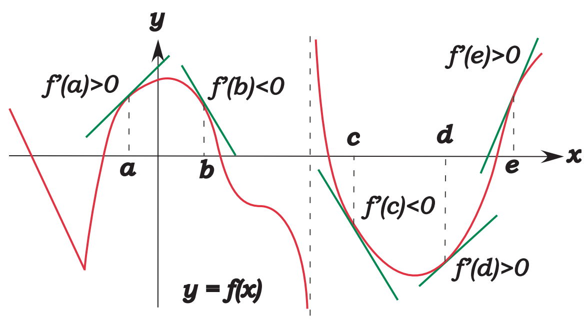
Math 10350 Fall 07 – Handout 7
(Sections 3.3 & 3.4)

1. (Review) Two cars start from the same intersection at the same time. Car A heads east at a constant speed of 40 miles per hour, and Car B heads north at a speed of 30 miles per hour. How fast is the distance between the cars changing? What if car A is speeding according to the position $s(t) = 10t^2$ miles per hour?

2. (Review) A point is moving on the curve $x^3 + y^3 = xy + 1$. If at the point $(1, -1)$, the velocity of the point in the x-direction is -2 units per minute, what is its velocity in the y-direction?

(First Derivative and monotonicity) Consider the graph of $f(x)$ below.



Q1: What does f' tell us about f?

A1: • If $f'(x) > 0$ for $\alpha < x < \beta$, then $f(x)$ is ____________ for $\alpha < x < \beta$.

• If $f'(x) < 0$ for $\alpha < x < \beta$, then $f(x)$ is ____________ for $\alpha < x < \beta$.

Remark: The only possible places (of x) where $f'(x)$ changes signs are at (i) ___________ or at (ii) where the graph has a ___________________ or undefined.

3. Find all values of x for which $f(x) = x^3 + 3x^2 - 9x + 3$ is increasing or decreasing with the steps outlined below. Classify all critical points using first derivative test.

Step 1: Find all **critical points** of f. (That is all points c in the domain where $f'(c) = 0$ or $f'(c)$ does not exist.)

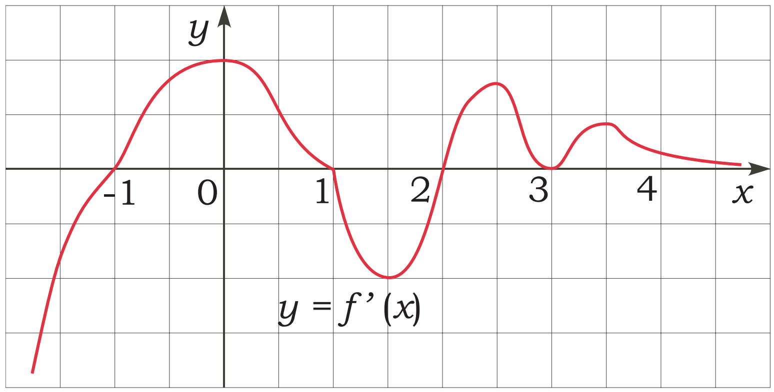
Step 2: Find points where f have a **vertical asymptote** or undefined. Answer: ____________

Step 3: Draw a number line, mark all points found in Steps 1 and 2, and find the sign of $f'(x)$ in each intervals between marked points.

Step 4: Write down the values of x for which f is increasing ($f'(x) > 0$) and those for which f is decreasing ($f'(x) < 0$).

Step 5: Classify all critical points using first derivative test.

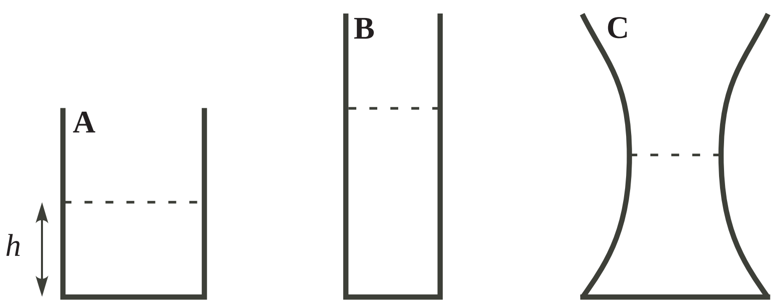
4. The graph of the **derivative** $f'(x)$ of $f(x)$ is given below. Find all critical points of $f(x)$ and use the derivative to determine where the function is increasing, where it is decreasing, and where it has a local maximum and minimum, if any. Sketch a possible graph of $f(x)$ for $-\infty < x \leq 1$ in the given axis below.



The First Derivative Test
Suppose $f(x)$ has a critical point at $x = c$. We classify the critical point as follows:

- if $f'(x)$ changes its sign from positive to negative at $x = c$, then there is a relative (local) ____________ at $x = c$.
- if $f'(x)$ changes its sign from negative to positive at $x = c$, then there is a relative (local) ____________ at $x = c$.
- if $f'(x)$ does not change its sign on both sides of $x = c$, then there is neither a relative (local) minimum nor a relative (local) maximum at $x = c$.

5. Water is filling up each of the following vessels at a constant rate of 1 cm^3/sec.



Let h be the height of the water level in the vessel at time t.

a. Sketch the graphs of h verses t for Vessels A and B in the axes for Figure 1. Indicate which graph belong to A and which to B.


Figure 1

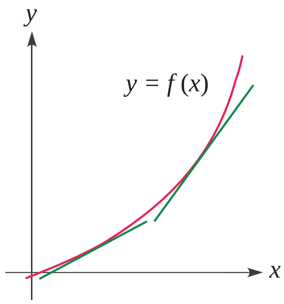

Figure 2

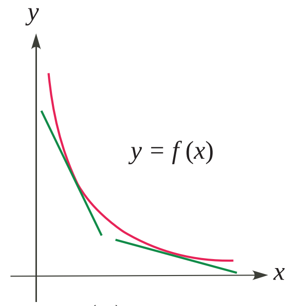
b. Sketch the graph of h verses time t for Vessel C in the axes for Figure 2.

c. Comment on how the "bending" (up or down) of the graph changes with $h'(t)$. Mark on the graph where the "bending" changes.

Characterize Concavity

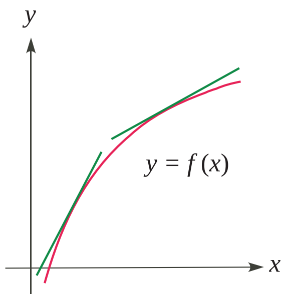
Case 1: For $a < x < b$, slope of the graph $f(x)$ is **increasing** as x increases i.e. $f'(x)$ is increasing. So $f''(x)$ is ______________ for $a < x < b$. (Portions of u-shape)

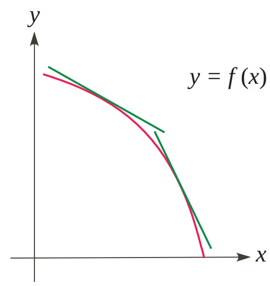




We say that the graph of $f(x)$ is _________________ for $a < x < b$.

Case 2: For $a < x < b$, slope of the graph $f(x)$ is **decreasing** as x increases i.e. $f'(x)$ is decreasing. So $f''(x)$ is ______________ for $a < x < b$. (Portions of n-shape)





We say that the graph of $f(x)$ is _________________ for $a < x < b$.

Definition (Points of Inflection) We say that $(c, f(c))$ is a point of inflection of $f(x)$ if $f'(c)$ exist (so graph has tangent line at $x = c$), and the graph of $f(x)$ changes concavity at $x = c$.

Remark: Graph changes concavity at points of inflection so to locate points of inflection we need to:

6. For $f(x) = 2 - 3x^2 + x^3$ carry out the following instructions:

a. Determine the monotonicity.

b. Find all critical points and determine whether they are local maximums or local minimums.

c. Determine the concavity and the inflection points.

d. Use the information from parts (a) through (c) to sketch the graph.

7. (Review) Verify that the function $f(x) = \dfrac{x}{x+2}$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$. Then find all numbers c that satisfies the conclusion of the Mean Value Theorem.

8. (Review) Find the second derivatives of the following functions:

(a) $y = 3\tan(3\theta)$ (b) $y = \dfrac{x}{(1-x)^3}$ (b) $y = (1-t^2)^{5/2}$

9. Suppose $f(x)$ is such that $f'(x) = \dfrac{x^2+3}{x-1}$. Find all points of inflection and concavity of $f(x)$.

Second Derivative Test

Let $f(x)$ be a function such that $f'(c) = 0$ and the function has a second derivative in an interval containing c.

- If $\boldsymbol{f''(c) > 0}$ then f has ____________________ at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) < 0}$ then f has ____________________ at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) = 0}$ then ____________________. Use first derivative test.

10. Find all relative extrema for the function $f(x) = x^3 - 9x^2 + 27x$. Use second derivative test to classify them.

11. (Review) Consider a cylindrical metal rod is heated up in a furnace. When the volume of the rod is 80π cm^3 and its height is 5 cm, the both radius and height is growing at a rate of 0.5 cm/min. At what rate is the volume is growing?

12. (Review) A triangle with **fixed** area of 50 cm^2 has its height decreasing at a rate of 0.1 cm/sec. At what rate is the base of the triangle changing at the instant when its height is 5 cm?