Math 10350 Fall 07 – Handout 4

Product Rule and Quotient Rule. Let $f(x)$ and $g(x)$ be differentiable functions. Let $P(x) = f(x) \cdot g(x)$ and $Q(x) = \dfrac{f(x)}{g(x)}$. Then

$P'(x) =$ $\qquad\qquad$ $Q'(x) =$ _____

1. Define the trigonometric functions:

$$\tan x = \frac{\sin x}{\cos x}\,, \quad \cot x = \frac{\cos x}{\sin x}\,, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

Use the fact that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$ to show that

a. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

b. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

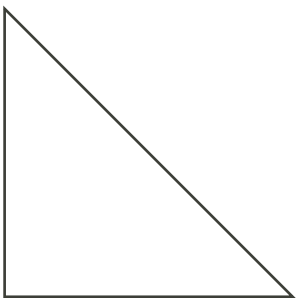
c. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

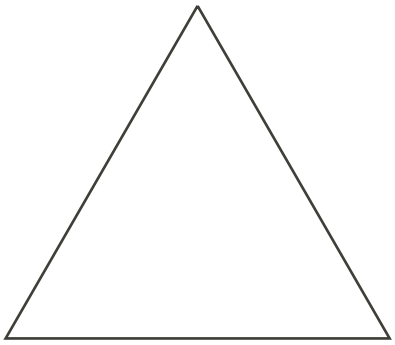
d. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

2. A purchasing officer gathers from a manufacturer that an order of x units of product X gives him a deal of $\dfrac{2}{3x+1}$ thousands of dollars per unit of X. Write down the total cost function $C(x)$ which gives the amount of money the officer has to pay. At what (instantaneous) rate is $C(x)$ changing when the order is 10 units?

3. The stationary points in the domain of a function $f(x)$ are the values of x such that $f'(x) = 0$. What can you say about the tangent line at stationary points? (a) Find the stationary points of $y = \dfrac{2x-1}{x^2+1}$. (b) Find the equation of the tangent line to the same curve at $x = -2$.

4. Using the equilateral triangle and right isosceles triangle, determine all trigonometric ratios of the special angles $\pi/6$, $\pi/4$ and $\pi/3$.





5. Find an equation of the tangent line to the curve $y = \sec x - 2\cos x$, at the point $(\pi/3, 1)$.

Definition 1. (The Composite Function) A function $h(x)$ is said to be a composite function of $g(x)$ followed by $f(x)$ if $h(x) = f(g(x))$. We may write: $\qquad h : x \overset{g}{\longmapsto}$ _____ $\overset{f}{\longmapsto}$ _____

6. Find functions $f(x)$ and $g(x)$, not equal x, such that $h(x) = f(g(x))$:

(a) $h(x) = (x^4 + 2x^2 + 7)^{21}$ $\qquad$ $h : x \overset{g}{\longmapsto}$ _____ $\overset{f}{\longmapsto}$ _____

Ans: $f(x) \overset{?}{=}$ _____ and $g(x) \overset{?}{=}$ _____

(b) $h(x) = \sin(x^2 + 1)$ $\qquad$ $h : x \longmapsto$ _____ $\longmapsto$ _____

Ans: $f(x) \overset{?}{=}$ _____ and $g(x) \overset{?}{=}$ _____

World Guinness Record Approved Text: "The razor-toothed piranhas of the genera Serrasalmus and Pygocentrus are the most ferocious freshwater fish in the world. In reality they seldom attack a human."

Think about it: In a competition for the title of "Fastest Text Messager", it is observed that Competitor A inputs text three times faster than B, and Competitor B inputs text two times faster than C. How much faster is Competitor A than Competitor C? Why?

$$A \;\overset{3}{--}\; \text{Rate of } A \text{ relative to } B \;--\; B \;\overset{2}{--}\; \text{Rate of } B \text{ relative to } C \;--\; C$$
$$\Big| \qquad\qquad ? \text{ Rate of } A \text{ relative to } C \text{ ?} \qquad\qquad \Big|$$

The Chain Rule. Suppose $y = f(g(x))$. To find a formula for $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))]$, we set $u = g(x)$ then $y = f(u)$.

$$y \;\overset{\frac{dy}{du}}{--}\; \text{Rate of } y \text{ relative to } u \;--\; u \;\overset{\frac{du}{dx}}{--}\; \text{Rate of } u \text{ relative to } x \;--\; x$$
$$\Big| \qquad\qquad \tfrac{dy}{dx} \overset{?}{=} \qquad\qquad \Big| \qquad\qquad ? \text{ Rate of } y \text{ relative to } x \text{ ?}$$

So we expect $\boxed{\dfrac{dy}{dx} =}$

Our guess is in fact correct, and the formula for $\dfrac{dy}{dx}$ is called the **Chain Rule** (in Leibniz notation).

But $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))] = [f(g(x))]'$, $\dfrac{dy}{du} = f'(u) = f'(g(x))$ and $\dfrac{du}{dx} = g'(x)$. Thus we also have:

$$\boxed{\frac{d}{dx}[f(g(x))] = [f(g(x))]' =}$$

7. Find the derivative of the following functions:

a. $f(x) = \sqrt[3]{1+x^3}$

b. $g(x) = \cot 5x$

c. $h(z) = \dfrac{1}{2}x^2\sqrt{16-x^2}$

e. $R = \csc^3(\pi x)$

f. $w = \left(\dfrac{t+1}{t^2+2}\right)^4$

g. $y = \tan^3(2x^2+1)$

8. Let $f(x) = (x^3 - 5x + 1)^2 g(x)$. Given that $f(2) = 2$, $g(1) = 3$, $f'(2) = -1$ and $g'(1) = -4$ find the values of $g(2)$ and $f'(1)$.

9. Find the equation of the tangent line at the point $P(1, 2)$ on the circle $x^2 + y^2 = 25$.