Math 10350 Fall 07 – Handout 14
(Review 3 & Sections 4.3 - 4.5)

1. (Review: Pg 243 Q33 - 39) Find the following limits:

a. $\displaystyle\lim_{x\to\infty}\frac{2x^2}{3x^2+5}$ **b.** $\displaystyle\lim_{x\to-\infty}\frac{3x^2}{x+5}$ **c.** $\displaystyle\lim_{x\to\infty}\frac{2x}{3x^2+5}$ **d.** $\displaystyle\lim_{x\to-\infty}\frac{6x}{x+\cos x}$

e. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+x}}{-2x}$ **f.** $\displaystyle\lim_{x\to\infty}\frac{5\cos x}{x}$ **g.** $\displaystyle\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+4}}$

Give graphical interpretations for each of the limits above.

(Ans: (a) 2/3; (b) $-\infty$; (c) 0; (d) 6; (e) 1/2; (f) 0; (g) 3. If the limit is finite, then there is a horizontal asymptote, eg. for (a), $y = 2/3$ is a horizontal asymptote.)

2. (Review) The profit of a company is given by the function $P(x) = -\frac{x^2}{2} + 2x + 200$ where x is the number of items sold. Using differential, estimate the change in the profit as sales increases from 4 to 4.5. Write down also the approximate percentage change in P. Compare your results with the exact change in P.

(Ans: $\Delta P \approx dP = -1$; approximate percentage change = -0.5%; Exact $\Delta P = -1.125$; Exact percentage change = -0.5625%)

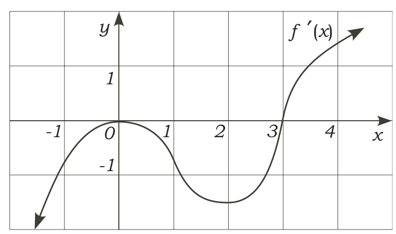
3. (Review) Find the linear approximate of the function $g(x) = \sqrt[3]{1+x}$ at $x = 7$ and use it to approximate the number $\sqrt[3]{7.95}$. Illustrate by graphing g and the tangent line.

(Ans: $g(x) \approx 2 + \frac{1}{12}(x-7)$ for x near 7; $\sqrt[3]{7.95} = \sqrt[3]{1+6.95} = g(6.95) \approx 1\frac{239}{240}$)

4. (Review) Let $f(x) = \frac{3x^2}{x^2-1}$; $x \neq 1, -1$. Determine the monotonicity and concavity of $f(x)$. Write down also the values of x for which there is a local maximum and minimum, and inflection points.

(Ans: Note $f'(x) = -\frac{6x}{(x^2-1)^2}$ and $f''(x) = \frac{6(3x^2+1)}{(x^2-1)^3} = \frac{6(3x^2+1)}{(x^2-1)^2}\cdot\frac{1}{(x^2-1)}$. Monotonicity: increasing on $(-\infty,-1)\cup(-1,0)$; decreasing on $(0,1)\cup(1,\infty)$. Concavity: Concave up on $(-\infty,-1)\cup(1,\infty)$; concave down $(-1,1)$. Local max at $x = 0$. No inflection points because $-1, 1$ are not in domain of $f(x)$.)

5. (Review) Discuss the monotonicity and concavity of the function $f(x)$ if the given graph is the graph of the derivative $f'(x)$ of $f(x)$. State all values of x for which there is a critical point or inflection point. Use second derivative test to give the nature of the critical point whenever possible otherwise use first derivative test.



(Monotonicity: $f(x)$ increasing when $f'(x) > 0$ so increasing on $(3,\infty)$; $f(x)$ decreasing when $f'(x) < 0$ so decreasing on $(-\infty,0)\cup(0,3)$. Concavity: Concave up when $f''(x) > 0$ i.e. $f'(x)$ is increasing. So concave up on $(-\infty,0)\cup(2,\infty)$. Concave down when $f''(x) < 0$ i.e. $f'(x)$ is decreasing. So concave down on $(0,2)$. Critical points at $x = 0, 3$. Local min at $x = 3$ since $f''(3) > 0$. Second derivative test falis for $x = 0$, so use first derivative test. $x = 0$ is neither max nor min. Inflection points $x = 0, 2$.)

6. (Review) Using Newton's method, find the smallest positive root of $\tan x = x$.

(Ans: Smallest positive root is between π and $3\pi/2$. Take $x_1 = 4.5$, use newton's formula to get $x = 4.49340946$)

7. (Review) Find the antiderivatives F of the following function f. In particular write down the antiderivative that satisfies given condition:

a. $f(x) = \dfrac{x^3 - 7x^{2/3} + 3}{\sqrt[3]{x}}$; $F(1) = 5$ **b.** $f(x) = \sin x - \cos x$; $F(0) = 1$.

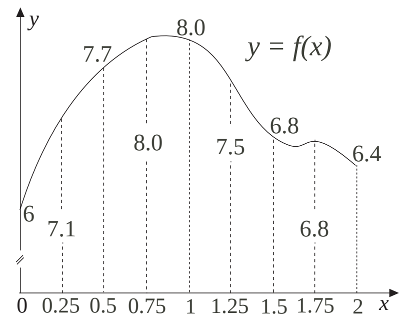
(Ans: Note $F'(x) = f(x)$ so $F(x) = \int f(x)dx$. (a) $F(x) = \frac{3}{11}x^{11/3} - \frac{21}{4}x^{4/3} + \frac{9}{2}x^{2/3} + \frac{41}{44}$; (b) $F(x) = -\cos x - \sin x + 2$)

8. (Review) A closed rectangular storage container is to have a volume 108 m^3. The base and the top of the container must be square. Material for the base and the top weighs $\frac{1}{2}$ g/m^2 while material for the sides weighs $\frac{1}{8}$ g/m^2. Find the dimension of the box that minimizes the weight of the material used to make the box.

(Ans: Set width = x and height = y. Minimize $W(x) = x^2 + \frac{54}{x}$ for $0 < x < \infty$; $x = 3$ and $y = 12$)

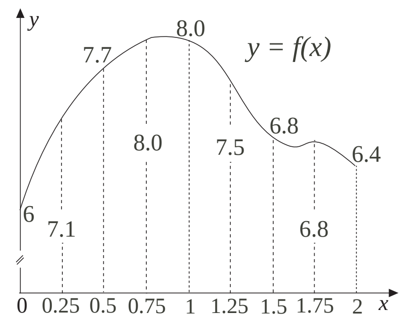
1. Estimate the area under the graph of $y = f(x)$ over $0 \le x \le 2$ using the following method.

(a) Upper sum with **4 segments**.



(b) Lower sum with **8 segments**.

$$\text{Area} \approx f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_8)\cdot\Delta x$$
$$= [f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)+f(x_8)]\cdot\Delta x$$



Computing the area under a curve using Riemann sum.

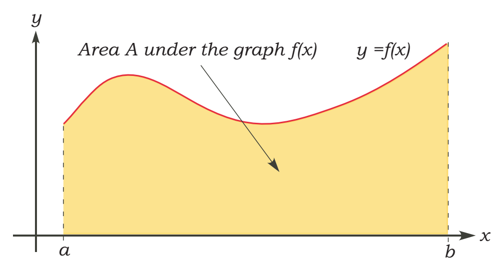
(1) Divide the interval $[a, b]$ into n equal subinterval (it does not have to be equal in size in general).

(2) Choose a point in each subinterval.

(3) Compute the area of the rectangle corresponding to each piece.

(4) Estimate the area of A by adding the areas of all rectangles.

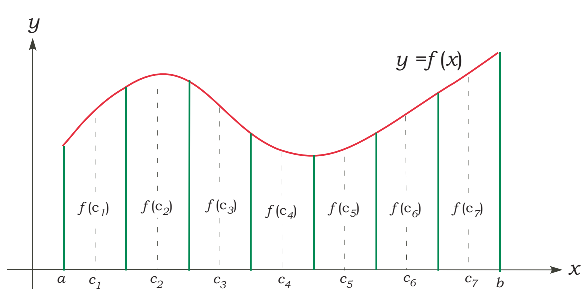
(5) Get the exact area by taking larger and larger n (smaller and smaller subintervals).



Let's look at each of the above steps in detail.

(a) Divide $[a, b]$ into n subintervals of equal width Δx and choose a point c_i in each subinterval.

(Usually this point is chosen to be either the left endpoint, the right endpoint, or the midpoint of the subinterval.)



$\Delta x =$ ________	← regular partition size

(b) Construct a rectangle over each subinterval with height $f(c_i)$ and compute the area of each rectangle.

Area of ith rectangle = height · base = ____________________.

(c) Estimate the area of A by adding the areas in (2).

area of $A \approx$	← **Riemann Sum**

(d) The approximation above gets more accurate as the rectangles get smaller. Allowing the number of segments $n \to \infty$ (or $\Delta x \to 0$), give us the exact area.

area of $A =$	← **Definite Integral**

2. Use geometry to compute the following definite integrals:

(a) $\displaystyle\int_1^3 (3x+2)\,dx$ (b) $\displaystyle\int_{-5}^0 \sqrt{25-x^2}\,dx$

Properties of Definite Integral. Let $a < b < c$ and k be real numbers. Let $f(x)$ and $g(x)$ be continuous functions. Then we have the following:

i. $\displaystyle\int_a^b [f(x)+g(x)]\,dx =$

ii. $\displaystyle\int_a^b k\cdot f(x)\,dx =$

iii. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$

We also define:

iv. $\displaystyle\int_a^a f(x)\,dx =$

v. $\displaystyle\int_b^a f(x)\,dx =$

3. Given that $\displaystyle\int_0^2 f(x)dx = \int_2^3 f(x)dx = 5$, find

a. $\displaystyle\int_0^3 f(x)dx$

b. $\displaystyle\int_0^2 [4f(x)+2]dx$

c. $\displaystyle\int_0^2 f(x)dx + \int_3^2 f(x)dx$

d. $\displaystyle\int_{-3}^0 f(x)dx \quad \text{if} \quad f(-x)=f(x)$

Fundamental Theorem of Calculus. Let $F(x)$ be an anti-derivative of $f(x)$. Then

$$\int_a^b f(x)\,dx = \rule{6cm}{0.4pt} = \rule{3.5cm}{0.4pt} \text{ (shorthand notation).}$$

In other words:

Total change in $F(x)$ over $[a,b]$ = ________________ = ____________________

4. Evaluate the following integrals:

a. $\displaystyle\int_{-1}^1 (1+3y-y^2)\,dy$

b. $\displaystyle\int_{\pi/2}^{\pi} \cos\theta\,d\theta$

c. $\displaystyle\int \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$

d. $\displaystyle\int \sqrt{x-1}\,dx$

e. $\displaystyle\int_1^5 \sqrt{x-1}\,dx$

f. $\displaystyle\int x(x^2+1)^{3/2}\,dx$

g. $\displaystyle\int_0^{\pi/4} \sin 4t\,dt$

h. $\displaystyle\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$

i. $\displaystyle\int \sec^3\theta\tan\theta\,d\theta$

5. Evaluate the limit $\displaystyle\lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{n}\sin\left[\pi\left(1+\frac{k}{n}\right)\right]\right)$.

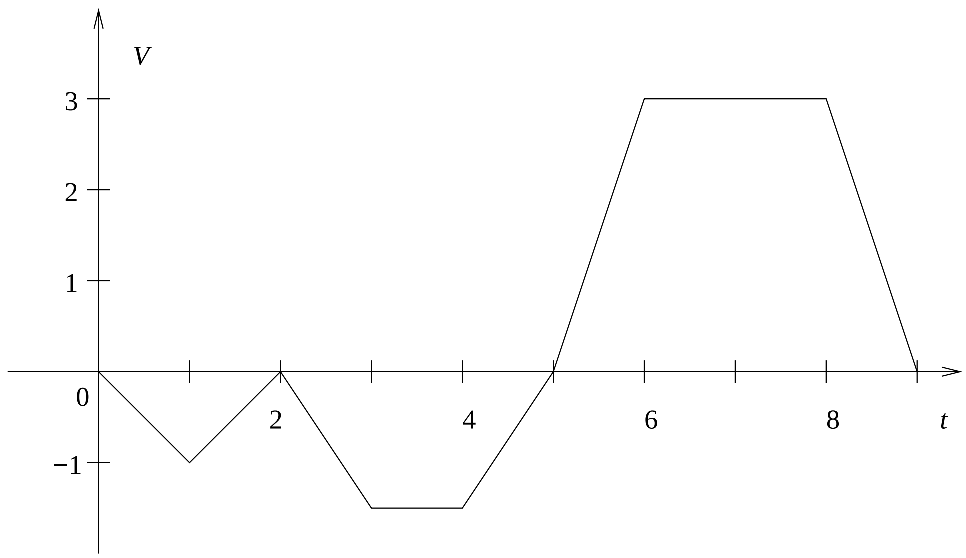
6. Find the area between the curves $y = 4x - x^2$ and $y = 2x$ for $0 \le x \le 4$.

7. The graph of the velocity V of a particle moving on a horizontal straight line is given below. Let $S(t)$ meters be the displacement of the particle after time t minutes. Assume that $S(0) = 0$. Find the exact value of the following quantities and express each of them as a definite integral.

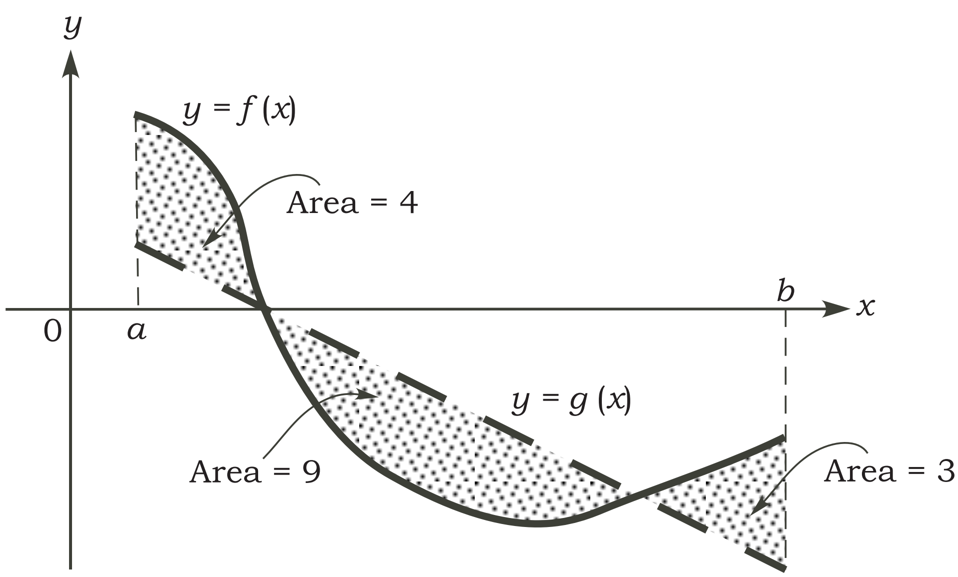
a. The displacement of the particle after 2 minutes.

b. Total distance covered by the particle.

c. The furthest distance covered by the particle to the left of its origin.



8. Using the graphs below, evaluate $\int_a^b [f(x) - g(x)]\,dx$.



Second Fundamental Theorem of Calculus If f is continuous on an open interval I containing a then, for all x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt = f(x)\right]$$

9. Find the derivative of each of the following functions

a. $g(y) = \int_2^y t^2 \sin t\,dt$

b. $F(x) = \int_x^2 \cos(t^2)\,dt$

c. $y = \int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt$

d. $y = \int_1^{\cos x} (u + \sin u)\,du$

10. Find the average value of $f(x) = \frac{1}{x+1}$ on the interval $x = 0$ to $x = 2$. Sketch a graph showing the function and the average value.