \infty$ and various values of $\mathbf{C_\#}$:
  - $|\xi'_n|$ plot shows unstable behavior for all $\mathbf{C_\#}$ values. Therefore the scheme is unconditionally unstable (as we previously found in our stability analysis). Therefore $|\xi'_n| > 1 \quad \forall \quad \left(\frac{\lambda}{\Delta x}\right) \cdot C_\#$
  - $\frac{|\xi'_n|}{|\xi_n|}$ ratio: This plot shows the ratio > 1.0. Thus the numerical solution damps less than the analytical solution.

- Figures L12.9/10: FD standard, fully explicit for $\mathbf{P_e} = 2$ and various values of $\mathbf{C_\#}$.
  - $|\xi'_n|$ plot shows stable behavior for $\mathbf{C_\#} \leq 1$ and unstable behavior for $\mathbf{C_\#} > 1$ (as our previous analysis indicated).
  - $|\xi'_n|/|\xi_n|$ plot; Shows that typically the ratio > 1 which indicates that solution components are typically damped less than the analytical solution.
  - Note that this case shows that $|\xi'_n| < 1.0$ (and thus stable) while we can have $|\xi'_n|/|\xi_n| > 1.0$. This just means that the numerical solution is damped less than the analytical solution. The numerical solution, however, is still damped enough to prevent unstable growth.

- Figures L12.11/12: FD upwinded and explicit for $\mathbf{P_e} = \infty$ and various values of $\mathbf{C_\#}$.
  - $|\xi'_n|$ plot shows stable behavior for $\mathbf{C_\#} \le 1.0$ (as predicted with stability analysis and numerical experiments).
  - $|\xi'_n|/|\xi_n|$ ratio plot exhibits very high damping for a very wide range of $\lambda_n/\Delta x$ values. This leads to severely overdamped solutions without any wiggles (since these are most heavily damped). However, at $\mathbf{C_\#} = 1$, we have a perfect solution (since $R_A = 1$ and $\Phi = 0$ for all values of $\lambda_n/\Delta x$).

- Figures L12.13/14: FD upwinded and explicit for $\mathbf{P_e} = 2$ and various values of $\mathbf{C_\#}$.
  - $|\xi'_n|$ plot shows stable behavior for $\mathbf{C_\#} \le 0.5$ (as predicted with stability analysis and numerical experiments).
  - $|\xi'_n|/|\xi_n|$ ratio plot again exhibits very high damping for most of the $\lambda_n/\Delta x$ range. This again leads to the highly overdamped solution seen in the numerical experiments (only considering the stable range $0 \le \mathbf{C_\#} \le 0.5$).

Lecture No. 13

References Leonard: 1. Comp. Meth. Appl. Mech. & Eng., 1974; 2. F.E.M. for Convection Dominated Flows AMD Vol. 34, 1979; 3. CTAC-83 Conference.

Solutions to C-D equation for Convection Dominated Flows

- Central schemes of any order applied to spatially resolve the convection operator of tend to lead to numerical oscillations (wiggles, noise, $2 \cdot \Delta x$ waves, etc.)
- First order upwinding of the convective terms in general introduces a truncation term which is identical to a diffusion type term. This introduces lots of "artificial diffusion" (or damping) which lowers peaks, causes excessive spread and destroys gradients. Even though the special case $\theta = 0.0$, $\mathbf{C}_{\#} = 1.0$, $\mathbf{P}_e = \infty$ leads to a perfect numerical solution, upwinding in general:
  - ***Is not robust. In fact we have to be very close to*** $\mathbf{C}_{\#} = 1.0$ ***and*** $\mathbf{P}_e = \infty$ ***to get it to work well!*** If we deviate from $\mathbf{C}_{\#} = 1.0$ and $\mathbf{P}_e = \infty$ the solution gets very bad and is overdamped.
  - Can not maintain $\mathbf{C}_{\#} = 1.0$ in 2-D general flows since Δx and V change in space.

Alternatives to Pure Upwinding

1. Optimal weighted upwinding

Approximate the convection term as:

$$\frac{\partial u}{\partial x} = \alpha_{up}\left[\frac{u_i - u_{i-1}}{\Delta x}\right] + (1-\alpha_{up})\left[\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right]$$

$$\alpha_{up} = \textit{upstream weighting factor}\left(\begin{array}{ll} 0 & \Rightarrow \textit{central differencing} \\ 1 & \Rightarrow \textit{full upwinding} \end{array}\right.$$

For steady problem we can derive:

$$\alpha_{up-opt} = \coth\left(\frac{\mathbf{P}_e}{2}\right) - \frac{2}{\mathbf{P}_e}$$

- For this selection of α_{up} all truncation terms are eliminated for <u>1-D steady state</u> problems.
- However this technique does have considerable limitations:
- Even for moderate $\mathbf{P_e} > 5 - 10$, $\alpha_{up-opt} = 1$. This works well for 1-D steady state problems. However for a <u>slightly</u> different problem we readily get overly diffusive (or damped) solutions. Also when applying this scheme to 2-D flows, we experience excessive damping perpendicular to the flow direction (referred to as cross wind diffusion).
- Very serious problems when flow direction changes
- α_{up-opt} has been derived for the steady state problem, it does <u>not</u> carry over <u>well</u> to the time dependent problem.

- Many prominent investigators are opposed to using standard upwinding methods (see Figure L13.1).
  - Leonard very much against 1st order upwinding
  - Gresho "Upwinding can be dangerous to your health"
  - Abbot "Like sedating a person with a nervous breakdown"
- For realistic flow computations upwinding invariably leads to over-diffused solutions!

<u>2. Matched Artificial Diffusivity Schemes (MAD)</u>

- Schemes which compute the artificial diffusivity introduced by upwinding and subtract it off of the physical diffusion term. This only works when $D_{num} < D_{phys}$. The procedure becomes unstable when $D_{num} > D_{phys}$ since we would be trying to add in negative diffusion.

From. F.E.M. for Convection Dominated Flows ASME N.Y. Dec. 1979 AMD Vol 34

A SURVEY OF FINITE DIFFERENCES OF OPINION ON NUMERICAL MUDDLING OF THE INCOMPREHENSIBLE DEFECTIVE CONFUSION EQUATION

B. P. Leonard

City University of New York

College of Staten Island, New York

ABSTRACT

Finite difference methods have been very successful for partial differential equations dominated by the Laplacian operator, such as those for diffusion and wave motion. However, when the first-derivative convection operator becomes important, standard central difference methods lead to incomprehensible wiggles or confusing nonconvergence -- and are therefore clearly defective. There have been various opinions on suitable remedies, one of the most popular being to use highly stable one-sided upstream differencing for the convection term. But the artificial diffusion of such methods leads to low accuracy. Nevertheless, proponents have "justified" upstream differencing (or upstream-central hybrids) by a variety of arguments -- unfortunately, all fallacious! Much confusion has stemmed from a muddling of the meaning of truncation error. When it is realized that "standard" central differencing for the Laplacian operator is a *third* order method, and that central difference methods (of any order) lack inherent stability for modelling *odd* order derivatives, the consistent third order convective differencing scheme is seen to be optimal in terms of accuracy, stability, and simplicity.

H.O L13-1b From Computers and Fluids

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DON'T SUPPRESS THE WIGGLES—THEY'RE TELLING YOU SOMETHING!†

PHILIP M. GRESHO and ROBERT L. LEE

Lawrence Livermore National Laboratory, University of California, Livermore, CA 94550, U.S.A.

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Abstract—The subject of oscillatory solutions (wiggles), which sometimes result when the conventional Galerkin finite element method is employed to approximate the solution of certain partial differential equations, is addressed. It is argued that there is an important message behind these wiggles and that the appropriate response to it usually involves a combination of: re-examination of the imposed boundary conditions, judicious mesh refinement (via isoparametric elements) in critical areas, and sometimes even admitting that the problem, as posed, is just too difficult to solve adequately on an "affordable" mesh. It is further argued that it is usually an inappropriate response to develop methods which *a priori* suppress these wiggles and thereby make claims that these unconventional FEM techniques are actually improvements and can be used to solve difficult problems on coarse meshes.

3. Higher Order Finite Difference Approximations

Leonard [1979: Comp. Meth. Appl. Mech. Eng.] proposed a 3rd order upwinded scheme for modeling the convective term. He uses the approximation:

$$\left(\frac{u_{i+1}-u_{i-1}}{2\Delta x}\right) - \left(\frac{u_{i+1}-3u_i+3u_{i-1}-u_{i-2}}{6\Delta x}\right) =$$

$$\frac{\partial u}{\partial x} + \frac{1}{12}\frac{\partial^4 u_i}{\partial x^4}\Delta x^3 + H.O.T.$$

$$\Rightarrow$$

$$\frac{2u_{i+1}+3u_i-6u_{i-1}+u_{i-2}}{6\Delta x} = \frac{\partial u}{\partial x} + \frac{1}{12}\frac{\partial^4 u_i}{\partial x^4}\Delta x^3 + H.O.T.$$

- The scheme is implemented with the above approximation for the convective terms, a central approximation for the diffusion term and an explicit scheme for time.
- ***"QUICKEST"*** which designates quadratic upstream interpolation for convective dynamics with estimated streaming terms.

Advantages of Quickest:

- Upwinding has a stabilizing influence on short wavelengths. i.e., they tend to get damped out.
- The spatial derivative on the leading order truncation term is higher than the modeled physical diffusion term. (i.e., fourth derivative truncation term). Thus a truncation term is no longer competing directly with a physically relevant term in the p.d.e.! Furthermore the leading order truncation term is of $O(\Delta x)^3$ as opposed to $O(\Delta x)$ as was the case in standard upwinding.

- An algorithm which uses the quickest approximation for convection and a central type approximation for diffusion will be exact for a cubic response (in terms of a polynomial) for any combination of convection and diffusion since the approximation for convection introduces a leading truncation term with a fourth derivative and the approximation to the diffusion term also includes a fourth derivative in the leading truncation term:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = \frac{\partial^2 u_i}{\partial x^2} + \frac{1}{12}\frac{\partial^4 u_i}{\partial x^4}(\Delta x)^2 + H.O.T.$$

- Let's examine the Fourier amplitude and amplitude ratio portraits to see why QUICKEST is successful. The amplification factor derived using Fourier analysis (Leonard, 1984, CTAC-83):

$$|\xi'_n| = 1 - (2\rho + \mathbf{C}_{\#}^2)(1 - \cos(\beta_n \Delta x)) - 2\mathbf{C}_{\#}\left[\frac{1}{6}(1 - C_{\#}^2) - \rho\right]$$

$$(1 - \cos(\beta_n \Delta x))^2 - \hat{\imath}\{\mathbf{C}_{\#}\sin(\beta_n \Delta x) + 2\mathbf{C}_{\#}\sin(\beta_n \Delta x)$$

$$\left[\frac{1}{6}(1 - \mathbf{C}_{\#}^2) - \rho\right](1 - \cos(\beta_n \Delta x))\}$$

Plot $|\xi'_n|$ and $\left[\frac{|\xi'_n|}{|\xi_n|}\right]^N$ versus $(\frac{\lambda_n}{\Delta x})$

- Figures L13.2/3, $\mathbf{P_e} = \infty$,
  - Figure L13.2: Shows that the method is stable at $\mathbf{P_e} = \infty$, $\mathbf{C}_{\#} \le 1.0$
  - Figure L13.3: Shows that damping occurs for $\frac{\lambda_n}{\Delta x} \le 10$.
  - However, for larger $\frac{\lambda_n}{\Delta x}$, there is almost no damping.
  - Therefore, the scheme damps very short wavelengths (these typically appear as wiggles due to phase propagation errors) while it doesn't damp long

wavelengths.

- Figures L13.4/5, $\mathbf{P_e} = 2$
  - Figure L13.4: shows that the method is stable for $\mathbf{C_\#} \leq 1.0$ and $\mathbf{P_e} = 2$.
  - Figure L13.5: Again shows that only <u>short</u> wavelengths are damped out (wiggles) whereas the longer wavelengths are not.

- QUICKEST damps very short wavelengths (wiggles) while it does not damp longer wavelengths.
- Furthermore phase lag (not shown) is significantly better than the standard central schemes.
- The above is about as good a set of conditions as one could hope for, short of having a "perfect" solution with no truncation terms.
- Stability of QUICKEST: Figure L13.6 shows the stability range for this scheme.

<u>General Remarks on Quickest</u>

- QUICKEST works very well (not perfectly) for a wide range of problems. It's an excellent scheme for FD solutions to equations with first spatial derivatives. Note that higher order versions of Quickest have also been developed. ~~recently~~.
- Even order upwinding schemes (e.g. 2nd order with 3 points, 4th order) doesn't work nearly as well as odd order upwinding schemes (e.g. 3rd order with ~~5~~ 4 points).
- Even order upwinded solutions tend to be wiggly. (See Figure L13.7b)

FIG L13-2

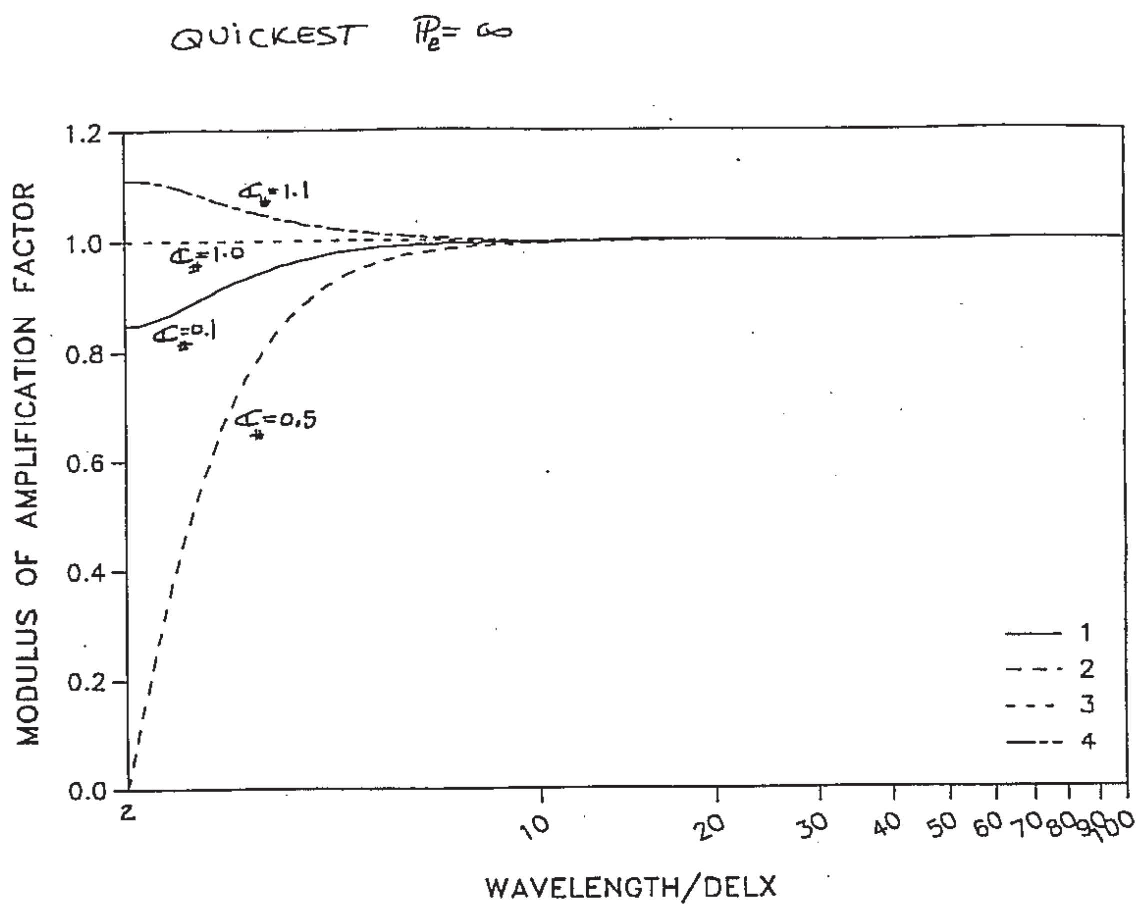


FIG L13-3

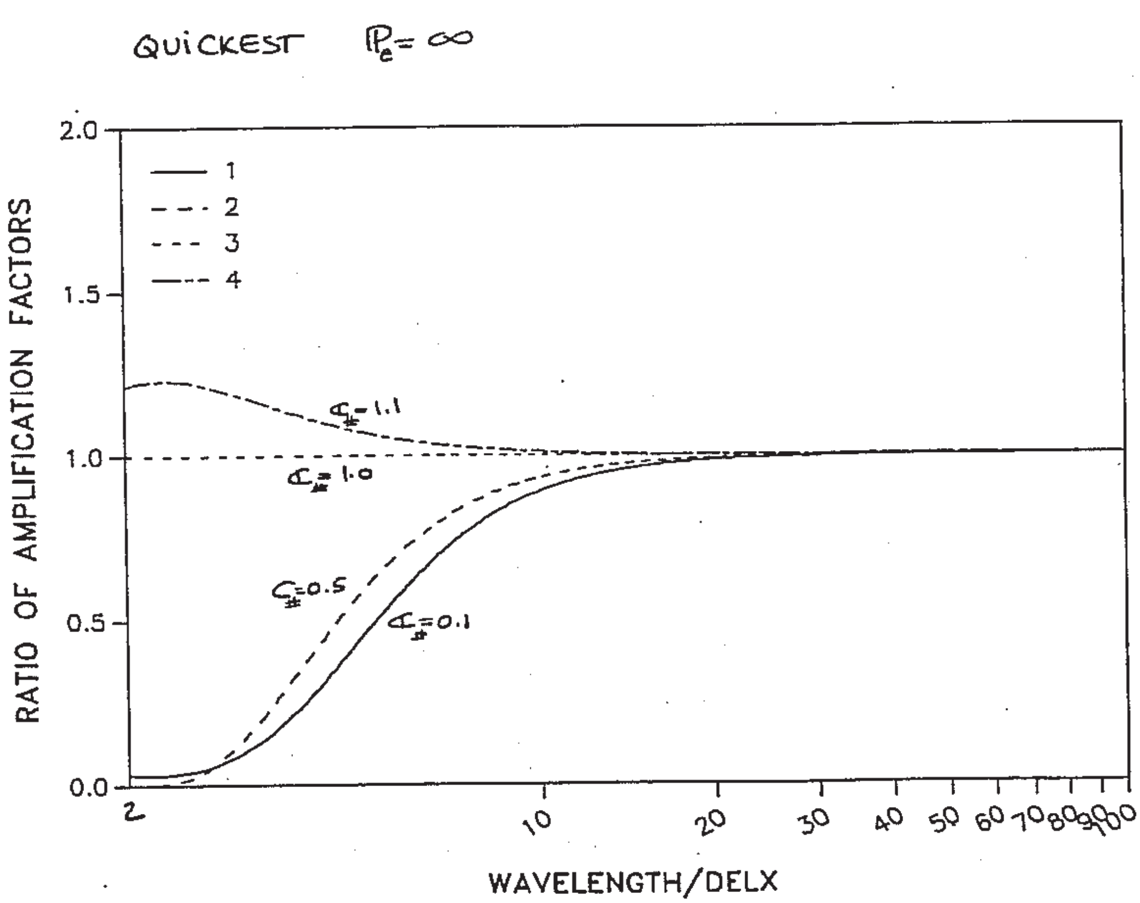


FIG L13-4

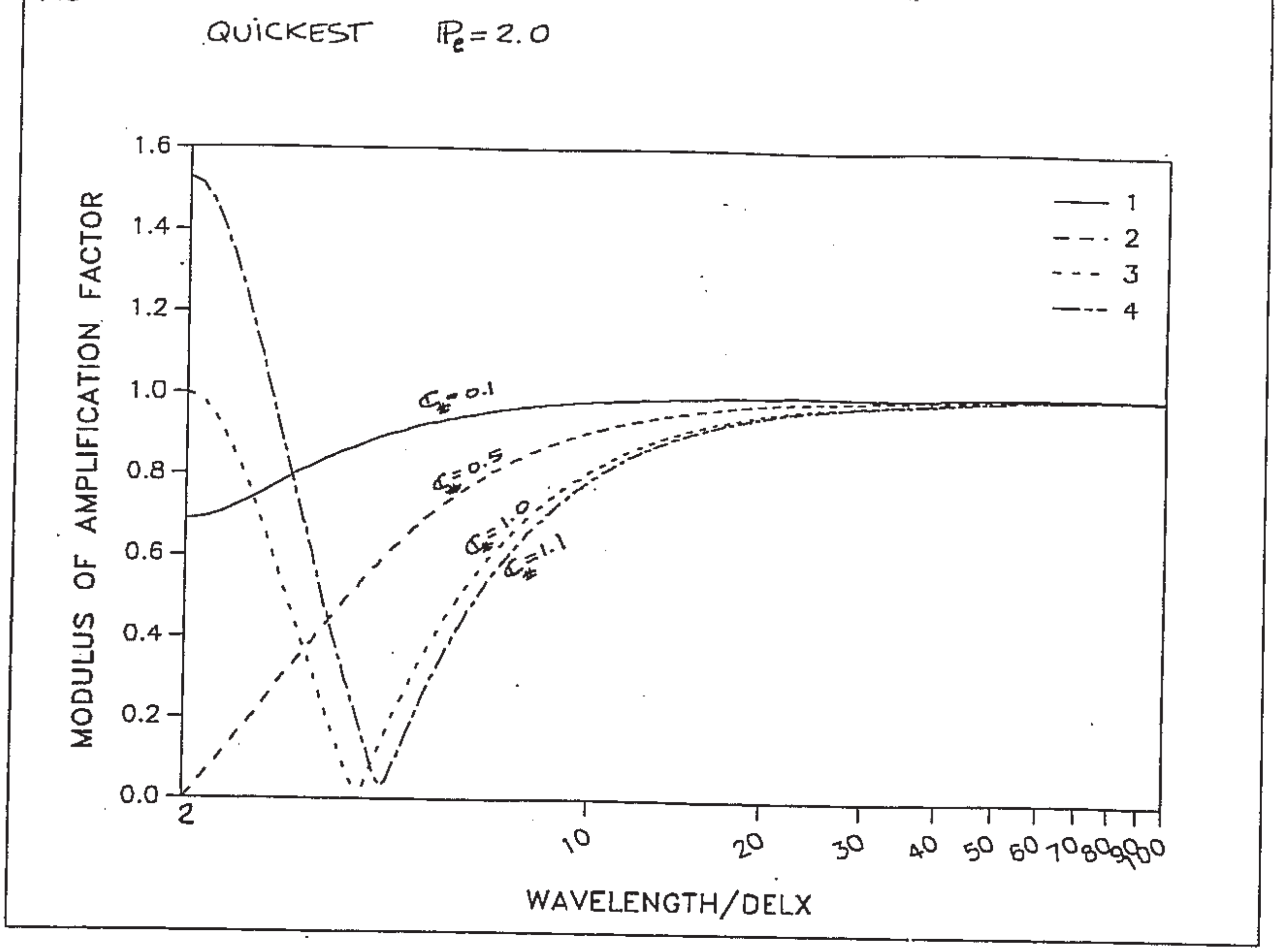


FIG L13-5

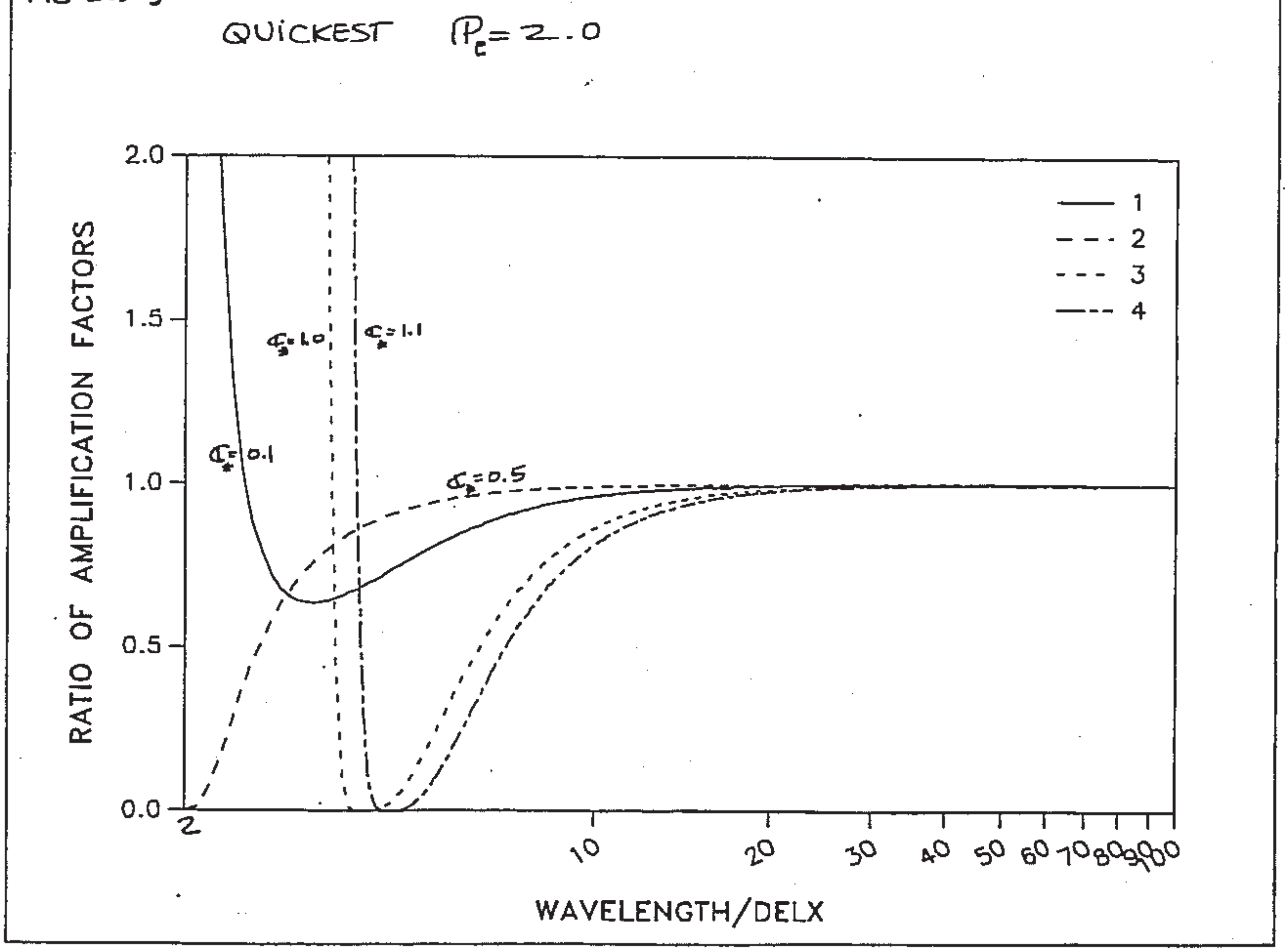
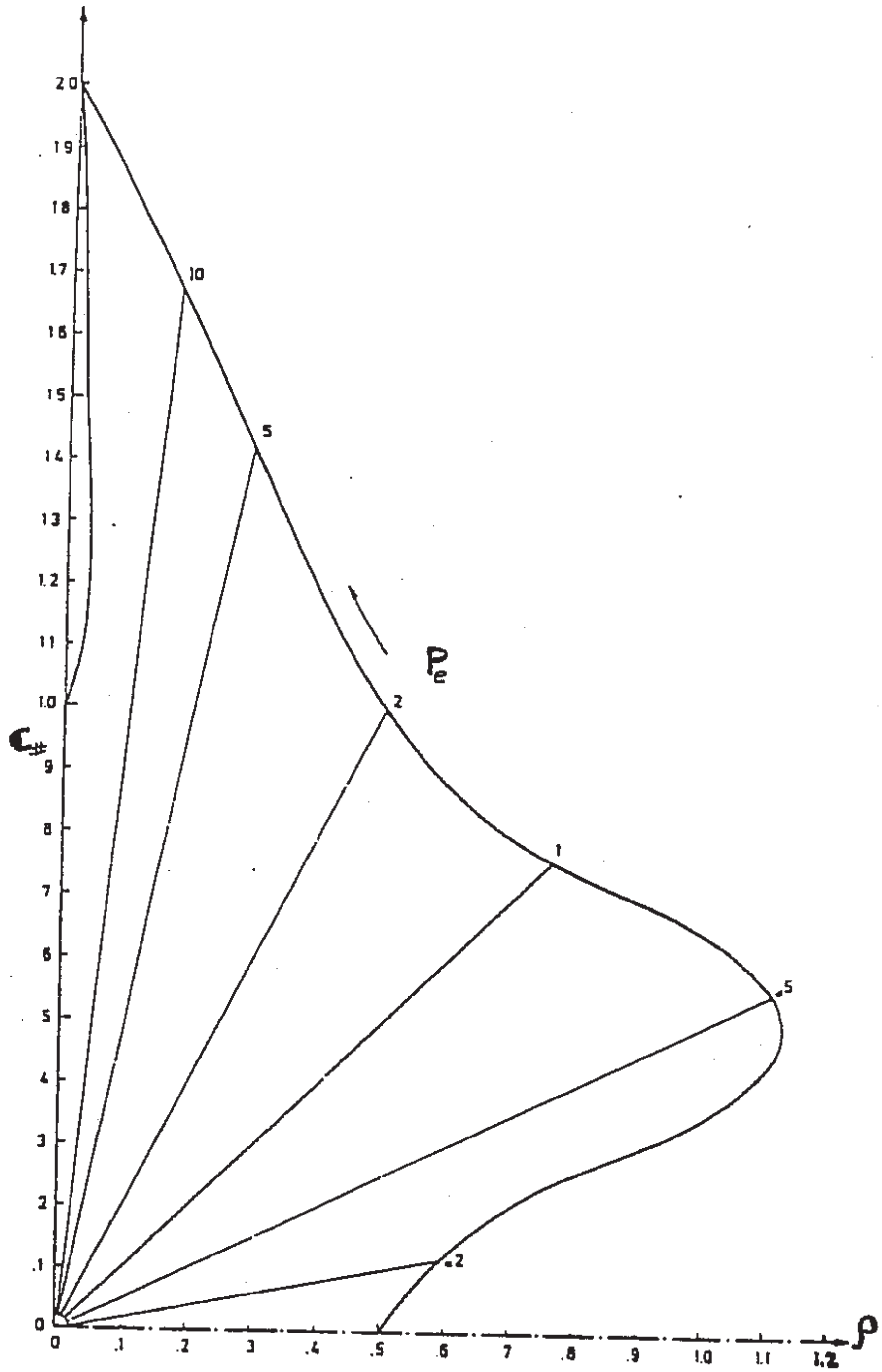
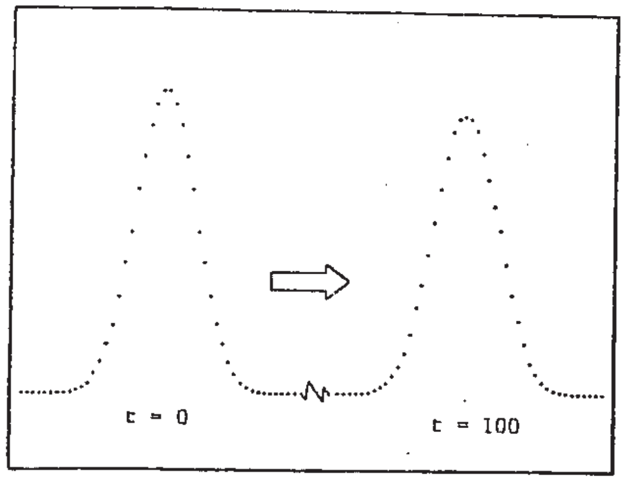


Fig L 13-6



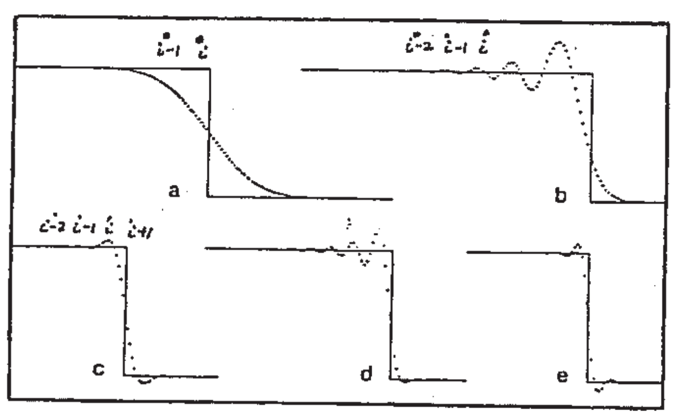
Stability Range of QUICKEST in $(\rho, C_{\#})$ plane as mapped by Leonard.
Note that scheme is unstable for $1 < C_{\#} < 2$ and again stable for $C_{\#} > 2$ when $\rho = 0$.

Fig L13-7a:



Pure convection of an initial Gaussian profile using the adjusted QUICKEST algorithm.

Fig L13-7b



Step front @ $t = 100$, $\Delta t = 0.1$ $C_{\#} = 0.5$ $P_e = \infty$

a. 1st order upwinding
b. 2nd " "
c. 3rd " "
d. 4th " "
f 5th " "