A DEFINABLE RELATION BETWEEN C.E. SETS AND IDEALS

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Abstract

by

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The Π^0_1 classes have become important structures in computability theory. Related to the study of properties of individual classes is the study of the lattice of all Π^0_1 classes, denoted $\mathcal{E}_\Pi$. We define a substructure of $\mathcal{E}_\Pi$, $G = [N, 2^\omega]$ for N nonprincipal, and a quotient structure of G, denoted $G^\diamondsuit$ and thought of as G modulo principal classes disjoint from N. Using the setting of computably enumerable ideals, we present basic results for G and $G^\diamondsuit$ and show that $G^\diamondsuit$ is isomorphic to $\mathcal{E}^*$, the lattice of computably enumerable sets modulo finite difference. This isomorphism allows us to transfer invariant classes from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$. However, it does not in general allow the transfer of orbits. We give the conditions under which an orbit could transfer and an example of one which does, and conclude with open questions related to degree-theoretic properties.

CONTENTS

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CHAPTER 1

INTRODUCTION

1.1 Π^0_1 Classes

A Π^0_1 class may be defined as the collection of infinite paths through a computable subtree of $2^{<\omega}$, the complete binary-branching tree. Π^0_1 classes have become a fundamental notion in computability theory because of their ability to code a wide range of constructions.

For example, the collection of ideals of a computably enumerable (c.e.) commutative ring forms a Π^0_1 class. In a commutative ring $(R, +, \cdot)$, an ideal is a set $I \subseteq R$, closed under addition, such that if $a \in I$, $b \in R$, then $a \cdot b \in I$. To make a Π^0_1 class of the ideals of R, we let $\{r_i\}_{i\geq 1}$ be an enumeration of R. In the tree, an infinite path f will correspond to an ideal I_f. The initial segment of length n of f terminates in a 1 when $r_n \in I_f$, and in a 0 when $r_n \notin I_f$. In order to make sure every I_f is actually an ideal of R, we exclude any path which breaks a rule of ideals. The tree is built one level at a time, with each node extended in both directions unless it becomes apparent that one or both extensions will violate a rule of ideals. At the same time, we enumerate two lists of triples, one of (a, b, c) such that $a + b = c$, and one such that $a \cdot b = c$. If the triple (r_i, r_j, r_k) appears in the addition enumeration, any node which has 1 at levels i and j but 0 at level k is no longer extended, since any infinite path extending such a node would correspond to a subset of R not closed under addition. Likewise, if (r_i, r_j, r_k) appears in the multiplication enumeration, any node with 1 at either level i or j and 0 at level k is no longer extended, although it remains on the tree.

The fact that the ring is c.e. allows us to make the tree computable. At stage s we have computably listed a finite number of addition and multiplication triples. We must check all the nodes of length s which are still being extended against each of the listed triples to make sure no node is now seen to be in violation of a rule for ideals. Since there are only a finite number of nodes and triples, and whether a given triple shows a given node to be in violation is simply a matter of observation, this is a computable procedure.

Notice that we cannot guarantee the tree will be without dead ends; that is, some nodes may be extended temporarily even though none of their infinite extensions can correspond to ideals. A large part of the flexibility of Π^0_1 classes is that while the tree must be computable, the set of extendible nodes, nodes which are initial segments of infinite paths, is not necessarily even computably enumerable.

The collection of prime ideals of a commutative c.e. ring also forms a Π^0_1 class. If the ring is commutative with unity, then its maximal ideals form a Π^0_1 class. Additional examples of objects which can be coded as Π^0_1 classes include the collection of completions of a computably axiomatizable theory, the collection of separating sets for two disjoint c.e. sets, and the set of fixed points of a computable continuous function on the real numbers. For the construction of these and other examples, as well as a survey of results about Π^0_1 classes, see [1], [3], and [5].

1.2 $\mathcal{E}_\Pi$ and G

The lattice of all Π^0_1 classes is called $\mathcal{E}_\Pi$, by analogy with $\mathcal{E}$, the lattice of computably enumerable (c.e.) sets. The properties of $\mathcal{E}$ have been extensively studied (for a survey, see [14], chapters X and XV). Research on Π^0_1 classes and $\mathcal{E}_\Pi$ is currently quite active, with many open questions (see [3] for a number of examples). However, while quite a bit is known about Π^0_1 classes as individual entities, substantially less is known about $\mathcal{E}_\Pi$ as a lattice. The goal of the research presented herein is to expand that knowledge, in particular by transferring information to $\mathcal{E}_\Pi$ from $\mathcal{E}^*$, the lattice of c.e. sets modulo finite difference.

A Π^0_1 class P is *principal* if there is a finite set of nodes of the complete binary tree such that an infinite path of the tree is in P if and only if it extends one of those nodes. Cholak, Coles, Downey, and Herrmann [7] showed that there were at most two non-isomorphic intervals of the form $[P, 2^\omega]$ in $\mathcal{E}_\Pi$: those where P is principal and those where it is nonprincipal. Cenzer and Nies [4] showed that these are in fact distinct cases.

Nies proceeded to define $G = [P, 2^\omega]$ for P nonprincipal. It is via G that we will transfer information from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$, and many of Nies' unpublished early results are reproduced in Chapter 2. Several of the results are directly proved in the setting of Π^0_1 classes. However, although the goal is to transfer information to $\mathcal{E}_\Pi$, we found it is generally more straightforward to approach G from a different perspective, that of c.e. ideals.

The set of nonextendible nodes of a computable binary-branching tree is computably enumerable. In fact, as we show in Claim 1.4.4, it is an ideal of $2^{<\omega}$, where ordering is given by extension. The collection of all such ideals, $\mathcal{I}(2^{<\omega})$, is isomorphic to $\mathcal{E}_\Pi$ by the map taking a Π^0_1 class with corresponding tree T to the ideal of nonextendible nodes of T. This map is well-defined, as discussed in §1.4, and order-reversing. We may define G inside $\mathcal{I}(2^{<\omega})$ as $[0, M]$, where M is nonprincipal (actually we will use a slightly different but also isomorphic setting; see §1.4).

1.3 G and $\mathcal{E}$

Since G in the ideal setting comprises c.e. objects, it is natural to compare it to $\mathcal{E}$. The two are not isomorphic; unlike $\mathcal{E}$, G does not have atoms, because every nonempty c.e. ideal can be split into two nonempty c.e. subideals. However, $\mathcal{E}$ can be embedded into G as an end segment, proven as Proposition 2.3.1.

Next we turn from $\mathcal{E}$ to $\mathcal{E}^*$, the c.e. sets modulo finite difference. That is, for two sets A and B, we say $A =^* B$ if the set of elements of ω which are in A or B, but not in both, is finite. The quotient $\mathcal{E}/{=^*}$ is denoted $\mathcal{E}^*$ (for a survey of results on $\mathcal{E}^*$, see [14]). Analogously, in §2.1 we put an equivalence relation on G, equality modulo "principal difference." Two ideals A and B are $\diamondsuit$-equivalent ($A =^{\diamondsuit} B$) if there is some element $m \in M$ (where M is the maximum element of G) such that the principal ideal generated by m contains all elements of M which are in A or B, but not in both. We denote $G/{=^{\diamondsuit}}$ by $G^{\diamondsuit}$. Note that in $\mathcal{E}^*$, all finite sets are trivial, and in $G^{\diamondsuit}$, all principal ideals are trivial. It is natural to take $\mathcal{E}^*$ as a quotient structure because the finite sets are closed under finite union and definable in $\mathcal{E}$. For ideals X and Y, the operation which corresponds to union is join, $X \vee Y$. The ideal $X \vee Y$ is the ideal generated by $X \cup Y$. As shown in [7], the principal ideals are definable in G and closed under finite join, so $G^{\diamondsuit}$ is a natural analogue to $\mathcal{E}^*$. Definitions and basic results for $G^{\diamondsuit}$ may be found in §2.1.

As we will see, the structure $G^{\diamondsuit}$ exhibits remarkable similarity to $\mathcal{E}^*$. André Nies and the author have translated several significant theorems of $\mathcal{E}^*$ to $G^{\diamondsuit}$, where they hold with similar proofs. Examples include the Owings splitting theorem (Theorem 2.3.5), which allows us to split a c.e. set preserving noncomplementation; the existence, for any initial segment, of sets maximal in that segment; the existence of major subsets of noncomplemented elements; and the existence of an orbit of creative sets. Proofs of the latter three results are unfortunately unavailable in published form, but all are corollaries of Theorem 3.7.1, below. These translations suggest a close relationship between $G^{\diamondsuit}$ and $\mathcal{E}^*$, and in fact in Chapter 3 we present the following, our main result:

Theorem 3.7.1. *$G^{\diamondsuit}$ is isomorphic to $\mathcal{E}^*$.*

The proof draws upon the Δ^0_3 automorphism machinery developed by Cholak, Soare, Harrington, and others (the specific format follows [10]; see also [6]), which will be fully developed in the exposition, construction, and verification in Chapter 3.

We will show that a class of $G^{\diamondsuit}$ which forms an orbit or is invariant under automorphisms gives a class of $\mathcal{E}_\Pi$ which is invariant. With the above isomorphism, then, we are able to translate an invariant class of $\mathcal{E}^*$ to one of $\mathcal{E}_\Pi$. Orbits, however do not necessarily survive the transition. In fact, we will show that any orbit of $G^{\diamondsuit}$ containing a Π^0_1 class of Cantor-Bendixson rank strictly less than ω_1^{CK} does not translate to an orbit of G. Invariance and orbit transfer results are presented in §4.1.

Unfortunately, the isomorphism does not allow us to automatically translate degree-theoretic information. In particular, we are interested in degree invariant classes, where a collection $\mathcal{D}$ of Turing degrees is invariant in $\mathcal{E}^*$ if there is a collection $\mathcal{C}$ of c.e. sets closed under automorphisms of $\mathcal{E}^*$, such that every set in $\mathcal{C}$ has a degree in $\mathcal{D}$ and every degree in $\mathcal{D}$ has a representative set in $\mathcal{C}$. The image of $\mathcal{C}$ under the isomorphism from $\mathcal{E}^*$ to $G^{\diamondsuit}$ does not necessarily correspond to the same degree collection $\mathcal{D}$, as we will discuss in §4.4. Degree-theoretic results and open questions may also be found in §4.1 and §4.3.

1.4 Preliminaries

We consider subsets of the natural numbers, ω, and say a set is *computably enumerable* (c.e.) if its elements can be listed out by an idealized computer program (one which is allowed arbitrary, though still finite, amounts of time and memory for each calculation); that is, a Turing machine. The set is *computable* if its elements can be listed in order; equivalently, if there is an idealized computer program which, on input n, tells whether n is in the set or not.

A *lattice* is a partially ordered set X with both a minimum and a maximum element, such that every pair of elements from X has a least upper bound and a greatest lower bound. The least upper bound is called the *join* of the elements, and the greatest lower bound is called the *meet*.

The collection of c.e. sets under inclusion forms a lattice denoted $\mathcal{E}$, with greatest element ω, least element $\emptyset$, and join and meet given by union and intersection, respectively. An important equivalence relation on $\mathcal{E}$ is $=^*$, equality up to finite difference. If A and B are c.e. sets, $A =^* B$ if $(A - B) \cup (B - A)$ is finite. The quotient structure $\mathcal{E}/{=^*}$ is denoted $\mathcal{E}^*$.

In general we are able to assume by coding techniques that all of the countable structures, functions, and relations we consider are actually subsets of ω. Therefore we also speak of them as c.e. or computable. *Effective* will be used as a synonym for computable, especially when speaking of functions or relations. Notation for functions and sets, and computability-theoretic terminology, will follow Soare [14].

The tree $2^{<\omega}$ is the collection of all finite strings of zeros and ones, ordered by substring inclusion. We refer to it as the complete binary-branching tree. A *subtree* of $2^{<\omega}$ is a subset closed under initial segment, so that, for example, if 101 is in the subtree, so are 1 and 10. An *infinite path* through a tree $T \subseteq 2^{<\omega}$ is an infinite binary string f such that every finite initial segment of f is in T. The collection of all infinite binary strings is called 2^ω.

We define a Π^0_1 class as the collection of infinite paths through a computable subtree of $2^{<\omega}$. The Π^0_1 classes are exactly the effectively closed subsets of 2^ω. The lattice of Π^0_1 classes ordered by inclusion forms is denoted $\mathcal{E}_\Pi$, after $\mathcal{E}$, and has least element $\emptyset$, greatest element 2^ω, and union and intersection, respectively, for join and meet. For a survey of results about Π^0_1 classes, see [1], [3], and [5].

We use Q to denote the the countable atomless Boolean algebra. We may view Q as a collection of propositional formulas modulo tautological equivalence, where the independent elements $\{p_i : i \in \omega\}$ generate Q. That is, letting $\epsilon_i p_i$ stand for either p_i or $\neg p_i$, a typical element of Q is a collection of logically equivalent formulas, each of the form

$$\bigvee_{j=1}^{n} \bigwedge_{k=1}^{m} \epsilon_{jk} p_{i_{jk}}.$$

We call elements of the set $\bigcup_{i\in\omega}\{p_i, \neg p_i\}$ *literals.* The ordering on Q is logical implication. Note that while in most logical statements and formulas the symbol & will be used for conjunction, within elements of Q conjunction will be denoted by the symbol $\wedge$.

Finite strings (elements of $2^{<\omega}$) will in general be denoted by lowercase Greek letters, especially σ and τ, and infinite strings (elements of 2^ω) by lowercase Roman letters, especially f and g. The notation for elements of Q will depend on the context. The empty string in $2^{<\omega}$ is denoted λ, and the length of a string σ is $|\sigma|$. String elements are numbered starting at zero; for a string σ of length n, the the first digit of σ is $\sigma[0]$ and the last is $\sigma[n-1]$. If τ extends σ, we write $\sigma \subseteq \tau$; if that extension is certainly proper, we write $\sigma \subset \tau$. The symbol $\perp$ indicates two elements which are *disjoint* or *incomparable.* In Q, $\varphi \perp \psi$ means $\varphi \not\to \psi$ and $\psi \not\to \varphi$. In $2^{<\omega}$, $\sigma \perp \tau$ means $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$. For a string $f \in 2^\omega$, $f \upharpoonright i$ is the *initial segment of f of length i*; that is, the unique string σ of length i in $2^{<\omega}$ such that $\sigma \subset f$. The concatenation of the string τ onto the end of the string σ will be denoted $\sigma^\frown\tau$.

For a string $\sigma \in 2^{<\omega}$, $[\sigma]$ is the *interval generated by* σ, which means either $\{f \in 2^\omega : \sigma \subset f\}$ or $\{\tau \in 2^{<\omega} : \sigma \subset \tau\}$, depending on context. Intervals are both closed and open in the topology of 2^ω and of $2^{<\omega}$, and so finite unions of intervals are also both closed and open, which will be abbreviated *clopen*.

Definition 1.4.1. *A subset* I *of* Q *is called an* ideal *if*

(i) $\sigma, \tau \in I \ \Rightarrow\ \sigma \vee \tau \in I$

(ii) $(\sigma \in I\ \wedge\ \tau \in Q)\ \Rightarrow\ \sigma \wedge \tau \in I$

The ideal I in the above definition is called a c.e. ideal if it is computably enumerable as a set. The ideal *generated* by a set X, denoted $\langle X \rangle$, is the closure of X under the above requirements. If an ideal may be generated by a finite subset of Q, it is called *principal*.

The collection of all c.e. ideals of Q is called $I(Q)$, and forms a lattice. The greatest element is Q, the least is 0 (the collection of logically contradictory formulas), the join of X and Y is $X \vee Y = \langle X \cup Y \rangle$, and the meet is $X \cap Y$.

We may also define ideals for $2^{<\omega}$, as follows.

Definition 1.4.2. *A subset* I *of* $2^{<\omega}$ *is called an* ideal *if*

(i) $\sigma^\frown 0, \sigma^\frown 1 \in I \ \Rightarrow\ \sigma \in I$

(ii) $(\sigma \in I\ \wedge\ \sigma \subseteq \tau)\ \Rightarrow\ \tau \in I$

Again, an ideal is called c.e. if it is c.e. as a set. In $2^{<\omega}$, an ideal X has a *root set*; that is, a collection of pairwise disjoint strings $\{\sigma_i\}_{i \in I}$ which generate X, with the minimality property that if $\tau \in X$ and $\tau \subseteq \sigma_i$, then $\tau = \sigma_i$. The root set is finite exactly when X is principal.

Corresponding with the notation $I(Q)$, the collection of all c.e. ideals of $2^{<\omega}$ is denoted $I(2^{<\omega})$, a lattice with the same definitions for join and meet as in $I(Q)$. In $I(2^{<\omega})$ the greatest element is $2^{<\omega}$ and the least element is $\emptyset$.

Lemma 1.4.3 ([7] 2.5, equivalent form). *$I(Q)$ and $I(2^{<\omega})$ are computably isomorphic in a natural way.*

Sketch of Proof. We map each string $\sigma \in 2^{<\omega}$ to a unique formula $\varphi_\sigma \in Q$. Consider the enumeration of positive literals of Q, $p_0, p_1, \ldots$, and let φ_σ be a conjunction of positive and negative literals, starting from p_0 and proceeding in order, where the literal p_n is positive if $\sigma[n] = 1$, and negated otherwise. For example:

$$101001 \longmapsto p_0 \wedge \neg p_1 \wedge p_2 \wedge \neg p_3 \wedge \neg p_4 \wedge p_5$$

The map from $I(2^{<\omega})$ to $I(Q)$ given by

$$X \mapsto \langle \varphi_\sigma : \sigma \in X \rangle$$

is an order-preserving bijection of ideals. $\square$

Next we associate $\mathcal{E}_\Pi$ with $I(Q)$, via $I(2^{<\omega})$. Let T be a computable subtree of $2^{<\omega}$, so that $[T]$ is a Π^0_1 class. A node of $2^{<\omega}$ with no extension in $[T]$ is called a *nonextendible* node of T (note that this set includes every node in $2^{<\omega} - T$).

Claim 1.4.4. *Let T be a computable binary-branching tree. The collection of all nonextendible nodes of T forms an ideal of $2^{<\omega}$; in fact, it is equal to $\langle \overline{T} \rangle$.*

Proof. Let the collection of all nonextendible nodes of T be called X. First we must verify the conditions of Definition 1.4.2. Suppose σ is such that $\sigma^\frown 0$ and $\sigma^\frown 1$ are both elements of X. Any infinite extension of σ must be an extension of either $\sigma^\frown 0$ or $\sigma^\frown 1$, none of which exist, so σ is also in X. For the second condition note that for $\sigma \subseteq \tau$, any infinite extension of τ is also an infinite extension of σ. Therefore, if σ is nonextendible, τ is also, and $\sigma \in X \Rightarrow \tau \in X$.

It is clear that $\langle \overline{T} \rangle \subseteq X$. For the reverse inclusion it is sufficient to show that a terminal node of T is in the ideal generated by $\overline{T}$; the claim then follows by induction. If σ is a terminal node of T, then $\sigma^\frown 0$ and $\sigma^\frown 1$ are both elements of $\overline{T}$ and therefore by the second condition in Definition 1.4.2, σ is in the ideal generated by $\overline{T}$. $\square$

Note that if T and T' are trees such that $[T] = [T']$, then $\overline{T}$ and $\overline{T'}$ generate the same ideal of $2^{<\omega}$, by the definition of nonextendible.

Claim 1.4.5. *Every ideal of $2^{<\omega}$ is the set of nonextendible nodes of some Π^0_1 class.*

Proof. Let I be an ideal of $2^{<\omega}$. We will build a computable tree T such that I is the collection of all nonextendible nodes of T. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be a computable enumeration of I; that is, a nested collection of uniformly computable sets such that $\bigcup_i I_i = I$. Without loss of generality, we may assume that if τ of length n is enumerated into I_k, for all $n < m \leq k$ the length-m extensions of τ are enumerated into I_k, and for all $m > k$ the length-m extensions of τ will be enumerated into some I_p, $p \leq m$. The tree T will consist of the length-n nodes in the complement of I_n. Since the I_n are uniformly computable, T is computable. To verify T is a tree, suppose $\sigma \in T$ and $\tau \subseteq \sigma$, with $|\tau| = n$ and $|\sigma| = m > n$; we must show that $\tau \in T$. If $\tau \notin T$, we know $\tau \in I_n$. Thus, by assumption, all length-m extensions of τ, including σ, will be in I_m, and thus not in T. Therefore T is closed under initial segment, and so is a computable tree.

It remains to show that I is exactly the nonextendible nodes of T. It is clear by construction that I contains the complement of T, and thus, as in the proof of Claim 1.4.4, I contains all nonextendible nodes of T. For the reverse inclusion, consider a node $\sigma \in T \cap I$. Since $\sigma \in I$, it will appear in some I_n, for $n > |\sigma|$, and by choice of the sequence (I_n), all of the length-n extensions of σ will also be enumerated into I_n, and therefore will not be in T. Thus σ is nonextendible in T. $\square$

Thus the map $T \mapsto \langle \overline{T} \rangle$ gives a well-defined bijective correspondence between ideals of $2^{<\omega}$ and Π^0_1 classes. In fact, it is a computable isomorphism, and therefore $I(Q)$ and $\mathcal{E}_\Pi$ are computably isomorphic as well. Notice that the isomorphism is order-reversing, since a larger Π^0_1 class has fewer nonextendible nodes and thus corresponds to a smaller ideal. In particular we have the following result.

Proposition 1.4.6. *Under the isomorphism above, a maximal ideal of* $2^{<\omega}$*, and thus of* Q*, corresponds to a singleton* Π^0_1 *class.*

Proof. It is clear that an ideal which corresponds to a singleton Π^0_1 class must be maximal. We show the converse by contradiction. Suppose M is maximal but the Π^0_1 class it corresponds to, P, is not a singleton. Then there are two intervals of $2^{<\omega}$, C_1 and C_2, which both have nonempty intersection with P and are disjoint from each other. The ideal generated by $M \cup C_1$ is c.e. and properly contains M, which means it must be equal to $2^{<\omega}$. However, since C_1 and C_2 are disjoint, C_1 cannot contain any predecessors to strings in C_2, nor can it contain $\sigma^\frown i$ for $\sigma \in C_2$, so the ideal generated by $M \cup C_2$ cannot contain any more of C_2 than M alone did. Therefore $M \cup C_1 \neq 2^{<\omega}$, and M is not maximal. $\square$

Corollary 1.4.7. *A maximal ideal of* $2^{<\omega}$ *has a computable root set.*

Proof. For M a maximal ideal, let f be the single path in the corresponding Π^0_1 class. Since $\{f\}$ is a Π^0_1 class, the set of initial segments of f must be computable. Then a computable root set for M is $\{\sigma^\frown i : \sigma^\frown(1-i) \subseteq f\}$. $\square$

There are some technical details of ideals to cover, in order to streamline matters later on. A sequence of elements $\{a_i\}_{i\in I}$ of Q (or $2^{<\omega}$) is called *pairwise disjoint* if for all $i, j, \in I$, $i \neq j$, $\langle a_i \rangle$ and $\langle a_j \rangle$ are disjoint. First note that given a c.e. generating sequence $\{a_i\}_{i\in\omega}$ for an ideal $A \in I(Q)$, one can construct a c.e. generating sequence $\{\hat{a}_i\}_{i\in\omega}$ which is pairwise disjoint. Let $\hat{a}_i = a_i \wedge \neg(\vee_{j<i} a_j)$. It is easy to see that sequence fulfills the requirements.

We want to standardize the enumeration of an ideal. Note first that any principal ideal is computable, so we may refer to it without using an enumeration. Given a c.e. generating sequence $\{a_i\}_{i\in\omega}$ as above for the ideal A, define A_s as the principal ideal generated by $\{a_i : i \leq s\}$. In Chapter 3 the enumeration will be defined differently, but unless otherwise stated $\{A_s\}$ is a nested sequence of principal ideals.

CHAPTER 2

G: DEFINITIONS AND PROPERTIES

In this chapter we introduce G, an initial segment of $I(Q)$ (equivalently, an end segment of $\mathcal{E}_\Pi$). We present basic results about G and its quotient structure $G^\diamondsuit$, including connections to $\mathcal{E}$ and $\mathcal{E}^*$.

2.1 Initial Definitions and Results for G

Recall that $I(Q)$ is the lattice of computably enumerable ideals of Q, where Q is the countable atomless Boolean algebra. The first theorem states that $I(Q)$ has no more than two isomorphism types of initial segments.

Theorem 2.1.1 ([7] 3.9, equivalent form). (i) *If $I \in I(Q)$ is nontrivial and principal, then $[0, I] \cong I(Q)$.*

(ii) *If $I, J \in I(Q)$ are nonprincipal, then $[0, I] \cong [0, J]$.*

(iii) *The isomorphisms above are computable.*

Herrmann conjectured that if $I \in I(Q)$ is principal and $J \in I(Q)$ is nonprincipal, then $[0, I] \ncong [0, J]$. His conjecture was proven by Cenzer and Nies.

Theorem 2.1.2 ([4] 4.1, equivalent form). *Let $I \in I(Q)$ be nonprincipal. Then $[0, I] \ncong I(Q)$.*

Definition 2.1.3 (Nies). *$G = [0, M] \subset I(Q)$, an initial segment of $I(Q)$ under inclusion, for any nonprincipal ideal M.*

By the theorems preceding the definition, all copies of G are isomorphic to each other but not to $I(Q)$. We will usually want to consider G where M is a maximal ideal of Q.

Define an equivalence relation $=^{\Diamond}$ on G by

$$A =^{\Diamond} B \iff (\exists m \in M)[A \vee \langle m \rangle = B \vee \langle m \rangle].$$

In other words, $A =^{\Diamond} B$ when their differences are contained in a principal subideal of M.

Notation. $G/{=^{\Diamond}}$ is denoted $G^{\Diamond}$.

Notice that $=^{\Diamond}$ depends on our choice of G. This will usually not be an issue, but when working with subintervals we might need to explicitly state which ideal we are defining $=^{\Diamond}$ from. In that case, for the ideal M we will denote our equivalence relation as $=^{\Diamond_M}$ and the equivalence class of A as $A^{\Diamond_M}$.

The structure $G^{\Diamond}$ is essentially G modulo principal ideals. The choice of principal ideals here is by analogy with $\mathcal{E}$, where $\mathcal{E}^*$ is obtained from modding out by finite sets. Finite sets are definable in $\mathcal{E}$, are closed under finite union, and determine initial segments nonisomorphic to $\mathcal{E}$. The principal ideals are likewise definable in G and closed under finite join. The initial segment of G determined by a principal ideal will be isomorphic to $I(Q)$ and thus not isomorphic to G itself.

The ordering on $G^{\Diamond}$ is set containment outside some principal ideal. The relationship $A^{\Diamond} \leq B^{\Diamond}$ holds if given representatives A, B, respectively,

$$(\exists m \in M)[A \vee \langle m \rangle \subseteq B \vee \langle m \rangle].$$

When we are considering representative ideals A, B of $A^{\Diamond}$, $B^{\Diamond}$, we will sometimes write $A \subseteq^{\Diamond} B$ for $A^{\Diamond} \leq B^{\Diamond}$, as we might write $A^{\Diamond} = B^{\Diamond}$ for $A =^{\Diamond} B$.

André Nies presented initial results on G in a talk at the San Diego Joint Mathematics Meetings in January, 2002. Afterward he defined $=^{\Diamond}$ and began to consider

$G^\diamondsuit$. Results attributed to Nies below were stated by him in the San Diego talk and during a visit to Notre Dame in May, 2002. Proofs have in most cases been fleshed out from sketches he provided while visiting.

Under duality, G may be considered as $[N, 2^\omega] \subset \mathcal{E}_\Pi$ for any nonprincipal Π^0_1 class N (note the order-reversal). As noted in Proposition 1.4.6, the case where the ideal M is maximal corresponds to N being a singleton. We may recast our previous definitions in the setting of Π^0_1 classes.

Here $=^\diamondsuit$ is the equivalence relation

$$P =^\diamondsuit Q \iff (\exists \text{ clopen } C)[N \subseteq C \ \wedge\ P \cap C = Q \cap C].$$

When N is a singleton $\{f\}$, $=^\diamondsuit$ simplifies to

$$P =^\diamondsuit Q \iff (\exists n)[P \cap [f \restriction n] = Q \cap [f \restriction n]].$$

In the singleton Π^0_1 class setting, $P^\diamondsuit \leq Q^\diamondsuit$ if given representatives P, Q, respectively,

$$(\exists n)[P \cap [f \restriction n] \subseteq Q \cap [f \restriction n]]. \tag{2.1.1}$$

The order relation for Π^0_1 classes, then, is eventual containment. Note that for P containing f, if there exists an $n \in \omega$ such that $[f \restriction n] \subseteq P$, then $P =^\diamondsuit 2^\omega$. Thus not only are the intermediate elements of $G^\diamondsuit$ nonprincipal, but indeed, they are nonprincipal in $[f \restriction n]$ for all n.

A third perspective we may use is that of c.e. ideals of the complete binary-branching tree, $2^{<\omega}$. Recall that $I(Q)$ and $I(2^{<\omega})$ are computably isomorphic. $I(2^{<\omega})$ has the advantage of being composed of c.e. (rather than co-c.e.) objects and having the same definition for $\subseteq^\diamondsuit$ as $I(Q)$. It is most helpful in being easy to visualize. However, as we shall see in Example 2.2.6, $2^{<\omega}$ is frequently too restrictive for our purposes.

While the Π^0_1 class and $2^{<\omega}$ settings will be useful to us at times, such as in the following proposition, from here on we are working from the $I(Q)$ perspective, unless otherwise stated.

Proposition 2.1.4. *The order relation in G is Π^0_2 complete, and the order relation of $G^\diamondsuit$ is Σ^0_3 complete.*

Proof. We will work in the Π^0_1 setting, specifically in $G = [\{0^\omega\}, 2^\omega]$. Since all copies of G are computably isomorphic, this will show the proposition for arbitrary G. Given P and Q in G, where T_P and T_Q are the corresponding computable trees, $P \subseteq Q$ if

$$(\forall \sigma \in T_P)(\exists k)(\forall |\tau| = k)[(\tau \succeq \sigma \to \tau \notin T_P) \vee \sigma \in T_Q]$$

which, since the innermost quantifier is bounded, is a Π^0_2 sentence. The sentence (2.1.1) defining ordering in $G^\diamondsuit$ is then Σ^0_3.

As shown in [14] (IV.3.2 and 3.5), the set $\mathrm{Tot} = \{e : W_e = \omega\}$ is Π^0_2 complete, and $\mathrm{Cof} = \{e : W_e \text{ is cofinite}\}$ is Σ^0_3 complete. First we show that Tot is reducible to the ordering of G.

Given a c.e. set W_e, define a tree T_e as follows: Begin to build a complete tree. At any point that you see $n \searrow W_e$, truncate all extensions in T_e of 0^n1. Then the index $e \in \mathrm{Tot}$ if and only if $[T_e] \subseteq \{0^\omega\}$.

The above construction also shows that Cof is reducible to the ordering on $G^\diamondsuit$, because $e \in \mathrm{Cof}$ if and only if there is some n such that all $m \geq n$ are in W_e. In that case, $[T_e] \cap [0^n] = \{0^\omega\}$, so $[T_e] \subseteq^\diamondsuit \{0^\omega\}$. □

Now we introduce examples of significant index sets for G.

Definition 2.1.5. *For a fixed copy $[0, M]$ of G, with M maximal, let W_e, $e \in \omega$, be an enumeration of all subideals of M. The following are three index sets for G:*

(i) $\mathrm{Prn} = \{e : W_e \textit{ is principal}\}$

(ii) Npr $= \{e : W_e$ *is nonprincipal*$\}$

(iii) Cop $= \{e : (\exists m \in M)[\overline{W}_e \subseteq \langle m \rangle]$, *that is,* W_e *is "co-principal"*$\}$

Theorem 2.1.6. *Prn is* Σ^0_2*, Npr is* Π^0_2*, and Cop is* Σ^0_3*.*

Proof. The ideal W_e is nonprincipal if every principal ideal of M omits some of W_e. The index set is Npr $= \{e : (\forall m \in M)(\exists x \in M)(\exists s)[x \in W_{e,s} \ \& \ x \notin \langle m \rangle]\}$, which is Π^0_2 because membership in M, $\langle m \rangle$, or $W_{e,s}$ is computable. Since every ideal is principal or nonprincipal but not both, this also shows that Prn is Σ^0_2.

W_e is co-principal if its complement is contained in a principal ideal of M; that is, if e is in the set Cop $= \{e : (\exists m \in M)(\forall x \in M)(\exists s)[x \in W_{e,s} \ \vee \ x \in \langle m \rangle]\}$, which is a Σ^0_3 set. □

Theorem 2.1.7. *Npr is* Π^0_2*-complete, and Prn is* Σ^0_2*-complete.*

Proof. We will work from the $2^{<\omega}$ perspective, specifically in $G = [0, M]$ where $M = 2^{<\omega} - \{0^n : n \in \omega\}$. As in Proposition 2.1.4, this will show the result for all copies of G. As shown in [14] (IV.3.2), Fin $= \{e : W_e$ is finite$\}$ is Σ^0_2-complete and Inf $= \{e : W_e$ is infinite$\}$ is Π^0_2-complete.

Given a c.e. set A, let I be the ideal generated by the set $\{0^n 1 : n \in A\}$. I is a c.e. ideal in G. If I is nonprincipal, the given set A is infinite, and if I is principal, A is finite. Therefore Fin reduces to Prn and Inf reduces to Npr, and Prn and Npr are Σ^0_2- and Π^0_2-complete, respectively. □

2.2 The Relationships Between G, $G^{\diamondsuit}$, and $I(Q)$

The relationships we are concerned with are about automorphisms, orbits and invariant classes, and translation of formulas from one structure to another. We begin with the automorphisms, specifically the relationships between $I(Q)$ and copies of G which terminate in a maximal ideal.

The property of being maximal is definable in $I(Q)$, as is the property of being principal (it is equivalent to being complemented; see [7]). The following claim shows that maximality defines not only an invariant class, but an orbit; in fact, a Δ^0_1 orbit.

Claim 2.2.1. *Any two maximal ideals of $I(Q)$ are computably automorphic.*

Proof. We work via $2^{<\omega}$. For two given maximal ideals of $I(Q)$, let M and $\widehat{M}$ be the corresponding ideals in $2^{<\omega}$. By Corollary 1.4.7, M and $\widehat{M}$ both have computable root sets. Let M's root set $\{\sigma_i : i \in \omega\}$ and $\widehat{M}$'s root set $\{\tau_i : i \in \omega\}$ both be ordered by length. After Proposition 1.4.6, let f and $\hat{f}$ be the paths in the complements of M and $\widehat{M}$, respectively. Then the automorphism $\Phi : 2^{<\omega} \to 2^{<\omega}$ taking M to $\widehat{M}$ is

$$\Phi(\nu) = \begin{cases} \tau_i{}^\frown\mu & \text{if } \nu = \sigma_i{}^\frown\mu \\ \hat{f} \restriction n & \text{if } \nu = f \restriction n \end{cases}$$

It is easily checked that Φ is an automorphism, and it is clearly computable. □

Claim 2.2.2. *For $G_M = [0, M]$ with M maximal, any automorphism of G_M extends to an automorphism of $I(Q)$ of the same Turing degree.*

Proof. As in Claim 2.2.1, work via $2^{<\omega}$ and let f be the path of 2^ω which is not in M. Let I be an ideal in $I(Q)$ and Φ an automorphism of G_M. We extend Φ to a map on $I(Q)$, Ψ, as follows.

$$\Psi(I) = \Phi(I \cap M) \vee \{I \cap \{f \restriction n : n \in \omega\}\}. \tag{2.2.1}$$

It is clear that this image is a c.e. ideal. We must show Ψ is an automorphism. First suppose that $I \subseteq J$ are ideals in $I(Q)$. Then $I \cap M \subseteq J \cap M$, so since Φ is an automorphism of G_M, $\Phi(I \cap M) \subseteq \Phi(J \cap M)$. Likewise, $\{I \cap \{f \restriction n : n \in \omega\}\} \subseteq \{J \cap \{f \restriction n : n \in \omega\}\}$ as well, so $\Psi(I) \subseteq \Psi(J)$ and Ψ preserves the ordering. To see that Ψ is injective, it suffices to show that the only preimage of the zero ideal is the zero ideal. In $2^{<\omega}$, the zero ideal is the empty set, so this is satisfied. Finally, we must show that Ψ is surjective. Let I be an ideal of $I(Q)$ for which we need a preimage. $I \cap M$ will also be a c.e. ideal, so it has a preimage under Φ; call it J_M. Now the use of diamond-equivalence greatly simplifies matters. If $I \subseteq M$, we are done; the preimage of I under Ψ is simply J_M. If not, there is some $N \in \omega$ such that $\{f \upharpoonright n : n \geq N\} \subseteq I$. That is, there is N such that $[f \upharpoonright N] \subseteq I$, which implies that $I \cap M =^{\diamond} M$. Therefore, $J_M =^{\diamond} M$ as well, so there is some $K \in \omega$ such that $[f \upharpoonright K] \cap M \subseteq J_M$. The preimage of I under Ψ is then $J = J_M \vee [f \upharpoonright K]$.

The automorphism Ψ has the same Turing degree as the original Φ since the right-hand set in the join in Equation (2.2.1) is computably enumerable. Note that an ideal $I \subseteq M$ has the same image under Ψ as it did under Φ. $\square$

Theorem 2.2.3 ([7] 6.1, equivalent form). *Every automorphism of $I(Q)$ is induced by a unique automorphism of Q.*

Corollary 2.2.4. *Every automorphism of G extends to an automorphism of $I(Q)$ which is induced by a unique M-preserving automorphism of Q.*

Corollary 2.2.5. *Every automorphism of G is induced by a unique automorphism of M.*

Note that we cannot use our simplifying view that M is a subset of $2^{<\omega}$ for the corollaries. Automorphisms of ideals of $2^{<\omega}$ require that elements of the root set map to each other and that the rest of the mapping respect not only substring relationships but distance from a root: if $r^\frown\tau$ maps to $\hat{r}^\frown\sigma$, where r and $\hat{r}$ are elements of the root set, then $2^{<\omega}$ requires $|\tau| = |\sigma|$. The following is an example of an automorphism for which $2^{<\omega}$ fails to be sufficient.

Example 2.2.6. Let $\{r_i\}_{i\in\omega}$ be the root set of M, considered as a subset of $2^{<\omega}$. Let f be a map on M which is the identity on every string extending a root r_n, $n > 2$, and elsewhere defined as follows (for $\tau \in 2^{<\omega}$):

$$f(r_0^\frown 0^\frown\tau) = r_1^\frown\tau$$

$$f(r_0^\frown 1^\frown \tau) = r_2^\frown \tau$$

$$f(r_1^\frown \tau) = r_0^\frown 0^\frown \tau$$

$$f(r_2^\frown \tau) = r_0^\frown 1^\frown \tau$$

It is easily checked that f induces an automorphism of G, since it is simply swapping two pairs of principal ideals (and their subideals). However, f is not an automorphism of $M \subseteq 2^{<\omega}$. Furthermore, the automorphism f induces on G cannot be induced by *any* automorphism of M as a subset of $2^{<\omega}$, because r_0 has no image or preimage. Therefore to be guaranteed an automorphism of M which generates the given automorphism of G, we must consider M as a subset of Q instead of $2^{<\omega}$.

Next we speak of orbits in the three structures. André Nies showed that an orbit in G induces an orbit in $I(Q)$.

Claim 2.2.7 (Nies). *For U an orbit of G, let U_M denote U's isomorphic copy in $G_M = [0, M]$. Then $EXT(U) = \bigcup\{U_M : M$ is a maximal ideal of $Q\}$ is an orbit of $I(Q)$ of the same complexity as U.*

Proof. Closure of $EXT(U)$ comes from the fact that containment in a maximal ideal is definable. Any automorphism Φ of $I(Q)$ must take any element $I \in G_M$ of $EXT(U)$ to an ideal which is contained in some maximal ideal. Since all copies of G terminating in a maximal ideal are automorphic in $I(Q)$, the image of I may be isomorphically mapped to an ideal in the original G_M. Since the composition of those two maps takes I to another element J of G_M, J must be an element of U_M, and the intermediate ideal is in an isomorphic copy of U_M. Therefore, the automorphism Φ maps $EXT(U)$ into itself.

To show transitivity, let I and J be two ideals in $EXT(U)$. If there is an M such that I and J are both in U_M, then by Claim 2.2.2, the automorphism of G_M which takes I to J extends to an automorphism of $I(Q)$ taking I to J. If not, suppose M and N are maximal ideals such that $I \in U_M$ and $J \in U_N$. There is some automorphism Φ of $I(Q)$ taking M to N. The ideal $\Phi(I)$ is an element of U_N, and so there is an automorphism Ψ of G_N taking $\Phi(I)$ to J. Let Ψ also denote the extension of Ψ to all of $I(Q)$. Then $\Psi \circ \Phi$ is an automorphism of $I(Q)$ taking I to J. The complexity does not increase because the automorphism taking M to N may be chosen to be computable, by Claim 2.2.1. $\square$

Unfortunately between G and $G^\diamondsuit$ the connection is not as strong. From $U^\diamondsuit$, an orbit or invariant class in $G^\diamondsuit$, define U, the collection of all the ideals in the equivalence classes making up $U^\diamondsuit$. The collection U must be invariant because any automorphism of G which takes an element in U to an element outside U will induce an automorphism of $G^\diamondsuit$ which does the same thing to $U^\diamondsuit$. However, U will not necessarily be an orbit even if $U^\diamondsuit$ was. See Corollary 4.1.6 for the exact result. The difficulty with automorphisms of $G^\diamondsuit$ is that, viewing the automorphism as a mapping on the ideals in G (assuming it is possible to do so), a principal ideal need not map to a principal ideal. It needs only map to an ideal which is contained inside a principal ideal, since an automorphism of $G^\diamondsuit$ is only concerned with the ideal's diamond-equivalence class. Likewise, the principal elements of G do not form an automorphism base for $G^\diamondsuit$ — the images of the principal ideals do not determine the automorphism.

Finally we turn our attention to formulas; in particular, the translation of formulas from $G^\diamondsuit$ to G and $I(Q)$, preserving truth. We can make the translation in two steps. Let I and J stand for ideals, members of G. A formula φ in $G^\diamondsuit$ translates to φ' in G, where φ' is obtained by expanding $=^\diamondsuit$ and $\subseteq^\diamondsuit$. That is, φ' is obtained by replacing all instances of $I = J$ in φ with $(\exists m)[I \vee \langle m\rangle = J \vee \langle m\rangle]$, and replacing all instances of $I \subseteq J$ with $(\exists m)[I \vee \langle m\rangle \subseteq J \vee \langle m\rangle]$.

A formula ψ in G corresponds to the formula $(\exists M)[M$ is maximal & $\psi']$ in $I(Q)$, where ψ' is obtained from ψ by replacing all instances of $(\exists I)$ with $(\exists I \subseteq M)$ and all instances of $(\forall I)$ with $(\forall I \subseteq M)$, and likewise for quantification over individual elements.

2.3 Comparing G and $G^{\diamondsuit}$with $\mathcal{E}$ and $\mathcal{E}^*$

First we ask if it is possible for any pair of these structures to be isomorphic. With G we obtain only negative results. G is not isomorphic to $\mathcal{E}$ because $\mathcal{E}$ has atoms (the singleton sets) and G does not; a nontrivial ideal always has proper subideals. G is, furthermore, not isomorphic to $\mathcal{E}^*$, because in $\mathcal{E}^*$ all nontrivial complemented elements share an orbit, whereas in G, the principal ideal $\langle m \rangle$, for example, does not share an orbit with its complement, $M \cap \overline{\langle m \rangle}$. Inside of M the complement of a principal ideal is nonprincipal, and being principal is definable.

Proposition 2.1.4 showed that G and $G^{\diamondsuit}$ have the same order relation complexities as $\mathcal{E}$ and $\mathcal{E}^*$, respectively. Theorem 2.1.7 showed that the index sets of principal and nonprincipal ideals in G correspond in complexity to the index sets of finite and infinite sets, respectively, in $\mathcal{E}$. There is another interesting connection between G and $\mathcal{E}$.

Proposition 2.3.1 (Nies). *G contains $\mathcal{E}$ as an end segment.*

Proof. Let $\{m_i\}$ be a disjoint list of generators for M, where G is the substructure $[0, M] \subset I(Q)$. Define (uniformly) a sequence M_i of maximal subideals of m_i, and let $C = \bigcup_i M_i$. Then we can map $\mathcal{E}$ to $[C, M] \subseteq G$ isomorphically by $V \in \mathcal{E} \mapsto C \cup < m_i : i \in V >$. $\square$

We are left to consider possible isomorphisms involving $G^{\diamondsuit}$, with $\mathcal{E}$, $\mathcal{E}^*$, or G. As we shall see in Chapter 3, $G^{\diamondsuit}$ is isomorphic to $\mathcal{E}^*$, so it is not isomorphic to either of the other two structures.

Initially, we believed the isomorphism was impossible, and so tried to construct substructures of $G^{\diamondsuit}$ which could not exist in $\mathcal{E}^*$, such as an end segment composed of three elements. André Nies proved that every initial segment of $G^\diamondsuit$ has coatoms, ideals maximal in the segment, so an end segment of two elements exists. However, we ultimately proved a translation of the Owings Splitting Theorem, Theorem 2.3.5 below, proving an end segment of three elements could not exist. Nies' coatom result and the translated Owings Splitting in $G^\diamondsuit$ are corollaries of the isomorphism between $\mathcal{E}^*$ and $G^\diamondsuit$ in Chapter 3. Owings Splitting is included here both because its G version (Corollary 2.3.6) does not come as a corollary of the isomorphism, and for its proof technique, as a warm-up for the proof of the isomorphism.

The Owings Splitting Theorem states that a c.e. set that is noncomplemented in an interval may be split into two disjoint c.e. sets that are noncomplemented in the same interval. In order to translate it we must consider complementation in G and $G^\diamondsuit$. Complements in G work as one would expect them to, with the complement to B in $[C, I]$ being $\widetilde{B}$ such that $\widetilde{B} \cap B = C$ and $\widetilde{B} \vee B = I$. Note, however, that unlike in $I(Q)$ as a whole, being complemented in G is not equivalent to being principal; for example, in $2^{<\omega}$ with M omitting the strings of only zeroes, consider the ideal generated by $\{0^{2n}1 : n \in \omega\}$. It is nonprincipal but is complemented by the ideal generated by $\{0^{2n+1}1 : n \in \omega\}$. Complementation in $G^\diamondsuit$ requires a definition.

Definition 2.3.2. *Let $C^\diamondsuit < B^\diamondsuit$ be elements of $G^\diamondsuit$. The equivalence class $\widetilde{B}^\diamondsuit$ is a complement of $B^\diamondsuit$ over $C^\diamondsuit$ if*

1. $(\exists m \in M)[(B \cap \widetilde{B}) \vee \langle m \rangle = C \vee \langle m \rangle]$

2. $(\exists n \in M)[B \vee \widetilde{B} \vee \langle n \rangle = M]$

For $B^\diamondsuit < I^\diamondsuit$, I a c.e. ideal, $\widetilde{B}^\diamondsuit$ is a complement of $B^\diamondsuit$ in $[C, I]^{\diamondsuit_M}$ if we replace (2) above with

2.′ $(\exists n \in M)[B \vee \widetilde{B} \vee \langle n \rangle = I \vee \langle n \rangle]$.

Unfortunately, unlike the case of $\mathcal{E}$ and $\mathcal{E}^*$, the complemented elements of G

and $G^{\diamondsuit}$ are not the same. Of course all elements complemented in G are complemented in $G^{\diamondsuit}$, but the converse is not true. As an example, consider $[0, M]$ where $M = 2^{<\omega} - \{0^n\}_{n\in\omega}$. Define

$$B = \langle 0^{2n}1, 1^n0 : n \geq 1\rangle.$$

That is, B contains all the intervals off of the path of all ones, and every other interval off the path of all zeroes. The ideal B is noncomplemented in G because its complement must contain $\{1^n : n \in \omega\}$. For any n, the ideal $\langle 1^n\rangle$ contains 1^n0, so every ideal containing $\{1^n : n \in \omega\}$ has nonempty intersection with B and is thus not a complement. However, in $G^{\diamondsuit}$, B is complemented by

$$\widetilde{B} = \langle 0^{2n-1}1 : n \geq 1\rangle$$

because $B \cap \widetilde{B} = \emptyset$ and $B \vee \widetilde{B} \vee \langle 1\rangle = M$.

There is no direct equivalence between complementation in G and $G^{\diamondsuit}$, but there is a correspondence.

Proposition 2.3.3. *An element of $G^{\diamondsuit}$ is complemented if and only if it contains a complemented element of G.*

Proof. The "if" direction is clear from the fact that a complement in G is a complement in $G^{\diamondsuit}$. We must show that every complemented element of $G^{\diamondsuit}$ contains a complemented element of G.

Let B be a c.e. ideal such that $B^{\diamondsuit}$ is complemented. Then there exists some $\widetilde{B}$ and $n, m \in M$ so $B \cap \widetilde{B} \subseteq \langle m\rangle$ and $B \vee \widetilde{B} \vee \langle n\rangle = M$. Let $C = \langle n\rangle \vee \langle m\rangle$ and $I = B \cap \overline{C}$. Then $I =^{\diamondsuit} B$ and I is complemented by $\widetilde{B} \vee C$ in G. $\square$

In $\mathcal{E}$, the Owings Splitting Theorem is a highly useful theorem relating to complementation. A translation of that theorem holds in both G and $G^{\diamondsuit}$. First we recall the statement of the theorem for $\mathcal{E}$.

Theorem 2.3.4 (Owings Splitting). *Let $C \subseteq B$ be c.e. sets such that $B - C$ is not co-c.e. (that is, B is not complemented over C). Then there exist c.e. sets A_0, A_1 such that*

1. $A_0 \cap A_1 = \emptyset$
2. $A_0 \cup A_1 = B$
3. $A_i - C$ *is not co-c.e. for* $i = 0, 1$
4. *For any c.e. set* W, $i = 0, 1$,

   $C \cup (W - B)$ *not c.e.* $\Rightarrow C \cup (W - A_i)$ *not c.e.*

Note that the last condition states for $W \supseteq B$ that if B is not complemented in $[C, W]$, then neither is $A_i \cup C$. For a proof of the theorem, see [14] (X.2.5).

First we will translate Owings Splitting to $G^\diamondsuit$; the proof of it there will give us the result for G as a corollary.

Theorem 2.3.5. *Let $C^\diamondsuit < B^\diamondsuit$ be elements of $G^\diamondsuit$ such that $B^\diamondsuit$ is noncomplemented over $C^\diamondsuit$. Then there exist c.e. ideals $A_0, A_1 \subseteq M$ such that*

1. $(\exists m \in M)[A_0 \cap A_1 \subseteq \langle m \rangle]$
2. $(\exists n \in M)[A_0 \vee A_1 \vee \langle n \rangle = B \vee \langle n \rangle]$
3. $A_i^\diamondsuit \vee C$ *is noncomplemented over* $C^\diamondsuit$, $i = 0, 1$
4. *For any c.e. ideal $I \subseteq M$, if $B^\diamondsuit < I^\diamondsuit$ and $B^\diamondsuit$ is noncomplemented in $[C, I]^{\diamondsuit_M}$, then $A_i^\diamondsuit \vee C$ is also noncomplemented in $[C, I]^{\diamondsuit_M}$ for $i = 0, 1$.*

Proof. Let B and C be representatives of $B^\diamondsuit$ and $C^\diamondsuit$, respectively. Let $\{b_i\}_{i\in\omega}$ be a c.e. list of disjoint generators for B and let $\{c_i\}_{i\in\omega}$ be any c.e. list of generators for C. We will let $\{m_i\}_{i\in\omega}$ be an enumeration of *all* of M. Define B_s, C_s, and M_s as usual, as the principal ideals generated by the first s elements of the enumeration. A_0 and

A_1 will be generated by subsets of the b_i; we will meet (1) and (2) by ensuring for every i, b_i is enumerated into exactly one of A_0, A_1. Since the b_i are disjoint, that will be sufficient.

As in the $\mathcal{E}$ setting, we will explicitly meet (3) and show that the process gives us (4) as well. Let W_e be an enumeration of all c.e. ideals. We have the requirement

$$R_{<e,j,i>} : W_e \vee A_i \vee C \vee \langle m_j \rangle \supseteq B \vee \langle m_j \rangle \implies W_e \cap A_i \not\subseteq C \vee \langle m_j \rangle.$$

$R_{<e,j,i>}$ *requires attention* at stage s if $W_{e,s} \vee A_{i,s} \vee C_s \vee \langle m_j \rangle \supseteq B_s \vee \langle m_j \rangle$ but $W_{e,s} \cap A_{i,s} \subseteq C_s \vee \langle m_j \rangle$, and we see some $x \in W_{e,s}$ such that $x \succeq b_s$ but $x \notin C_s \vee \langle m_j \rangle$. If $R_{<e,j,i>}$ is the highest-priority requirement requiring attention, we put b_s into A_i and $R_{<e,j,i>}$ is satisfied; temporarily if x later enters C and permanently if not. For each $R_{<e,j,i>}$ we have a computable function $g(<e,j,i>, s)$, the position of a movable marker at the end of stage s.

Construction

Stage $s = 0$: $b_0 \searrow A_0$; for all e, i, $g(<e,j,i>, 0) = 0$.

Stage $s + 1$:

1. For all $<e,j,i>$ less than $s+1$, if $W_{e,s} \vee A_{i,s} \vee C_s \vee \langle m_j \rangle \supseteq B_s \vee \langle m_j \rangle$, look for an $x \in W_{e,s} \vee A_{i,s}$ such that $x \notin C_s \vee \langle m_j \rangle$. If $W_{e,s} \vee A_{i,s} \vee C_s \vee \langle m_j \rangle \not\supseteq B_s \vee \langle m_j \rangle$, or if $x \succeq b_j$ with $|b_j| \leq g(<e,j,i>, s)$, then let $g(<e,j,i>, s+1) = g(<e,j,i>, s)$. If the b_j that x extends is longer than $g(<e,j,i>, s)$, or there is no such x, let $g(<e,j,i>, s+1) = s+1$. The function $g(<e,j,i>, s)$ then bounds the length of generators of B we must look at to find a witness for $R_{<e,j,i>}$.

2. Choose the least $<e,j,i>$ such that $R_{<e,j,i>}$ requires attention, and enumerate b_{s+1} into A_i. If no such triple exists, then $b_{s+1} \searrow A_0$.

Verification

We prove (4), of which (3) is a special case. The contrapositive of (4) is that for any c.e. ideal $I \subseteq M$, if $B \leq^{\diamondsuit} I$ and $(A_i \vee C)^{\diamondsuit}$ is complemented in $[C, I]^{\diamondsuit_M}$ for

$i = 0$ or 1, then $B^{\Diamond}$ is complemented in $[C, I]^{\Diamond_M}$. Assume there exist e, m, i, I such that m witnesses that $W_e^{\Diamond}$ is a complement to $(A_i \vee C)^{\Diamond}$ in $[C, I]^{\Diamond_M}$. Let n be such that $C \vee \langle n \rangle \subseteq B \vee \langle n \rangle \subseteq I \vee \langle n \rangle$, and let $m_j = m \vee n$. Since m witnesses that W_e is a complement to A_i, so does m_j. Choose s' large enough that for all $<e', j', i'>$ less than $<e, j, i>$, if $\lim_s g(<e', j', i'>, s)$ exists, it is equal to $g(<e', j', i'>, s')$. Let z be the maximum of all the settled marker values, and let $s'' \geq s'$ be such that everything enumerated into B_s is of length greater than z for all $s \geq s''$.

Since by assumption $W_e \vee A_i \vee C \vee \langle m_j \rangle \supseteq B \vee \langle m_j \rangle$ and $W_e \cap A_i \subseteq C \vee \langle m_j \rangle$, the requirement $R_{<e,j,i>}$ will never be permanently satisfied. That is, for any witness $x \in W_{e,s} \vee A_{i,s}$ such that $x \notin C_s \vee \langle m_j \rangle$ there will be a stage $t > s$ such that $x \in C_t$. Therefore $\lim_s g(<e, j, i>, s) = \infty$, meaning that every x which enters W_e after stage s'' meets at least one of the following four conditions: (a) x has already been enumerated into B, (b) $x \notin B$, (c) $x \in \langle m_j \rangle$, or (d) $x \searrow C$ at a later stage.

Define the c.e. ideal $\widetilde{W_e}$, a subideal of W_e generated by the set

$$\{x : (\exists s \geq s'')[x \in W_{e,s} - B_s]\}.$$

Note that $\widetilde{W_e} \cap \overline{B} = W_e \cap \overline{B}$. From that and the choice of m_j, since $W_e \vee A_i \vee C \vee \langle m_j \rangle = I \vee \langle m_j \rangle$, it follows that $\widetilde{W_e} \vee B \vee C \vee \langle m_j \rangle = I \vee \langle m_j \rangle$.

Now suppose that there is some $x \in \widetilde{W_e} \cap B$ such that $x \notin C \vee \langle m_j \rangle$. Let s be such that $x \in W_{e,s} - B_s$ and $x \succeq b_{s+1}$. Since $s \geq s''$, all the requirements of higher priority than $R_{<e,j,i>}$ which achieve permanent satisfaction have done so, and never require attention again. Therefore b_{s+1} must be chosen by some $R_{<e',j',i'>}$ such that $<e', j', i'>$ is less than or equal to $<e, j, i>$ and $g(<e', j', i'>, s) \to \infty$. That implies that x is not a permanent witness for $R_{<e,j,i>}$, so $x \searrow C$ at some stage after $s + 1$. This is a contradiction, so $\widetilde{W_e} \cap B \subseteq C \vee \langle m_j \rangle$. Therefore m_j witnesses that $\widetilde{W_e} \vee C$ is a complement to B in $[C, I]^{\Diamond_M}$, and (4) holds. □

Corollary 2.3.6. *The Owings Splitting Theorem also holds in G. That is, if $C \subseteq B$ are elements of G such that B is noncomplemented over C, there exist c.e. ideals*

$A_0, A_1 \subseteq M$ *such that*

1. $A_0 \cap A_1 = 0$
2. $A_0 \vee A_1 = B$
3. $A_i \vee C$ *is noncomplemented over* C*,* $i = 0, 1$
4. *For any c.e. ideal* $I \subseteq M$*, if* $B \subseteq I$ *and* B *is noncomplemented in* $[C, I]$*, then* $A_i \vee C$ *is also noncomplemented in* $[C, I]$ *for* $i = 0, 1$*.*

Proof. In the proof of Theorem 2.3.5, let B and C be the chosen representatives of $B^\diamondsuit$ and $C^\diamondsuit$, respectively. Let $\hat{A}_0$ and $\hat{A}_1$ be the splitting of B obtained. Since containment and complementation are more restrictive in G than in $G^\diamondsuit$, properties (3) and (4) are already satisfied. In fact, any representatives of $\hat{A}_0^\diamondsuit$ and $\hat{A}_1^\diamondsuit$ will satisfy (3) and (4) in G. Therefore we must find representatives which are a split of B in G. Suppose $m \in M$ is such that $\hat{A}_0 \vee \hat{A}_1 \vee \langle m \rangle \subseteq B \vee \langle m \rangle$ and additionally $\hat{A}_0 \cap \hat{A}_1 \subseteq \langle m \rangle$. Note that $\hat{A}_i \vee \langle m \rangle \subseteq B \vee \langle m \rangle$ for $i = 0, 1$.

Let $A_0 = \hat{A}_0 \cap \overline{\langle m \rangle}$. It is immediate that $A_0 =^\diamondsuit \hat{A}_0$, $A_0 \subseteq B$, and $A_0 \cap \hat{A}_1 = 0$. Now we alter $\hat{A}_1$ so it is a complement to A_0 in B. Let $A_1 = B \cap (\hat{A}_1 \vee \langle m \rangle)$. Clearly $A_0 \cap A_1 = 0$ and $A_0 \vee A_1 = B$. We must show $A_1 =^\diamondsuit \hat{A}_1$. The witness is simply m. Note $A_1 \vee \langle m \rangle = (B \cap \hat{A}_1) \vee \langle m \rangle = (B \vee \langle m \rangle) \cap (\hat{A}_1 \vee \langle m \rangle)$. Since $\hat{A}_1 \vee \langle m \rangle \subseteq B \vee \langle m \rangle$, that last ideal is simply $\hat{A}_1 \vee \langle m \rangle$, which is clearly in $\hat{A}_1^\diamondsuit$. □

CHAPTER 3

AN ISOMORPHISM BETWEEN $G^{\diamondsuit}$ AND $\mathcal{E}^*$

The goal of this chapter is to construct an isomorphism between $G^{\diamondsuit}$ and $\mathcal{E}^*$. To do so, we will establish a correspondence between two enumerations of c.e. ideals and two enumerations of c.e. sets. This chapter builds on the Δ^0_3 automorphism machinery as developed by Cholak, Soare, Harrington, and others ([6], [10]), especially as presented in Harrington and Soare [10]. I have tried to keep as closely as possible to their notation, and the construction and verification are laid out nearly identically. This chapter is designed to be self-contained, so some definitions are repeated from §1.4 and §2.1.

3.1 Summary

This section is directed at the reader who is familiar with the Δ^0_3 automorphism method and whose primary interest is in where the isomorphism method differs. We refer specifically to Harrington and Soare [10]; all construction step and lemma numbers said to be from "the original construction" are from that paper.

3.1.1 Definitions and basic changes

Denote the countable atomless Boolean algebra by Q, and the lattice of c.e. ideals of Q by $I(Q)$. The structure G is $[0, M]$ for any nonprincipal c.e. ideal $M \subset Q$, an initial segment of $I(Q)$. Define the equivalence relation $=^\diamondsuit$ on G by

$$A =^\diamondsuit B \iff (\exists m \in M)[A \vee \langle m \rangle = B \vee \langle m \rangle].$$

The quotient structure $G/{=^\diamondsuit}$ is denoted $G^\diamondsuit$. For this construction, we will fix a copy of G with M maximal.

We replace ω with M, letting $\widehat{\omega}$ be as before. Player RED builds an enumeration of c.e. ideals, $\{U_n\}_{n\in\omega}$ and one of c.e. sets, $\{V_n\}_{n\in\omega}$. Player BLUE builds sets $\{\widehat{U}_n\}_{n\in\omega}$ and ideals $\{\widehat{V}_n\}_{n\in\omega}$. State is defined as before, where enumeration of ideals is as follows.

Ideal enumeration. Suppose we have already determined J_s for J an ideal. During stage $s+1$, we may enumerate some finite collection of elements of M into J_s; call that collection X. At the end of stage $s+1$ we will close the ideal J with respect to $Y_{\lambda,s+1}$, the set of all elements on the tree. That is, we let $J_{s+1} = \langle J_s \cup X \rangle \bigcap Y_{\lambda,s+1}$. Since there are only a finite number of elements on the tree at any stage, J_{s+1} will be finite for every $s \in \omega$. Note that closing the ideals is effective, because membership in a principal ideal is computable.

Principal versus nonprincipal. Note that at any given stage we will only be dealing with a finite subset of M. This set generates an infinite principal ideal. An important characteristic of M is that it is nonprincipal, so no finite subset of M can generate all of M; there is always an element outside the principal ideal generated by the current set of elements on the construction tree. Since every nontrivial ideal is infinite, we must amend our concept of "almost every" $x \in M$. Instead of saying almost every $x \in M$ have a property φ if the set $\{x : \neg\varphi(x)\}$ is finite, we require $\{x : \neg\varphi(x)\}$ be contained in a principal ideal.

$$(\text{a.e. } x)[\varphi(x)] \Leftrightarrow (\exists m \in M)(\forall x \in M)[\neg\varphi(x) \Rightarrow x \in \langle m \rangle].$$

Likewise, we must replace "there exist infinitely-many x" ($\exists^\infty x$) with something more discerning. We say "there exists a nonprincipal collection of x" ($\exists^{np}x$) as shorthand for $(\forall m \in M)(\exists x \notin \langle m \rangle)$. That is, there is an element outside every principal ideal of M. Membership in a maximal ideal or a principal ideal is computable, so $\exists^{np}x$ is of the same complexity as $\exists^{\infty}x$.

Complexity of sets of states. For a state to be well-visited with respect to ideals means there is a nonprincipal collection of elements which have that state at some time during the construction. Almost every element leaves a non-well-resided state by the end of the construction, in the new sense of "almost every." The only changes from the original definitions are from $\exists^{\infty}x$ to $\exists^{np}x$ in each case. The fact that $\exists^{\infty}x$ and $\exists^{np}x$ have the same complexity means the properties of a state being well-visited (§3.4.1) and non-well-resided (§3.4.3) are still Π^0_2 and Σ^0_3, respectively.

Restrictions on the movement of x. In the original construction, the size of a number $n \in \omega$ is used for a number of restrictions on n's movement and enumeration. In this construction, there are two possible replacements for size. When all we need is a linear order on the elements of M, we use a fixed enumeration, essentially letting the "size" of x be the stage at which it is enumerated into M. In some cases, however, we need a stronger restriction. In that case we require x be outside the principal ideal generated by some initial segment of the enumeration of M. The key to our success is that M is nonprincipal, so there is always an element of M independent of any given finite initial segment.

3.1.2 Specific alterations

The chief point at which the stronger replacement for size is used is in defining k^+_β, the bound on the set of elements which have non-well-visited α-states for $\alpha^- = \beta$ (Equation (3.4.7)). It is still a number, but now instead of requiring $x > k^+_\beta$ in Steps 1 and 2, we require x be outside the principal ideal generated by the first

k_β^+ elements of M enumerated. This makes the pockets of nodes $\alpha \subset f$ contain a principal collection of elements rather than just a finite collection. Pockets of nodes to the left of the true path still contain a finite number of elements, and pockets to the right are emptied every time the final step (here, Step 6) is applied.

All mention of α-witnesses has been removed, as it is not necessary for our purposes that the isomorphism have any additional special properties. Correspondingly, we do not split S_α, R_α, and Y_α into S_α^0, S_α^1, and so on. Besides that, the only change to Step 2 (moving elements down one level) is the change in k_β^+ above. Step 1 (prompt pulling from the right to ensure $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$) has the additional restriction that the chosen x is independent of the elements we have already seen in R_α; that is, $x \notin \langle Y_{\alpha,s} \rangle$. Steps 3, 4, 5, and 6 (formerly 11) are unchanged.

In Lemmas 5.1 and 5.5 of the original construction, independence considerations must also be added. Lemma 5.1 (now Lemma 3.7.2) lists the ways elements may move on the tree and be enumerated into sets; in order to retain the usefulness of the lemma, we must restrict to enumerations into ideals such that the element is independent from what was already in the ideal. Lemma 5.5 (now Lemma 3.7.6) cannot assert each element is enumerated into only finitely many ideals, because Step 6 will enumerate any given element into an infinite number of ideals, so the enumerations Lemma 3.7.6 considers are restricted in the same way as in Lemma 3.7.2. This change is sufficient for the use of the lemma, which is in asserting Steps 1-5 and $\hat{1}$-$\hat{5}$ act finitely often between applications of Step 6. The only other change in the statement of a lemma is in Lemma 5.8 (here, Lemma 3.7.9), which now shows that for $\alpha \subset f$, $R_{\alpha,\infty} =^\diamond Y_\lambda =^\diamond M$.

3.2 Framework

Any terminology and notation not explicitly defined here may be found in §3.3. Given two enumerations, $\{U_n\}_{n\in\omega}$ of ideals and $\{V_n\}_{n\in\omega}$ of sets, we build two enumerations, $\{\widehat{U}_n\}_{n\in\omega}$ of sets and $\{\widehat{V}_n\}_{n\in\omega}$ of ideals. The $\widehat{U}_n$ are intended as images for the U_n, and the $\widehat{V}_n$ are intended as preimages for the V_n. We think of the correspondence in terms of states, where a state ν is a collection of indices such that, for x with state ν, x is in a set or ideal if and only if the index for that set or ideal is in ν. The exact definition is as follows.

Definition 3.2.1. *Let $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ be two sequences of c.e. sets or ideals. The* final e-state *of x with respect to (w.r.t.) $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ is $\nu(e,x) = \langle e, \sigma(e,x), \tau(e,x)\rangle$, where*

$$\sigma(e,x) = \{i : i \le e \ \&\ x \in X_i\}, \text{ and}$$

$$\tau(e,x) = \{i : i \le e \ \&\ x \in Y_i\}.$$

If a correspondence of ideals (U_n, $\widehat{V}_n$) and sets ($\widehat{U}_n$, V_n) is to be an isomorphism, it must certainly satisfy the following condition.

$$\begin{aligned} &(\forall \nu)(\exists^{np} x \in M)[\nu(e,x) = \nu \text{ w.r.t. } \{U_n\}_{n\in\omega} \text{ and } \{\widehat{V}_n\}_{n\in\omega}] \\ &\iff (\exists^{\infty} \hat{x} \in \omega)[\nu(e,\hat{x}) = \nu \text{ w.r.t. } \{\widehat{U}_n\}_{n\in\omega} \text{ and } \{V_n\}_{n\in\omega}]. \end{aligned} \tag{3.2.1}$$

That is, the state corresponds to a nonprincipal ideal in G if and only if the state corresponds to an infinite set in $\mathcal{E}$.

We would like, then, to talk about the *well-resided* states. A state is well-resided on the M side if the collection of elements which have that state at the end of the construction is not contained in any principal ideal (on the ω side, the collection must be infinite). However, we have the limitation that our construction be Δ^0_3, while being well-resided is Π^0_3. This necessitates worrying about the states *as elements have them* during the construction. Therefore we split the definition into two:

Definition 3.2.2. *A state ν is* well-visited *on the M side if the collection of elements which have state ν during the construction is not contained in any principal ideal. On the ω side, ν is well-visited if the collection of elements which have state ν during the construction is infinite.*

Definition 3.2.3. *A state ν is* non-well-resided *on the M side (ω side) if it is well-visited, but at the end of the construction, the collection of elements with state ν is contained in a principal ideal (is finite).*

Well-visited is a Π^0_2 property (see the definition of $\mathcal{F}_\alpha$ in §3.4.1). Non-well-resided is the complement of well-resided inside the set of well-visited states. It does not immediately appear to be an improvement over well-resided, since it is still Σ^0_3 (see the definition of $\mathcal{N}_\alpha$ in §3.4.3), but we may approximate it with Π^0_2 predicates which essentially say "after this (fixed) value, nothing which enters the state stays."

Since states are disjoint, what we need to know to have an automorphism is that the well-resided states coincide on the M side and the ω side (Requirement (3.2.1)), which we will accomplish by ensuring the well-visited states and the non-well-resided states coincide.

3.3 Initial Definitions

3.3.1 Enumerations, ideals, and the construction tree

Fix a maximal ideal M. We map from M to ω, but for clarity we rename the image $\widehat{\omega}$. Designate elements of M by lowercase Roman letters (x, y, $\ldots$), and natural numbers by hatted lowercase Roman letters ($\hat{x}$, $\hat{y}$, $\ldots$). On the M side we have two indexings of the computably enumerable subideals of M, $\{U_n\}_{n\in\omega}$ and $\{\widehat{V}_n\}_{n\in\omega}$. On the $\widehat{\omega}$ side we likewise have two indexings of the c.e. sets, $\{\widehat{U}_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$. The enumerations $\{U_n\}$ and $\{V_n\}$ are given; the enumerations $\{\widehat{U}_n\}$ and $\{\widehat{V}_n\}$ are built in response as images and preimages, respectively. Note that the hats on the V ideals and sets are reversed with respect to which side they live in; this is the only place where such reversal takes place. We view the construction as a game between two players. Player 1 (RED) controls the U ideals and V sets, and Player 2 (BLUE) controls the $\widehat{U}$ ideals and $\widehat{V}$ sets.

The notation for ideals will be as follows. We fix an enumeration $\{m_0, m_1, \ldots\}$ of M to use throughout the construction. Given that fixed enumeration, let $x \lhd y$ indicate x is enumerated before y. Let $P_{\unlhd x}$ be the principal ideal generated by all elements of M enumerated up to and including x, and $P_{\lhd x}$ the ideal generated by all elements of M enumerated up to but not including x. When we know which $m_i \in M$ we are working with, we have the shorthand $P_{<i} := P_{\lhd m_i}$ and $P_{\leq i} := P_{\unlhd m_i}$. For $X \subseteq M$, we will also use the notation $\langle X \rangle$ to mean the ideal generated by X. When X is an explicitly-listed finite set $\{x_1, x_2, \ldots, x_n\}$ we may omit the curly braces and say $\langle x_1, x_2, \ldots, x_n \rangle$.

In this chapter we will use a slightly different definition of enumeration for ideals. In past chapters, to enumerate an ideal J, at stage $s+1$ we enumerated an additional element x into J_s and let J_{s+1} be the principal ideal $\langle J_s \cup \{x\} \rangle$. In this construction we will only consider a finite initial segment of M at each stage, called $Y_{\lambda,s}$. During stage $s+1$, we may enumerate some finite collection of elements of M into J_s; call that collection X. At the end of stage $s+1$ we will close the ideal J with respect to $Y_{\lambda,s+1}$. That is, we let $J_{s+1} = \langle J_s \cup X \rangle \bigcap Y_{\lambda,s+1}$, which will be finite for every $s \in \omega$.

By analogy with $(\exists^\infty x) \equiv (\forall n)(\exists x > n)$ for ω, we define $(\exists^{np} x)[\varphi(x)]$ as $(\forall m \in M)(\exists x \in M)[x \notin \langle m \rangle \ \& \ \varphi(x)]$. Since M is maximal, and membership in a maximal or principal ideal is computable, there is no complexity increase over $(\exists^\infty x)$. Verbally this will be described as a *nonprincipal collection*; one which may not itself be a nonprincipal ideal, but which cannot be contained in any principal ideal. Likewise, if "almost every" (a.e.) $x \in M$ has a property φ, it means that the collection of x which do not have φ is contained in a principal ideal. Recall that for A and B, two subideals of M, $A =^\diamondsuit B$ if there is some $m \in M$ such that $A \vee m = B \vee m$. We will extend that notion to situations where A and B are not necessarily ideals but simply sets of elements, so, for example, $A =^\diamondsuit \emptyset$ means the elements of A are contained in a principal subideal of M. We will abuse terminology to refer to such a set A as "principal," and to a nonprincipal collection A as simply "nonprincipal."

The construction takes place on a tree T, which we think of as a subset of ω^ω, using coding. The tree T grows downward with its root, λ, at the top. Each node α of T will control part of the construction. For example, it may build a pair U_α, $\widehat{U}_\alpha$, where for some n_α determined by the length of α, U_α is intended as an approximation to U_{n_α} and $\widehat{U}_\alpha$ as its image $\widehat{U}_{n_\alpha}$. Likewise, some nodes control V, $\widehat{V}$ pairs, and some perform other tasks; see §3.3.3. T will be computable, and will have a *true path* f. If the above node α is on the true path, then $U_\alpha =^\diamondsuit U_{n_\alpha}$ and $\widehat{U}_\alpha$ is the correct candidate for $\widehat{U}_{n_\alpha}$. In this construction f is not in general computable but instead is $\emptyset''$-computable, which means the sequences of images and preimages will have only a $\emptyset''$-computable (that is, Δ^0_3) presentation. The definitions of f and T are in §3.5.

We use the notation for trees found in [14]. The set of all infinite paths through T is denoted $[T]$. Let nodes on the tree be designated by lowercase Greek letters (α, β, γ, δ, $\ldots$), where $\beta \subseteq \alpha$ ($\beta \subset \alpha$) indicates α extends (properly extends) β. When neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$ is true, we write $\alpha \perp \beta$. For two strings α and β, whether they are finite or infinite, $\alpha \cap \beta$ denotes the longest string which is a substring of both α and β. Let λ denote the empty string. Let $|\alpha|$ denote the length of α, and α^- be the immediate predecessor of α if $\alpha \neq \lambda$. Let $\alpha^\frown\beta$ denote the string formed by concatenating β to the end of α. When β is the string composed of only one element b, we may write $\alpha^\frown b$ for $\alpha^\frown\beta$.

Definition 3.3.1. *Let $\alpha, \beta \in T$.*

(i) *For $\alpha \perp \beta$, α is to the left of β ($\alpha <_L \beta$) if*

$$(\exists a, b \in \omega)(\exists \gamma \in T)[\gamma^\frown a \subseteq \alpha \;\&\; \gamma^\frown b \subseteq \beta \;\&\; a < b].$$

(ii) *$\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$.*

(iii) *$\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.*

(iv) *If $h \in [T]$, we say $\alpha <_L h$ ($h <_L \alpha$, $\alpha < h$, $h < \alpha$) if there exists $\beta \subset h$ such that $\alpha <_L \beta$ ($\beta <_L \alpha$, $\alpha < \beta$, $\beta < \alpha$, respectively).*

3.3.2 Elements of M and $\widehat{\omega}$ on the tree

We think of each element of M and each natural number as being painted on a ball. At each node α we place a pocket, called S_α, which can hold no more than a principal collection of M-balls, and a pocket called $\hat{S}_\alpha$ which can hold finitely-many $\widehat{\omega}$-balls. During the construction we pour balls into the tree, always starting from the top, S_λ ($\hat{S}_\lambda$). The balls will move on the tree, sometimes being retrieved to a higher pocket but in general moving downward. The $\widehat{\omega}$-ball marked $\hat{x}$ may move no lower than the level with nodes of length $\hat{x}$, and there may be other restrictions on the movement of $\hat{x}$. On the M side there are similar limitations on x, described for both M and $\widehat{\omega}$ in Steps 1 and 2 of the construction in §3.6. For $\alpha \subset f$, however, the collection of x ($\hat{x}$) which are not at or below α will be principal (finite).

The function $\alpha(x, s)$ ($\hat{\alpha}(\hat{x}, s)$) will designate the location of ball x ($\hat{x}$) at the end of stage s. We will guarantee in the construction that $\alpha(x) = \lim_s \alpha(x, s)$ ($\hat{\alpha}(\hat{x}) = \lim_s \hat{\alpha}(\hat{x}, s)$) exists. For each stage s we define

$$S_{\alpha,s} = \{x : \alpha(x, s) = \alpha\},$$

$$R_{\alpha,s} = \{x : \alpha(x, s) \supseteq \alpha\},$$

$$Y_{\alpha,s} = \bigcup\{R_{\alpha,t} : t \leq s\},$$

and likewise the hatted versions. The pocket S_α is called an *α-section*, and R_α an *α-region*. The region R_α consists of all elements in pockets at or below node α. We will prove that an element x can enter R_α at most once; however, it might not remain, so $R_{\alpha,\infty}$ (defined below) will be a d.c.e. set. Therefore we define the c.e. set $Y_\alpha = \bigcup_s Y_{\alpha,s}$ of all elements which are in R_α at any point during the construction. Another set we will find useful is

$$Y_{<\alpha} = \bigcup\{Y_\delta : \delta <_L \alpha\},$$

the collection of all elements which ever enter the pockets of nodes to the left of α.

Let $S_{\alpha,\infty} = \{x : \alpha(x) = \alpha\}$, and $R_{\alpha,\infty} = \{x : \alpha(x) \supseteq \alpha\}$. We will ensure that if $\alpha \subset f$, $R_{\alpha,\infty} =^\diamond Y_\alpha =^\diamond M$ ($\hat{R}_{\alpha,\infty} =^* \hat{Y}_\alpha =^* \omega$), by guaranteeing that $R_{\alpha,\infty}$ is empty if $f <_L \alpha$ and finite if $\alpha <_L f$, and that every $S_{\alpha,\infty}$ is principal or finite.

We will also guarantee that balls move into R_α from R_{α^-}, so that $Y_\alpha \setminus Y_{\alpha^-} = \emptyset$ (recall that $A \setminus B$ is $A - B$ together with the elements of $A \cap B$ which are enumerated into A before entering B). During the construction, the true path will be approximated by a computable sequence of finite strings $\{f_s\}_{s\in\omega}$, such that $f = \liminf_s f_s$. This approximation to the true path will restrict the movement of elements on the tree.

Definition 3.3.2. *If $f_s <_L \alpha$ at some stage s such that $x \lhd m_s$ ($\hat{x} \leq s$), the element x ($\hat{x}$) is* α-ineligible *at all stages $t \geq s$.*

If x is α-ineligible at stages $t \geq s$, we will require $x \notin S_{\alpha,t}$ ($\hat{x} \notin \hat{S}_{\alpha,t}$) for all $t \geq s$. The true path is defined in such a way that if $\alpha \subset f$, the number of times we see $f_s <_L \alpha$ is finite, so only a finite number of elements become α-ineligible.

3.3.3 States and the duties of α

Any given node α will either be building a U, $\widehat{U}$ pair, building a $\widehat{V}$, V pair, or thinking about non-well-resided α-states (Definition 3.3.3, below). Accordingly, we must spread out the U and V indices. Which nodes do what will depend on their length, so we assign to each node α indices e_α, $\hat{e}_\alpha$ which depend on $|\alpha|$. If α is building U_α, for instance, it will attempt to ensure $U_\alpha =^\diamondsuit U_{e_\alpha}$. We begin by defining $e_\lambda = \hat{e}_\lambda = -1$, and continue inductively according to $|\alpha|$ as follows:

n Activity at α for $|\alpha| = n \pmod 4$

0 Build U_α and $\widehat{U}_\alpha$ (goal: $\alpha \subset f \Rightarrow U_\alpha =^\diamondsuit U_{e_\alpha}$)

V_α, $\widehat{V}_\alpha$ undefined

$e_\alpha = e_{\alpha^-} + 1$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$

1 Build $\widehat{V}_\alpha$ and V_α (goal: $\alpha \subset f \Rightarrow \widehat{V}_\alpha =^\diamondsuit \widehat{V}_{e_\alpha}$)

U_α, $\widehat{U}_\alpha$ undefined

$e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-} + 1$

2 Consider new α-states ν believed to be non-well-resided on Y_α (see §3.4.3)

U_α, $\widehat{U}_\alpha$, V_α, $\widehat{V}_\alpha$ undefined

$e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$

3 Consider new α-states $\hat{\nu}$ believed to be non-well-resided on $\hat{Y}_\alpha$ (see §3.4.3)

U_α, $\widehat{U}_\alpha$, V_α, $\widehat{V}_\alpha$ undefined

$e_\alpha = e_{\alpha^-}$; $\hat{e}_\alpha = \hat{e}_{\alpha^-}$

Since it only makes sense to think about whether x is in U_α, say, when $|\alpha| = 0 \pmod 4$ (that is, when $e_\alpha = e_{\alpha^-} + 1$), we adjust our concept of e-state to α-state.

Definition 3.3.3. (i) *The* α-state *of x at stage s is*

$$\nu(\alpha, x, s) = \langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s)\rangle, \text{ where}$$

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \& \ e_\beta > e_{\beta^-} \ \& \ x \in U_{\beta,s}\}, \text{ and}$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \& \ \hat{e}_\beta > \hat{e}_{\beta^-} \ \& \ x \in \widehat{V}_{\beta,s}\}.$$

(ii) *The* final α-state *of x is*

$$\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x)\rangle,$$

where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ *and* $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

(iii) *The only λ-state is* $\nu_{-1} = \langle \lambda, \emptyset, \emptyset\rangle$.

The α-state of $\hat{x}$ has the dual definition to the above.

For ease of discussion, we define some orderings and operations on states.

Definition 3.3.4. *Given α-states $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$ and $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$, we define the following inequalities, with the strict version of each defined as expected.*

(i) $\nu_0 \leq_B \nu_1$ *if* $\sigma_0 = \sigma_1$ *and* $\tau_0 \subseteq \tau_1$ *(BLUE claims more $\widehat{V}$ ideals).*

(ii) $\nu_0 \leq_R \nu_1$ *if* $\sigma_0 \subseteq \sigma_1$ *and* $\tau_0 = \tau_1$ *(RED claims more U ideals).*

(iii) $\hat{\nu}_0 \leq_B \hat{\nu}_1$ *if* $\hat{\sigma}_0 \subseteq \hat{\sigma}_1$ *and* $\hat{\tau}_0 = \hat{\tau}_1$ *(BLUE claims more $\widehat{U}$ sets).*

(iv) $\hat{\nu}_0 \leq_R \hat{\nu}_1$ *if* $\hat{\sigma}_0 = \hat{\sigma}_1$ *and* $\hat{\tau}_0 \subseteq \hat{\tau}_1$ *(RED claims more V sets).*

Note that considering $\hat{\nu}_0$ and $\hat{\nu}_1$ to be ν_0 and ν_1 read with respect to $\widehat{U}$ and V rather than U and $\widehat{V}$, we get the following correspondence:

$$[\nu_0 \leq_R \nu_1 \Leftrightarrow \hat{\nu}_0 \leq_B \hat{\nu}_1] \ \& \ [\nu_0 \leq_B \nu_1 \Leftrightarrow \hat{\nu}_0 \leq_R \hat{\nu}_1] \tag{3.3.1}$$

Definition 3.3.5. *Given $\alpha \in T$, $\beta \subseteq \alpha$, and an α-state $\nu_0 = \langle \alpha, \sigma_0, \tau_0\rangle$, a set of α-states $\mathcal{C}_\alpha$, or a finite set of α-states $\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\}$:*

(i) $\nu_0 \restriction \beta = \langle \beta, \sigma_1, \tau_1 \rangle$, *where* $\sigma_1 = \sigma_0 \cap \{0, \ldots, e_\beta\}$ *and* $\tau_1 = \tau_0 \cap \{0, \ldots, \hat{e}_\beta\}$.

(ii) $\mathcal{C}_\alpha \restriction \beta = \{\nu \restriction \beta : \nu \in \mathcal{C}_\alpha\}$.

(iii) $\nu_1 \preceq \nu_0$ *("ν_0 extends ν_1") if* $\exists \beta$ *such that* $\nu_0 \restriction \beta = \nu_1$.

(iv) $\bigcup\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\} = \langle \alpha, \sigma, \tau \rangle$ *where* $\sigma = \bigcup\{\sigma_i : i \in I\}$ *and* $\tau = \bigcup\{\tau_i : i \in I\}$.

3.4 Keeping Track of the Residedness of States

3.4.1 Well-visited states

For each $\alpha \in T$ we define a number of sets of α-states. The set $\mathcal{F}_\alpha$ is the collection of α-states ν which are well-visited by elements x while they are in R_α. Adding the restriction that x must have the state ν when it first appears in R_α (which is to say, when it first appears in S_α) gives the set $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$. Each of these sets also has a dual. The explicit definitions are

$$\mathcal{E}_\alpha = \{\nu : (\exists^{np} x)(\exists s)[x \in S_{\alpha,s} - \bigcup\{S_{\alpha,t} : t < s\} \;\&\; \nu(\alpha, x, s) = \nu]\}$$

$$\mathcal{F}_\alpha = \{\nu : (\exists^{np} x)(\exists s)[x \in R_{\alpha,s} \;\&\; \nu(\alpha, x, s) = \nu]\}$$

where the duals are obtained by hatting appropriately and replacing $(\exists^{np} x)$ with $(\exists^\infty \hat{x})$.

To meet the automorphism requirement (3.2.1), we must have

$$\widehat{\mathcal{F}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{F}_\alpha\} \tag{3.4.1}$$

for $\alpha \subset f$. To achieve (3.4.1), each node α will also have an associated set $\mathcal{M}_\alpha$, the set of α-states α believes to be well-visited. At every node α we require $\mathcal{M}_\alpha \restriction \alpha^- = \mathcal{M}_{\alpha^-}$. For $\alpha \subset f$, we will prove that $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$ and $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ to get $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$. Depending on the length of α, $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ will either be proved directly or by proving the following three conditions.

$$\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha \tag{3.4.2}$$

$$\text{(a.e. } x)[\text{if } x \in Y_{\alpha,s}, \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha, \text{ and BLUE causes enumeration of } x \text{ so that } \nu(\alpha, x, s+1) = \nu_1, \text{ then } \nu_1 \in \mathcal{M}_\alpha] \tag{3.4.3}$$

$$\text{(a.e. } x)[\text{if } x \in Y_{\alpha,s}, \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha, \text{ and RED causes enumeration of } x \text{ so that } \nu(\alpha, x, s+1) = \nu_1, \text{ then } \nu_1 \in \mathcal{M}_\alpha] \tag{3.4.4}$$

Condition (3.4.2) will be met by exerting tight control of the entry of elements into S_α. Condition (3.4.3) will be met by ensuring $\mathcal{M}_\alpha$ is sufficiently closed with respect to BLUE's possible enumerations; that is, by making sure $\alpha \subset f$ is $\mathcal{M}$-*consistent.*

Definition 3.4.1. *A node α is $\mathcal{M}$-*inconsistent *if $e_\alpha > e_{\alpha^-}$ and there exist α-states $\nu_0 <_B \nu_1$ such that $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \upharpoonright \alpha^- \in \mathcal{M}_{\alpha^-}$, but $\nu_1 \notin \mathcal{M}_\alpha$. Otherwise α is* $\mathcal{M}$-consistent.

The dual notion is

Definition 3.4.2. *A node α is $\widehat{\mathcal{M}}$-*inconsistent *if $\hat{e}_\alpha > \hat{e}_{\alpha^-}$ and there exist α-states $\hat{\nu}_0 <_B \hat{\nu}_1$ such that $\hat{\nu}_0 \in \widehat{\mathcal{M}}_\alpha$, $\hat{\nu}_1 \upharpoonright \alpha^- \in \widehat{\mathcal{M}}_{\alpha^-}$, but $\hat{\nu}_1 \notin \widehat{\mathcal{M}}_\alpha$. Otherwise α is* $\widehat{\mathcal{M}}$-consistent.

Condition (3.4.4) will be met via the dual case. By (3.4.2), $\mathcal{M}_\alpha$ contains many of the well-visited states: every one which is witnessed sufficiently by elements as they enter R_α. Together (3.4.3) and (3.4.4) guarantee that all of the states which are witnessed to be well-visited by elements which are already in R_α are also in $\mathcal{M}_\alpha$, giving $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$.

The dual $\widehat{\mathcal{M}}_\alpha$ is defined as

$$\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}. \tag{3.4.5}$$

In the verification we will prove that $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$ as well, so that (3.4.1) is satisfied and the well-visited α-states coincide on the M and $\widehat{\omega}$ sides.

3.4.2 Avoiding circularity

Although the intention for $\mathcal{M}_\alpha$ is that it be equal to $\mathcal{F}_\alpha$, we must be able to determine from the node α^- which extension to take. Since $\mathcal{F}_\alpha$ is dependent on the particular α chosen, we now define a set which depends only on α^-. For $\beta = \alpha^-$, the new set $\mathcal{F}_\beta^+$ will be such that for $\alpha \subset f$, $\mathcal{M}_\alpha = \mathcal{F}_\beta^+ = \mathcal{F}_\alpha$.

Fix $\alpha \in T$ such that $e_\alpha > e_\beta$ for $\beta = \alpha^-$. Define the c.e. set $Z_{e_\alpha} = \bigcup_s Z_{e_\alpha,s}$ where

$$Z_{e_\alpha,s+1} = \{x : x \in U_{e_\alpha,s+1} \ \& \ x \in Y_{\alpha^-,s}\}.$$

Define a new α-state $\nu^+(\alpha,x,s)$ exactly as for $\nu(\alpha,x,s)$ (Definition 3.3.3) but with $Z_{e_\alpha,s}$ in place of $U_{\alpha,s}$. Note that we are only changing (possibly) the last place of $\nu(\alpha,x,s)$. Define $\mathcal{F}_\beta^+$ and k_β^+ as follows.

$$\mathcal{F}_\beta^+ = \{\nu : (\exists^{np} x)(\exists s)[x \in Y_{\beta,s} \ \& \ \nu^+(\alpha,x,s) = \nu]\}. \tag{3.4.6}$$

$$\begin{aligned} k_\beta^+ = \min\{y : (\forall x \rhd m_y)(\forall s)[[x \notin P_{<y} \ \& \\ x \in Y_{\beta,s} \ \& \ \nu^+(\alpha,x,s) = \nu_1] \longrightarrow \nu_1 \in \mathcal{F}_\beta^+]\}. \end{aligned} \tag{3.4.7}$$

The value k_β^+ is the bound on the set of elements which have non-well-visited states (since there are only a finite number of α-states, only a principal collection of elements can have non-well-visited states). The object is to keep elements in $P_{<k_\beta^+}$ out of Y_α. We also define $\widehat{\mathcal{F}}_\beta^+ = \{\hat{\nu} : \nu \in \mathcal{F}_\beta^+\}$. If $\alpha \in T$, $\beta = \alpha^-$ are such that $\hat{e}_\alpha > \hat{e}_\beta$, we define $\widehat{\mathcal{F}}_\beta^+$ and $\hat{k}_\beta^+$ using the duals to (3.4.6) and (3.4.7).

Along with $\mathcal{M}_\alpha$, every $\alpha \in T$ will have a k_α such that if $\alpha \subset f$, $k_\alpha = k_\beta^+$. If $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$, we define $\mathcal{F}_\beta^+ = \mathcal{F}_\beta$, $k_\beta^+ = k_\beta$, and likewise for the duals. We allow x to enter Y_α only if $x \notin P_{<k_\alpha}$ (to enter $\hat{Y}_\alpha$, $\hat{x}$ must be greater than $\hat{k}_\alpha$). Therefore if there is an element allowed into Y_α which has a state α considers non-well-visited, we have a witness that k_α is wrong.

Definition 3.4.3. *If* $(\exists x)(\exists s)[x \in Y_{\alpha,s}$ & $\nu(\alpha, x, s) \notin \mathcal{M}_\alpha]$*, then* α *is* provably incorrect *at all stages* $t \geq s$.

Nodes α which are provably incorrect are kept off the true path.

3.4.3 Non-well-resided states

As with well-visited states, we define several sets of states related to non-well-residedness for each node α. The set of non-well-resided α-states is

$$\mathcal{N}_\alpha = \{\nu_1 : \neg(\exists^{np} x)[x \in Y_\alpha \;\&\; \nu(\alpha, x) = \nu_1]\}.$$

Likewise we define $\widehat{\mathcal{N}}_\alpha$. As with the well-visited states in requirement (3.4.1), we must show for all $\alpha \subset f$ that

$$\widehat{\mathcal{N}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{N}_\alpha\}. \tag{3.4.8}$$

While $\mathcal{F}_\alpha$ and $\mathcal{E}_\alpha$ are Π^0_2, and so can be guessed at (almost) directly in the construction, $\mathcal{N}_\alpha$ is Σ^0_3 and so requires approximation. The Π^0_2 approximation will be the disjoint union of two sets $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$, which correspond to states α believes are non-well-resided and emptied by RED or BLUE respectively.

We define $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and their duals inductively. Fix $\alpha \in T$ and assume $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$, $\widehat{\mathcal{R}}_\gamma$, and $\widehat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. We define all four sets as disjoint unions, e.g.,

$$\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha.$$

Define

$$\mathcal{R}^{<\alpha}_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha \;\&\; \nu \restriction \alpha^- \in \mathcal{R}_{\alpha^-}\}.$$

The set $\mathcal{B}^{<\alpha}_\alpha$ is defined as above but with $\mathcal{B}_{\alpha^-}$ in place of $\mathcal{R}_{\alpha^-}$, and $\widehat{\mathcal{B}}^{<\alpha}_\alpha$ and $\widehat{\mathcal{R}}^{<\alpha}_\alpha$ are defined likewise, with appropriate hatting. If $|\alpha| \not\equiv 2 \pmod 4$, we set

$$\mathcal{R}^\alpha_\alpha = \widehat{\mathcal{B}}^\alpha_\alpha = \emptyset;$$

they might be nonempty otherwise. Note that when $|\alpha| \equiv 2 \pmod 4$, $\mathcal{R}_\alpha^{<\alpha}$ depends only on nodes up to α^- because at such an α, $e_\alpha = e_{\alpha^-}$ and $\hat{e}_\alpha = \hat{e}_{\alpha^-}$, so $\mathcal{M}_\alpha = \mathcal{M}_{\alpha^-}$.

If $|\alpha| \equiv 2 \pmod 4$, we define the Π_2^0 predicate

$$F(\alpha^-, \nu) \equiv (\forall x)[x \in Y_{\alpha^-} \;\longrightarrow\; (\nu(\alpha, x) \neq \nu \;\vee\; x \in P_{\leq|\alpha^-|})].$$

$F(\alpha^-, \nu)$ says that any element with state ν at the end of the construction is in the ideal generated by the first $|\alpha^-|$ elements of M enumerated. That is, α^- witnesses that ν corresponds to a principal ideal and is thus non-well-resided. Note also that as with $\mathcal{F}_\beta^+$, $F(\alpha^-, \nu)$ avoids circularity, since α-state depends only on $|\alpha|$. Having defined $F(\alpha^-, \nu)$, we let $\mathcal{R}_\alpha^\alpha$ be nonempty, allowing $\alpha \subset f$ only if

$$\mathcal{R}_\alpha^\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}_\alpha^{<\alpha} \cup \mathcal{B}_\alpha^{<\alpha}) \;\&\; F(\alpha^-, \nu)\}.$$

Also for $|\alpha| \equiv 2 \pmod 4$, we define

$$\widehat{\mathcal{B}}_\alpha^\alpha = \{\hat{\nu} : \nu \in \mathcal{R}_\alpha^\alpha\}.$$

If $|\alpha| \not\equiv 3 \pmod 4$, we set

$$\widehat{\mathcal{R}}_\alpha^\alpha = \mathcal{B}_\alpha^\alpha = \emptyset.$$

If $|\alpha| \equiv 3 \pmod 4$, we allow $\widehat{\mathcal{R}}_\alpha^\alpha \neq \emptyset$, defining the predicate $\hat{F}(\alpha^-, \hat{\nu})$ as follows.

$$\hat{F}(\alpha^-, \hat{\nu}) \equiv (\forall \hat{x})[[\hat{x} > |\alpha^-| \;\&\; \hat{x} \in \hat{Y}_{\alpha^-}] \;\rightarrow\; \hat{\nu}(\alpha, \hat{x}) \neq \hat{\nu}]$$

Again, the requirement is that for $\alpha \subset f$,

$$\widehat{\mathcal{R}}_\alpha^\alpha = \{\hat{\nu} : \hat{\nu} \in \widehat{\mathcal{M}}_\alpha - (\widehat{\mathcal{R}}_\alpha^{<\alpha} \cup \widehat{\mathcal{B}}_\alpha^{<\alpha}) \;\&\; \hat{F}(\alpha^-, \hat{\nu})\}$$

and we define

$$\mathcal{B}_\alpha^\alpha = \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}_\alpha^\alpha\}.$$

It will be BLUE's responsibility to change the state of elements x such that $\nu(\alpha, x, s) \in \mathcal{B}_\alpha$, for $x \in R_\alpha$, which takes care of half of the approximation. For $\mathcal{R}_\alpha$, we know that if $\alpha \subset f$, $\mathcal{R}_\alpha$ will in fact be non-well-resided, so

$$(\forall \nu \in \mathcal{R}_\alpha)(\text{a.e. } x \in Y_\alpha)(\forall s)[\nu(\alpha, x, s) = \nu \longrightarrow (\exists t > s)[\nu(\alpha, x, t) \neq \nu]]. \tag{3.4.9}$$

Therefore BLUE can wait for RED to move elements out of states in $\mathcal{R}_\alpha$. This leads to the definition of another kind of consistency. Since $\alpha \subset f$ means that all states in $\mathcal{R}_\alpha$ must be emptied by RED, for every state in $\mathcal{R}_\alpha$ there must be a state reachable in RED moves which is not non-well-resided. Furthermore, since there are only a finite number of α-states, at least one such state must also be well-visited. This is another closure property of $\mathcal{M}_\alpha$, as was $\mathcal{M}$-consistency.

Definition 3.4.4. *A node* $\alpha \in T$ *is* $\mathcal{R}$-consistent *if*

$$(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1 \in \mathcal{M}_\alpha)[\nu_0 <_R \nu_1]$$

and $\mathcal{R}$-inconsistent *otherwise.*

The dual notion is

Definition 3.4.5. *A node* $\alpha \in T$ *is* $\widehat{\mathcal{R}}$-consistent *if*

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{R}}_\alpha)(\exists \hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha)[\hat{\nu}_0 <_R \hat{\nu}_1]$$

and $\widehat{\mathcal{R}}$-inconsistent *otherwise.*

As with $\mathcal{M}$-consistency, we will require $\alpha \subset f$ be $\mathcal{R}$-consistent. Therefore for $\alpha \subset f$, using (3.3.1) and the definition of $\widehat{\mathcal{B}}_\alpha$ we know

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha)(\exists \hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha)[\hat{\nu}_0 <_B \hat{\nu}_1]. \tag{3.4.10}$$

Note that Equation (3.4.10) and Definition 3.4.4, via repeated application, guarantee after some number of iterations of moves by BLUE or RED we can get out of $\widehat{\mathcal{B}}_\alpha$

or $\mathcal{R}_\alpha$, respectively. Thus, for every state α believes to be emptied by BLUE, there must be a state which α believes to be well-visited and not emptied by BLUE which is reachable by the BLUE moves. That motivates the following definition.

Definition 3.4.6. *A function* $\hat{h}_\alpha : \widehat{\mathcal{B}}_\alpha \to (\widehat{\mathcal{M}}_\alpha - \widehat{\mathcal{B}}_\alpha)$ *is a* target function *if*

$$(\forall \hat{\nu} \in \widehat{\mathcal{B}}_\alpha)[\hat{\nu} <_B \hat{h}_\alpha(\hat{\nu})].$$

Dually, $h_\alpha : \mathcal{B}_\alpha \to (\mathcal{M}_\alpha - \mathcal{B}_\alpha)$ *is a target function if* $(\forall \nu \in \mathcal{B}_\alpha)[\nu <_B h_\alpha(\nu)]$.

The notes preceding the definition assert the existence of such an h_α for $\alpha \subset f$. We will require that for almost every $x \in \mathcal{B}_\alpha$, BLUE must move x to the target state $h_\alpha(\nu(\alpha, x, s))$.

Since $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$ are approximations, we must make sure that using them we empty exactly the states in $\mathcal{N}_\alpha$. By the use of $F(\alpha^-, x)$ in the definition of $\mathcal{R}_\alpha$, we know $\mathcal{R}_\alpha \cup \mathcal{B}_\alpha \subseteq \mathcal{N}_\alpha$. In order to guarantee we empty all states in $\mathcal{N}_\alpha$, it is sufficient to make sure that if $\alpha \subset f$ and $\nu_0 \in \mathcal{N}_\alpha$, there is some $\gamma \supseteq \alpha$ such that $\gamma \subset f$ and for all $\nu_1 \in \mathcal{M}_\gamma$ which extend ν_0, $\nu_1 \in \mathcal{R}_\gamma \cup \mathcal{B}_\gamma$. Removing references to $\mathcal{N}_\alpha$, the statement we must prove is

$$\begin{aligned}(\forall \alpha \subset f)(\forall \nu_0 \in \mathcal{M}_\alpha)(\neg \exists^{np} x)[x \in Y_\alpha \;\&\; \nu(\alpha, x) = \nu_0] \Longrightarrow \\ (\exists \gamma)[\alpha \subseteq \gamma \subset f \;\&\; \{\nu_1 \in \mathcal{M}_\gamma : \nu_1 \restriction \alpha = \nu_0\} \subseteq \mathcal{R}_\gamma \cup \mathcal{B}_\gamma]\end{aligned} \tag{3.4.11}$$

along with its dual. To check this, fix some $\alpha \subset f$ and $\nu_0 \in \mathcal{M}_\alpha$ such that the hypothesis of (3.4.11) holds. Since $\alpha \subset f$ we know $Y_\alpha =^\diamond M$, so we can find some i such that for all $x \in M$, $x \notin P_{\leq i} \Rightarrow \nu(\alpha, x) \neq \nu_0$. Choose $\gamma \subset f$ such that $\alpha \subseteq \gamma$, $|\gamma| > i$, and $|\gamma| \equiv 2 \pmod 4$. Consider any $\nu_1 \in \mathcal{M}_\gamma$ such that $\nu_1 \restriction \alpha = \nu_0$. If ν_1 is not in $\mathcal{R}_\gamma^{<\gamma} \cup \mathcal{B}_\gamma^{<\gamma}$, then $F(\gamma^-, \nu_1)$ holds, so by definition of $\mathcal{R}_\gamma^\gamma$ for $\gamma \subset f$, $\nu_1 \in \mathcal{R}_\gamma^\gamma$. The dual statement is proved likewise.

Finally, we note that since $\mathcal{B}_\alpha$ and $\widehat{\mathcal{B}}_\alpha$ are defined as duals to $\mathcal{R}_\alpha$ and $\widehat{\mathcal{R}}_\alpha$, again using (3.3.1), to show all of the states in these four sets are emptied it suffices to prove

$$(\forall \nu_0 \in \mathcal{B}_\alpha)[\{x : \nu(\alpha, x) = \nu_0\} =^{\diamondsuit} \emptyset] \tag{3.4.12}$$

and its dual.

3.5 The Definition of the Tree and the True Path

First we collect our notions of consistency, allowing a node on the tree to have successors only if it satisfies all such notions.

Definition 3.5.1. *A node $\alpha \in T$ is* consistent *if it is $\mathcal{M}$-, $\widehat{\mathcal{M}}$-, $\mathcal{R}$-, and $\widehat{\mathcal{R}}$-consistent.*

In the following definition, the intended meanings of $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α have already been explained. The number $c_\alpha \in \omega$ is an additional empty symbol that will guess a Σ^0_3 predicate; its function is explained below, in Definition 3.5.5 and the remarks that follow it.

Definition 3.5.2 ($\boldsymbol{T}$, the construction tree). *Put $\lambda \in T$, and let $\mathcal{M}_\lambda$, $\mathcal{R}_\lambda$, and $\mathcal{B}_\lambda$ all be empty. Define $k_\lambda = e_\lambda = \hat{e}_\lambda = -1$. If $\beta \in T$, put $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha\rangle$ in T provided it meets the following conditions:*

(i) *β is consistent.*

(ii) *$\mathcal{M}_\alpha$ is a set of α-states; $\mathcal{R}_\alpha, \mathcal{B}_\alpha \subseteq \mathcal{M}_\alpha$; $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$.*

(iii) *$\mathcal{M}_\alpha \restriction \beta = \mathcal{M}_\beta$.*

(iv) *$(e_\alpha = e_{\alpha^-}\ \&\ \hat{e}_\alpha = \hat{e}_{\alpha^-}) \Rightarrow \mathcal{M}_\alpha = \mathcal{M}_\beta$.*

(v) *$\mathcal{R}^{<\alpha}_\alpha \subseteq \mathcal{R}_\alpha$; $\mathcal{B}^{<\alpha}_\alpha \subseteq \mathcal{B}_\alpha$.*

(vi) *$\mathcal{R}^{\alpha}_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 2\ (mod\ 4)$; $\mathcal{B}^{\alpha}_\alpha \neq \emptyset \Rightarrow |\alpha| \equiv 3\ (mod\ 4)$.*

In addition, each $\alpha \in T$ has associated dual sets $\widehat{\mathcal{M}}_\alpha$, $\widehat{\mathcal{R}}_\alpha$, and $\widehat{\mathcal{B}}_\alpha$, determined from $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{B}_\alpha$, respectively, as well as integers e_α and $\hat{e}_\alpha$ depending only on $|\alpha|$. Recall that we are associating $\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha \rangle$ with an integer under some effective coding so that we may regard T as a subset of $\omega^{<\omega}$.

Definition 3.5.3. *The* true path $f \in [T]$ *is defined by induction on* n. *If* $\beta = f \restriction (n-1)$ *has been defined and is consistent, then* $f \restriction n$ *is the* $<_L$*-least length-*n *extension* α *of* β *such that the following hold:*

(i) $n \equiv 0 \ (mod\ 4) \Longrightarrow \mathcal{M}_\alpha = \mathcal{F}^+_\beta$ *and* $k_\alpha = k^+_\beta$.

(ii) $n \equiv 1 \ (mod\ 4) \Longrightarrow \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$ *and* $k_\alpha = k^+_\beta$.

(iii) $n \equiv 2 \ (mod\ 4) \Longrightarrow \begin{array}{c} \mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cap \mathcal{B}^{<\alpha}_\alpha) \ \& \ F(\beta,\nu)\} \\ \text{and } \widehat{\mathcal{B}}^\alpha_\alpha = \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\}. \end{array}$

(iv) $n \equiv 3 \ (mod\ 4) \Longrightarrow \begin{array}{c} \widehat{\mathcal{R}}^\alpha_\alpha = \{\hat{\nu} : \hat{\nu} \in \widehat{\mathcal{M}}_\alpha - (\widehat{\mathcal{R}}^{<\alpha}_\alpha \cap \widehat{\mathcal{B}}^{<\alpha}_\alpha) \ \& \ \hat{F}(\beta,\nu)\} \\ \text{and } \mathcal{B}^\alpha_\alpha = \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}^\alpha_\alpha\}. \end{array}$

(v) *Unless otherwise specified above,* $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, *and* k_α *have the values* $\mathcal{M}_\beta$, $\mathcal{R}_\beta$, $\mathcal{B}_\beta$, *and* k_β, *respectively, as in Definition 3.5.2.*

(vi) *The set* C_α, *defined below in Definition 3.5.5, is infinite.*

For a consistent $\beta = f \restriction n$, note that $\mathcal{F}^+_\beta$ is just a finite set of states and k^+_β is an integer, so we may find α satisfying Conditions (i)-(v) of the definition. In fact, it is clear that there are unique $\mathcal{M}_\alpha$ and k_α satisfying the conditions. To see the same for $\mathcal{R}_\alpha$, recall from §3.4.3 that $\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha$, where $\mathcal{R}^{<\alpha}_\alpha$ depends only on β, and $\mathcal{R}^\alpha_\alpha$ is uniquely determined by Conditions (iii) and (v). Likewise, $\mathcal{B}_\alpha$ is uniquely determined by Conditions (iv) and (v). We will show that of the α meeting (i)-(v), there is a unique α meeting (vi) (see Definition 3.5.5 and the remarks that follow). Hence, as long as every node on f is consistent, which will be proved in Lemmas 3.7.10 and 3.7.12, f is infinite.

Condition (vi) of Definition 3.5.3 is included so we may approximate the true path during the isomorphism construction. We will now define C_α. Recalling the remarks in §3.2 and §3.4, we see that Conditions (i)-(v) of Definition 3.5.3 are uniformly Δ^0_3 in β, and thus also uniformly Σ^0_3 in β. The following lemma is a modification of Lemma 2.35 in Cholak [6], which is an easy modification of Theorem IV.3.4 in Soare [14]. Define

$$\mathcal{A} = \{(\alpha, \beta) : \alpha \text{ satisfies Conditions (i)-(v) of Definition 3.5.3 with respect to } \beta\}.$$

The set $\mathcal{A}$ is uniformly Δ^0_3 and hence Σ^0_3.

Lemma 3.5.4. *Let $\mathcal{A}$ be defined as above. Since $\mathcal{A}$ is Σ^0_3, then there is a computable function g such that*

$$x \in \mathcal{A} \iff (\exists! c)[|W_{g(x,c)}| = \infty]$$

and

$$x \notin \mathcal{A} \iff (\forall c)[|W_{g(x,c)}| < \infty].$$

Definition 3.5.5. *Let g be the function given by Lemma 3.5.4. For $x = (\alpha, \beta)$, where $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, c_\alpha\rangle$, the* chip set *$C_\alpha$ is the set $W_{g(x,c_\alpha)}$.*

We will use the chip sets in §3.6, Step 6A, to define the true path approximation, a computable sequence of finite strings $\{f_s\}_{s\in\omega}$ such that $f = \liminf_s f_s$. For any consistent β, there are unique $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α satisfying Conditions (i)-(v) of Definition 3.5.3, and hence there is a unique α such that C_α is infinite. Therefore the chip sets form a computable sequence of c.e. sets, $\{C_\alpha\}_{\alpha\in T}$, such that $\alpha \subset f$ iff $\beta = \alpha^-$ is on the true path and $|C_\alpha| = \infty$.

We included c_α in the node α so we could attach a particular chip set to each node of the tree. Once c_α is included in the node, there are an infinite number of paths through the tree that satisfy Conditions (i)-(v) of Definition 3.5.3 at every level. Condition (vi) is then included to ensure the uniqueness of the true path f.

Given the sequence $\{C_\alpha\}_{\alpha\in T}$, fix a simultaneous computable enumeration $\{C_{\alpha,s}\}_{\alpha\in T, s\in\omega}$ for use in §3.6, Step 6A.

To ensure $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$, we define $\mathcal{L}$, a list of elements of the form $\langle \alpha, \nu_1\rangle$, such that $\nu_1 \in \mathcal{M}_\alpha$. Loosely speaking, we allow an element x into $S_{\alpha,s+1}$ only when there is an unused entry $\langle\alpha,\nu_1\rangle \in \mathcal{L}_s$ such that x may be enumerated in such a way as to give $\nu(\alpha, x, s+1) = \nu_1$. In such a case we *mark* the entry $\langle\alpha,\nu_1\rangle$. $\mathcal{L}_s$ is augmented with new elements beginning with α at any stage s such that it and $\widehat{\mathcal{L}}_s$ are both *α-marked*; that is, all entries of the form $\langle\alpha,\nu_1\rangle$ on $\mathcal{L}$ ($\langle\alpha,\hat{\nu}_1\rangle$ on $\widehat{\mathcal{L}}_s$) have been marked. The value $m(\alpha, s)$ is the number of times $\mathcal{L}$ and $\widehat{\mathcal{L}}$ have been α-marked by the end of stage s. It does not have a hatted version.

3.6 The Construction

Steps 1-5 below, their duals $\hat{1}$-$\hat{5}$, and a final Step 6, produce the isomorphism. The duals should be clear; in cases where there may be ambiguity, it is explicitly noted. The "purpose of" remarks after some steps may contain statements to be proved in §3.7. There is one remaining definition we need for the construction.

Definition 3.6.1. *To* initialize *a node α means to remove every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and put x into $S_{\beta,s}$ ($\hat{x}$ into $\hat{S}_{\beta,s}$) for $\beta = \alpha \cap f_{s+1}$.*

Stage *s*=0: For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$ and $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = Y_{\lambda,0} = \emptyset$ and $f_0 = \lambda$.

Stage *s*+1: Find the least $n < 6$ such that Step n's hypotheses are satisfied for some $x \in Y_{\alpha,s}$ and perform Step n's action. If there is no such n, find the least $n < 6$ such that some Step $\hat{n}$ applies. If all of those fail to apply, apply Step 6. At the end of every step, close all ideals U_α, $\widehat{V}_\alpha$ with respect to $Y_{\lambda,s+1}$.

In the following steps, let $x \in Y_{\lambda,s}$ ($\hat{x} \in \hat{Y}_{\lambda,s}$) and $\alpha \in T$, $\alpha \neq \lambda$, be arbitrary, and let $\beta = \alpha^-$. Recall that $x \lhd y$ means in the fixed enumeration of M, x is enumerated before y. $P_{\unlhd x}$ is the ideal generated by all elements of M enumerated up to and including x, and $P_{\lhd x}$ is the ideal generated by all elements of M enumerated up to but not including x. Letting $M = \{m_0, \ldots, m_i, \ldots\}$, we have the shorthand $P_{<i} := P_{\lhd m_i}$ and $P_{\leq i} := P_{\unlhd m_i}$.

Step 1: Let $\langle \alpha, \nu_1 \rangle$ be the first unmarked entry of $\mathcal{L}$ ($\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$; note that by the definition of $\mathcal{L}$, $\nu_1 \in \mathcal{M}_\alpha$). Look for x meeting the following conditions.

Size

1. $x \notin P_{<k_\alpha}$ (in $\hat{1}$, $\hat{x} > \hat{k}_\alpha$), $x \rhd m_{|\alpha|}$

2. x is α-eligible

3. $x \rhd m_{m(\alpha,s)}$

Location

4. $x \in R_{\beta,s} - Y_{\alpha,s}$

5. $\neg(\alpha(x,s) <_L \alpha)$

State and Independence

6. $\nu(\beta, x, s) = \nu_1 \upharpoonright \beta$

7. $e_\alpha > e_\beta \Rightarrow \nu^+(\alpha, x, s) = \nu_1$

8. $x \notin \langle Y_{\alpha,s} \rangle$ (absent from $\hat{1}$)

Choose the least such x (with respect to $\lhd$) and perform the following actions.

9. mark the list entry $\langle \alpha, \nu_1 \rangle$.

10. put x into S_α

11. if $e_\alpha > e_\beta$ and $e_\alpha \in \sigma_1$, then put x into $U_{\alpha,s+1}$

12. if $\hat{e}_\alpha > \hat{e}_\beta$ and $\hat{e}_\alpha \in \tau_1$, then put x into $\widehat{V}_{\alpha,s+1}$

Purpose of Step 1: If $\alpha \subset f$ and $\nu_1 \in \mathcal{M}_\alpha$, then $\mathcal{L}$ will have an infinite number of entries of the form $\langle \alpha, \nu_1 \rangle$ put on it and later marked. Each time such an entry is marked, an element x, which is not in the principal ideal of $Y_{\alpha,s}$, is put into S_α for the first time and given state ν_1. Since that happens an infinite number of times, ν_1 is well-visited by independent elements when they first appear in S_α; i.e., $\nu_1 \in \mathcal{E}_\alpha$ and $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$.

Step 2: Look for x and α meeting the following conditions.

1. $x \in S_{\beta,s}$

2. $x \rhd m_{|\alpha|}$, $x \notin P_{<k_\alpha}$ (in $\hat{1}$, $\hat{x} > \hat{k}_\alpha$)

3. x is α-eligible

4. $x \lhd m_{m(\alpha,s)}$ (contrast with 1.3)

5. α is the leftmost $\gamma \in T$ such that when you put γ in for α and γ^- in for β, #1-4 are satisfied

Choose the least such pair $\langle \alpha, x \rangle$ (with respect to node length and $\lhd$) and move x from S_β to S_α.

Purpose of Step 2: If $\alpha \subset f$, this ensures $R_\alpha =^* \omega$. We control with condition 2.4 in order to slow the flow down the tree. This keeps us from throwing too many elements down a path which is not f; $m(\alpha, s) \to \infty$ iff $\alpha \subset f$, so we bound what can move down into nodes α which are not on f.

Step 3: Look for x and α meeting the following conditions.

1. $e_\alpha > e_\beta$

2. $x \in S_{\alpha,s}$

3. $\nu(\alpha, x, s) = \nu_0 \in \mathcal{M}_\alpha$

4. $(\exists \nu_1)[\nu_0 <_B \nu_1 \ \& \ \nu_1 \upharpoonright \beta \in \mathcal{M}_\beta \ \& \ \nu_1 \notin \mathcal{M}_\alpha]$

Choose the least such pair $\langle \alpha, x \rangle$ (with respect to node length and $\lhd$) and enumerate x into $\widehat{V}_{\delta,s+1}$ for all $\delta \subset \alpha$ such that $e_\delta \in \tau_1$.

Purpose of Step 3: If α is $\mathcal{M}$-inconsistent (which means exactly conditions 3.1, 3.3, 3.4), witnessed by $x \in S_\alpha$ (condition 3.2), then we give x the state ν_1 to make α provably incorrect (which means there is an element in the region, in particular, x, which has a state α considers non-well-visited). This knocks α off of f.

Step 4: Look for $x \in R_{\alpha,s}$ meeting the following conditions.

1. $e_\alpha > e_\beta$

2. $x \notin U_{\alpha,s}$

3. $x \in Z_{e_\alpha,s}$

Choose the least such pair $\langle \alpha, x \rangle$ (with respect to node length and $\lhd$) and enumerate x into $U_{\alpha,s+1}$.

Step 5: Look for x and α satisfying the conditions of one of the following two cases.

Case 1

1. $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$

2. $x \in S_{\alpha,s}$

3. α is $\mathcal{M}$-consistent and $\mathcal{R}$-consistent

Case 2

1. $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$

4. $x \in S_{\delta,s}$ where $\delta^- = \alpha$

5. δ is either $\mathcal{M}$-inconsistent or $\mathcal{R}$-inconsistent

In either case, choose the least such $\langle \alpha, x \rangle$ (with respect to node length and $\lhd$). Let $\nu_1 = h_\alpha(\nu_0)$, which will be BLUE-greater than ν_0 (by definition of h_α), $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Enumerate x into $\widehat{V}_\gamma$ for all $\gamma \subseteq \alpha$ such that $\hat{e}_\gamma > \hat{e}_{\gamma^-}$ and $e_\gamma \in \tau_1 - \tau_0$ (that is, we considered a new $\widehat{V}$ set at γ and it was in the chosen extension to x's α-state). This makes $\nu(\alpha, x, s+1) = \nu_1$.

Step 6:

6A. Define δ_t by induction for $t \leq s+1$. Let $\delta_0 = \lambda$. Given δ_t, let $v \leq s$ be maximal such that $\delta_t \subseteq f_v$ if such a v exists, and let $v = 0$ otherwise (v is the most recent stage at which the true path appeared to go through δ_t). Choose the $<_L$-least $\alpha \in T$ such that $\alpha^- = \delta_t$ and $C_{\alpha,s} \neq C_{\alpha,v}$. If such an α exists, define $\delta_{t+1} = \alpha$. If not, define $\delta_{t+1} = \delta_t$. That is, look for the leftmost node which extends δ_t by one and which has increased its chip set since the last time you were at that node (because $\alpha \subset f \iff |C_\alpha| = \infty$) and go through there.

Define $f_{s+1} = \delta_{s+1}$.

6B. For every $\alpha \subseteq f_{s+1}$ such that both $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ are α-marked (every entry beginning with α is marked), do the following:

1. define $m(\alpha, s+1) = m(\alpha, s) + 1$

2. add to the bottom of $\mathcal{L}_s$ a new unmarked α-entry $\langle \alpha, \nu \rangle$ for every $\nu \in \mathcal{M}_\alpha$. Do likewise for $\widehat{\mathcal{L}}_s$.

After doing the above for all relevant α, let $\mathcal{L}_{s+1}$ be the augmented $\mathcal{L}_s$ and likewise for $\widehat{\mathcal{L}}_{s+1}$. If no such α exists, let the stage $s+1$ version of everything equal the stage s version.

6C. Empty R_α to the right of f_{s+1}: initialize all α to the right of f_{s+1}. That is, pull all the balls in α's pockets up to where α branches off from f_{s+1}.

6D. Add balls to the machine: choose the $\lhd$-least $x \notin Y_{\lambda,s}$ such that $x \lhd m_s$ ($<$-least $\hat{x} \notin \hat{Y}_{\lambda,s}$ such that $x < s$) and put x into S_λ ($\hat{x}$ into $\hat{S}_\lambda$). For each $x \in Y_{\lambda,s+1}$, let $\alpha(x, s+1)$ denote the unique γ such that $x \in S_{\gamma,s+1}$, and likewise for all $\hat{x}$.

3.7 The Isomorphism Theorem and Verification

Theorem 3.7.1 (Isomorphism Theorem). *Suppose c.e. ideals* $\{U_\alpha\}_{\alpha\in T}$ *and* $\{\widehat{V}_\alpha\}_{\alpha\in T}$ *and c.e. sets* $\{\widehat{U}_\alpha\}_{\alpha\in T}$ *and* $\{V_\alpha\}_{\alpha\in T}$ *are enumerated by the construction in §3.6 using Steps 1-5,* $\hat{1}$*-*$\hat{5}$*, and 6. Then the correspondence* $U_\alpha \leftrightarrow \widehat{U}_\alpha$*,* $\widehat{V}_\alpha \leftrightarrow V_\alpha$*,* $\alpha \subset f$*, defines an isomorphism between* $G^\diamondsuit$ *and* $\mathcal{E}^*$*.*

The proof of the theorem is split into the following thirteen lemmas. Lemmas 3.7.3, 3.7.7, 3.7.8, 3.7.11, and 3.7.14 have no duals. The remaining lemmas have a dual case whose proof should be clear from the proof as written.

Lemma 3.7.2. *At stage* $s+1$*,*

(i) *if* x *enters* R_α*,* $\alpha \neq \lambda$*, it is via Step 1 or Step 2 applying to* α *and* x*;*

(ii) *if* x *moves from* S_α *to* S_δ*, it is via one of the following three steps:*

  (a) Step 1 applies to δ *and* x *(*$\delta <_L \alpha$ *or* $\delta^- = \alpha$*);*

  (b) Step 2 applies to δ *and* x *(*$\delta^- = \alpha$*);*

   (c) *Step 11C applies to* α $(f_{s+1} <_L \alpha)$*;*

(iii) *if* $x \in S_{\alpha,s}$ *is enumerated in a RED set* $U_{\alpha,s+1}$ *such that* x *is not generated by the elements in* $U_{\alpha,s}$*, it is via Step 1 or Step 4 applying to* α *and* x*;*

(iv) *if* $x \in S_{\alpha,s}$ *is enumerated in a BLUE set* $\widehat{V}_{\alpha,s+1}$ *such that* x *is not generated by the elements in* $\widehat{V}_{\alpha,s}$*, it is via one of the following three steps:*

   (a) *Step 1 applies to* x *and* α $(\hat{e}_\alpha > \hat{e}_\beta)$*;*

   (b) *Step 3 applies to* x *and some* $\delta \supset \alpha$*;*

   (c) *Step 5 applies to* x *and some* $\delta \supseteq \alpha$ $(\hat{e}_\alpha > \hat{e}_\beta)$*.*

Proof. Clear from the construction. □

Lemma 3.7.3 (True Path Lemma). $f = \liminf_s f_s$.

Proof. This is immediate from the definitions of C_α and f in §3.5, and f_s in Step 6A. □

Lemma 3.7.4. *For all* $\alpha \in T$*,*

(i) $f <_L \alpha \Rightarrow R_{\alpha,\infty} = \emptyset$*;*

(ii) $\alpha <_L f \Rightarrow Y_\alpha =^* \emptyset$*;*

(iii) $\alpha \subset f \Rightarrow Y_{<\alpha} =^* \emptyset$*.*

Proof. Given x, choose s such that $x \lhd m_s$ and $f_s <_L \alpha$. Step 6C will initialize all nodes in R_α the next time Step 6 acts, emptying the region. Now, x is γ-ineligible for all $t \geq s$ and all $\gamma \supseteq \alpha$, so x cannot be in any such $S_{\gamma,t}$. Steps 1 and 2 will not act on x and α, by construction conditions 1.2 and 2.3, so $x \notin R_{\alpha,t}$, giving (i).

For (ii), assume $\alpha <_L f$. Since by definition $|C_\alpha| < \infty$, we will only see $\alpha \subset f_s$ a finite number of times. Step 6B will act finitely-often on α and therefore there will be only a finite number of entries $\langle \alpha, \nu \rangle$ on $\mathcal{L}$. Since Step 1 marks a list entry each time it acts, only finitely-many x can enter S_α under Step 1; also, $\mathcal{L}$ can be α-marked only finitely many times, so $\lim_s m(\alpha, s) < \infty$. Step 2, by condition 2.4, will move only finitely-many x into R_α, and by Lemma 3.7.2, those are the only ways for x to enter R_α. Therefore $Y_\alpha =^* \emptyset$.

Part (iii) is immediate from (ii) since $<_L$ is a well-order. □

Lemma 3.7.5. *For every $\alpha \in T$, if $\alpha \neq \lambda$ and $\beta = \alpha^-$, then*

(i) $Y_\alpha \setminus Y_\beta = \emptyset$ *and* $Y_\alpha \subseteq Y_\beta$*;*

(ii) $(\forall x)(\exists^{\leq 1} s)[x \in R_{\alpha,s+1} - R_{\alpha,s}]$*;*

(iii) $U_\alpha \setminus Y_\alpha = \widehat{V}_\alpha \setminus Y_\alpha = \emptyset$*;*

(iv) $\alpha \subset f \Rightarrow (\exists v_\alpha)(\forall x)(\forall s \geq v_\alpha)[x \in R_{\alpha,s} \rightarrow (\forall t \geq s)[x \in R_{\alpha,t}]]$.

Proof. To see (i), note the only way for x to enter Y_α is by Step 1 or 2 moving it there, both of which require $x \in R_\beta \subseteq Y_\beta$.

For (ii), suppose $x \in R_{\alpha,s+1} - R_{\alpha,s}$ and $x \in R_{\alpha,t} - R_{\alpha,t+1}$ for some $t > s$ (i.e., it leaves again). Then by 6D, we know $x \lhd m_s$. By Lemma 3.7.2 (ii), at stage $t+1$ either

1. Step 6C applies to α and x
2. Step 1 applies to δ and x for some $\delta <_L \alpha$, $\delta = \alpha(x, t+1)$

In case (1), we know $f_{s+1} <_L \alpha$, so x is γ-ineligible for all stages $v \geq t+1$ and $\gamma \supseteq \alpha$, so x cannot re-enter R_α. In case (2), by Lemma 3.7.2 (ii), construction condition 1.5, and induction on $v \geq t$, there are two possibilities. The first is that for all $v \geq t$, $\alpha(x, v) <_L \alpha$ so $x \notin R_{\alpha,v}$, which happens if the only steps which apply to x are 1 and 2. The second possibility is that at some stage v, Step 6C applies to x and some $\eta <_L \alpha$ ($\eta = \alpha(x, v-1)$). In that event, we know $f_v <_L \eta <_L \alpha$, so as in case (1) $x \notin R_{\alpha,w}$ for all $w \geq v$.

Enumeration of x into U_α or $\widehat{V}_\alpha$ can take place in Step 1, 3, 4, or 5. Step 1 also puts x into Y_α, Steps 3 and 5 require $x \in S_\alpha$, and Step 4 requires $x \in R_\alpha$, so (iii) holds.

For (iv), assume $\alpha \subset f$ and choose v_α such that $\forall s \geq v_\alpha$, $f_s \not<_L \alpha$, and such that no $\beta <_L \alpha$ acts at stage s (which we can assure by Lemma 3.7.4 (iii)), so $Y_{<\alpha,s} = Y_{<\alpha}$. As in (ii), the only ways for x to leave R_α are by Step 1 or 6C. Step 1 would pull x to S_γ for some $\gamma <_L \alpha$, but by assumption γ is no longer acting. Step 6C would have to pull x from R_α to the left, onto the true path, but again by assumption, the true path never again appears to be to the left of α. Thus x must remain in $R_{\alpha,s}$ for all $s \geq v_\alpha$. □

Lemma 3.7.6. *For all x,*

(i) $\alpha(x) := \lim_s \alpha(x, s)$ *exists;*

(ii) *x is enumerated into at most finitely-many c.e. ideals U_γ, $\widehat{V}_\gamma$ such that for such an ideal X, $x \in X_{s+1}$ but $x \notin \langle X_s \rangle$ (that is, x is independent from X_s).*

Proof. For (i), if $x \in S_\alpha$, we may assume $x \rhd m_{|\alpha|}$ because both Step 1 and Step 2 require that, and they are the only ways for x to enter S_α originally. Fix x and suppose it is m_i in the enumeration of M. Let $\gamma = f \upharpoonright i$, and let v_γ be defined as in Lemma 3.7.5 (iv). Choose $s > v_\gamma$ such that $\gamma \subset f_s$. Let $\delta_0 = \alpha(x, s)$. Either $\delta_0 <_L \gamma$ or $\delta_0 \subseteq \gamma$ (our choice of s prohibits $\gamma <_L \delta_0$, and $|\alpha(x, s)| < i = |\gamma|$ prevents $\gamma \subset \delta_0$). By choice of s, x can only be moved by Step 1 or 2, not by 6C. By induction on $t \geq s$, if $\delta_1 = \alpha(x, t)$ and $\delta_2 = \alpha(x, t+1)$ are nonequal, then either $\delta_2 <_L \delta_1$ or $\delta_2 \supset \delta_1$. However, there is no infinite sequence $\{\delta_0, \ldots\}$ allowed for x such that $\forall k(\delta_{k+1} <_L \delta_k \vee \delta_{k+1} \supset \delta_k)$, because x can go no lower on the tree than level i, and $<_L$ is a well-order.

To see (ii), note that by Lemma 3.7.2, the only ways for independent x to be enumerated into U_γ or $\widehat{V}_\gamma$ are via Steps 1, 3, 4, and 5. Step 1 requires x be moved on the tree, and by part (i) that can only happen finitely-many times. Steps 3, 4, and 5 require that x be in a specific pocket or region, and again by part (i), x only changes pockets a finite number of times. With x at a particular location α, each of those three steps can only enumerate x into ideals U_γ, $\widehat{V}_\gamma$ for $\gamma \subseteq \alpha$, of which there are finitely-many. Therefore x is only enumerated into X such that $x \in X_{s+1}$ but $x \notin \langle X_s \rangle$ a finite number of times. $\square$

Lemma 3.7.7. (i) *Step 6 applies infinitely often;*

(ii) *If the hypotheses of some Step 1-5 ($\hat{1}$-$\hat{5}$) remain satisfied, then that step eventually applies.*

Proof. If Step 6 applies at stage s, then Y_λ remains the same from stage s until the next time Step 6 applies; in particular, it is finite. Steps 1-5 may move balls on the tree or enumerate elements into ideals U_α or $\widehat{V}_\alpha$, where the element enumerated is not generated by the elements already in the ideal, but by Lemma 3.7.6 this happens only finitely many times. Therefore, Step 11 applies again at some stage $t > s$ and (i) holds.

By the design of the construction, Step 6 cannot occur if the hypotheses for some Step 1-5 ($\hat{1}$-$\hat{5}$) are satisfied, so by (i) a step whose hypotheses remain satisfied must eventually apply, and (ii) holds. $\square$

Lemma 3.7.8. *If $\alpha \subset f$, $\alpha \neq \lambda$, and $\beta = \alpha^-$, then*

(i) $(\forall \gamma <_L f)[m(\gamma) := \lim_s m(\gamma, s) < \infty]$*;*

(ii) $m(\alpha) := \lim_s m(\alpha, s) = \infty$*;*

(iii) $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha = \mathcal{F}^+_\beta$*;*

(iv) $\widehat{\mathcal{E}}_\alpha \supseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$*;*

Proof. If $\gamma <_L f$, then $\gamma \subseteq f_s$ for only finitely-many s. Thus only finitely-many γ-entries are ever added to $\mathcal{L}$, so $\mathcal{L}$ is necessarily γ-marked only finitely often, giving (i).

To see (ii), fix $\alpha \subset f$, $\alpha \neq \lambda$, and let $\beta = \alpha^-$. By definition of f, $\alpha \subset f$ implies $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ and $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$. Suppose $m(\alpha) < \infty$; say $m(\alpha, s) = n$ for all $s \geq s_0$.

Claim: Every α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ ($\langle \alpha, \hat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$) is eventually marked. (Proved below.)

Using the claim, find $s > s_0$ such that $\alpha \subset f_{s+1}$ and every α-entry on $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ is marked. But then by Step 6B, $m(\alpha, s+1) > m(\alpha, s) = n$, which contradicts the choice of s_0.

Proof of Claim: Suppose $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$ is never marked. By Step 6B, then, there are only finitely-many entries on $\mathcal{L}$. Choose $s_1 \geq s_0$ such that (1) every α-entry on $\mathcal{L}$ and every entry on $\mathcal{L}$ preceding $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$ which will ever be marked has been marked by stage s_1; (2) $Y_{<\alpha, s_1} = Y_{<\alpha}$; and (3) for all $x \trianglelefteq m_n$, $x \in Y_{\alpha, s_1} \Leftrightarrow x \in Y_\alpha$. Such a state exists by (1) assumption, (2) Lemma 3.7.4 (iii), and (3) Lemma 3.7.6 (i). Then $Y_\alpha = Y_{\alpha, s_1}$, because no $x \vartriangleright m_n$ can enter R_α under Step 2, and no x can later enter R_α under Step 1 because it must mark an α-entry on $\mathcal{L}$. We know $\nu_1 \in \mathcal{M}_\alpha$ because $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$, and $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ since $\alpha \subset f$. Then, by the definition of $\mathcal{F}^+_\beta$,

$$(\exists^{np} x)(\exists s > s_1)[x \in Y_{\beta, s} \;\&\; \nu^+(\alpha, x, s) = \nu_1]$$

Almost every such x also satisfies the hypotheses of Step 1, so some such element is moved to S_α under Step 1 at some stage $s + 1 > s$, which marks an entry $\langle \alpha, \nu_1 \rangle$, contradicting the assumption.

The dual proof establishes the claim for $\widehat{\mathcal{L}}$. $\dashv$

By (ii), since $\alpha \subset f$, $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are α-marked an infinite number of times. Thus for every $\nu_1 \in \mathcal{M}_\alpha$, an infinite number of entries $\langle \alpha, \nu_1 \rangle$ are added to $\mathcal{L}$. Each entry is later marked by Step 1 when at some stage $s+1$, some x is moved into S_α, where x is not generated by the elements of $Y_{\alpha, s}$. Hence, $\nu_1 \in \mathcal{E}_\alpha$ and (iii) holds. Part (iv) holds by the same proof as (iii), with Step $\hat{1}$. $\square$

Lemma 3.7.9. $\alpha \subset f \Rightarrow R_{\alpha, \infty} =^\diamond Y_\alpha =^\diamond Y_\lambda = M$.

Proof. By Lemma 3.7.7 (i), Step 6 applies infinitely-many times, so it must eventually put every $x \in M$ into Y_λ. By induction, assume $R_{\beta,\infty} =^\diamond Y_\beta =^\diamond M$ for $\beta = \alpha^-$. By Lemma 3.7.4, $Y_{<\alpha} =^* \emptyset$, and almost every $x \in R_\beta$ not yet in R_α must eventually lie in S_β. By Lemma 3.7.8, $m(\alpha) = \infty$ and $m(\gamma) < \infty$ for $\gamma <_L \alpha$ with $\gamma^- = \beta$. Therefore almost every $x \in R_\beta$ not yet in R_α must eventually satisfy the conditions of Step 2 of the construction and be moved to S_α. By Lemma 3.7.5 (iv), cofinitely many such x will remain in R_α forever, so $R_{\alpha,\infty} =^\diamond Y_\alpha =^\diamond M$. $\square$

Lemma 3.7.10. *$\alpha \subset f \Rightarrow \alpha$ is $\mathcal{M}$-consistent.*

Proof. Let $\alpha \subset f$ such that $\beta = \alpha^-$ and α is not $\mathcal{M}$-consistent. That is, $e_\alpha > e_\beta$, and for some $\nu_0 \in \mathcal{M}_\alpha$, there is $\nu_1 >_B \nu_0$ such that $\nu_1 \restriction \beta \in \mathcal{M}_\beta$ but $\nu_1 \notin \mathcal{M}_\alpha$. By the definition of T, α is a terminal node, so $R_\alpha = S_\alpha$. Since $\alpha \subset f$, Lemma 3.7.9 says $S_{\alpha,\infty} =^\diamond M$ and Lemma 3.7.5 (v) gives a stage v_α such that no $x \in S_{\alpha,s}$ $(s > v_\alpha)$ ever leaves S_α. By Lemma 3.7.8, $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha$, so

$$(\exists^{np} x)(\exists s)[x \in S_{\alpha,s+1} - S_{\alpha,s} \;\&\; \nu(\alpha, x, s+1) = \nu_0]. \tag{3.7.1}$$

Choose any such x and $s > v_\alpha$. Step 1 cannot move x, as it would cause x to leave R_α, and neither can Step 2, as there is no γ with $\gamma^- = \alpha$. Therefore Step 3 has the first chance to act on any such x. Almost every x satisfying (3.7.1) meets the conditions of Step 3, so Step 3 must apply to some $x \in S_{\alpha,s+1} - S_{\alpha,s}$, $t > s$, such that $\nu(\alpha, x, t) = \nu_0$. The action of Step 3 will cause $\nu(\alpha, x, t+1) = \nu_1$, with the result that α is provably incorrect for all stages $v \geq t + 1$, so $\alpha \not\subset f$. $\square$

Lemma 3.7.11. *If $\alpha \subset f$, then*

(i) $\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}\}$*;*

(ii) $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$*;*

(iii) $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$.

Proof. Part (i) is true by definition of $\widehat{\mathcal{M}}$. For (ii) and (iii), fix $\alpha \subset f$ and let $\beta = \alpha^-$. Assume by induction that the lemma holds for β. By definition, we know $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$, and by Lemma 3.7.8, we know $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$, and likewise on the hatted side. Therefore it suffices to show that $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ and $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Case 1. $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$.

In this case $\mathcal{M}_\alpha = \mathcal{M}_\beta$, and since $Y_\alpha \subseteq Y_\beta$, we know $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. By induction, $\mathcal{M}_\beta = \mathcal{F}_\beta$, so $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta = \mathcal{M}_\beta = \mathcal{M}_\alpha$.

Before Cases 2 and 3, we need a technical sublemma.

Technical Sublemma: If $e_\alpha > e_\beta$, $\nu_2 \in \langle \alpha, \sigma_2, \tau_2 \rangle \in \mathcal{F}^+_\beta$, and $\nu_1 = \langle \alpha, \sigma_1, \tau_2 \rangle$, where $\sigma_1 = \sigma_2 - \{e_\alpha\}$, then $\nu_1 \in \mathcal{F}^+_\beta$ also.

Proof. Suppose $\nu_2 \in \mathcal{F}^+_\beta$. Then $\nu_3 = \nu_2 \restriction \beta \in \mathcal{F}^+_\beta$ also, and $\mathcal{F}_\beta = \mathcal{E}_\beta$ by the inductive hypothesis. Therefore,

$$(\exists^{np} x)(\exists s)[x \in Y_{\beta,s} - Y_{\alpha,s-1} \;\&\; \nu(\beta, x, s) = \nu_3].$$

But for each such x and s, $x \notin Z_{e_\alpha,s} = \{x : x \in U_{e_\alpha,s} \;\&\; x \in Y_{\beta,s-1}\}$. Therefore $\nu^+(\alpha, x, s) = \nu_1$, and so a nonprincipal collection of x have ν^+ state ν_1 and $\nu_1 \in \mathcal{F}^+_\beta$. The dual proof shows the sublemma holds for $\mathcal{F}^+_\beta$. $\dashv$

Case 2. $e_\alpha > e_\beta$.

Part (ii) may be proved directly. Suppose $\nu_1 \in \mathcal{F}_\alpha$, and let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Then

$$(\exists^{np} x)(\exists s)[x \in Y_{\alpha,s} \;\&\; \nu(\alpha, x, s) = \nu_1] \tag{3.7.2}$$

by definition of $\mathcal{F}_\alpha$. Note that $Y_{\alpha,s} \subseteq Y_{\beta,s}$ and $\nu(\alpha, x, s) \leq_R \nu^+(\alpha, x, s)$ because $U_{\alpha,s} \subseteq Z_{e_\alpha,s}$. Suppose

$$(\exists^{np} x)(\exists s)[x \in Y_{\alpha,s} \;\&\; \nu^+(\alpha, x, s) = \nu_1]. \tag{3.7.3}$$

Then by the definition of $\mathcal{F}^+_\beta$, $\nu_1 \in \mathcal{F}^+_\beta$ since $Y_\alpha \subseteq Y_\beta$ and $\alpha \subset f$ gives $\mathcal{F}^+_\beta = \mathcal{M}_\alpha$.

If (3.7.3) fails, then for almost every x in (3.7.2), $\nu^+(\alpha, x, s) = \nu_2 >_R \nu_1$, so $\nu_2 = \langle \alpha, \sigma_2, \tau_1 \rangle$ where $e_\alpha \notin \sigma_1$ and $\sigma_2 = \sigma_1 \cup \{e_\alpha\}$. Again, by definition $\nu_2 \in \mathcal{F}^+_\beta$, so by the sublemma, $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$.

Part (iii) is proved using the following three claims.

Claim 1. $\widehat{\mathcal{E}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Claim 2. If $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ (where v_α is defined as in Lemma 3.7.5 (iv)), and RED causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Claim 3. If $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ (where v_α is defined as in Lemma 3.7.5 (iv)), and BLUE causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Claim 1 says that the states well-visited by elements when they first enter R_α are in $\mathcal{M}_\alpha$. Claims 2 and 3 together say that after stage v_α, every state attained by an element after it is already in R_α is also in $\mathcal{M}_\alpha$, so in particular, the well-visited states are in $\mathcal{M}_\alpha$. These three suffice to show $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Proof of Claim 1. Suppose $\hat{\nu}_1 \in \widehat{\mathcal{E}}_\alpha$. Then

$$(\exists^{np}\hat{x})(\exists s)[\hat{x} \in \hat{S}_{\alpha,s} - \hat{Y}_{\alpha,s-1} \;\&\; \nu(\alpha, \hat{x}, s) = \hat{\nu}_1]$$

For every such $\hat{x}$ and s, $\hat{x}$ must have entered $\hat{S}_{\alpha,s}$ under Step $\hat{1}$ or Step $\hat{2}$. If it was via Step $\hat{1}$, we must have marked an entry $\langle \alpha, \hat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$ by the definition of $\widehat{\mathcal{L}}$. If Step $\hat{2}$ acted we know $\hat{x} \notin \widehat{U}_{\alpha,s}$ because Lemma 3.7.5 (iv) gives $\widehat{U}_\alpha \setminus \hat{Y}_\alpha = \emptyset$, so $\hat{x} \notin \hat{Y}_{\alpha,s-1} \Rightarrow \hat{x} \notin \widehat{U}_{\alpha,s-1}$ and Step $\hat{2}$ does not enumerate. Thus by (3.3.1), $e_\alpha \notin \sigma_1$, where $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Let $\nu_3 = \nu_1 \upharpoonright \beta$. By the inductive hypothesis, $\hat{\nu}_3 \in \widehat{\mathcal{F}}_\beta = \widehat{\mathcal{M}}_\beta$, so $\nu_3 \in \mathcal{M}_\beta = \mathcal{F}_\beta$. The set $\mathcal{F}^+_\beta$ must contain a state extending ν_3, so either $\nu_1 \in \mathcal{F}^+_\beta$ or $\nu_2 \in \mathcal{F}^+_\beta$, where $\nu_2 = \langle \alpha, \sigma_1 \cup \{e_\alpha\}, \tau_1 \rangle$. If $\nu_2 \in \mathcal{F}^+_\beta$, then $\nu_1 \in \mathcal{F}^+_\beta$ by the sublemma. If not, $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$ anyway, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$. ⊣

Proof of Claim 2. Suppose RED enumerates $\hat{x}$ such that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, where $\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, and $s > v_\alpha$ (where v_α is defined as in Lemma 3.7.5 (iv)). Then $\hat{\nu}_1 <_R \hat{\nu}_2$, so $\nu_1 <_B \nu_2$. Since $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$, $\nu_1 \in \mathcal{M}_\alpha$. Since $\alpha \subset f$, so α is $\mathcal{M}$-consistent, $\nu_2 \in \mathcal{M}_\alpha$, and thus $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$. ⊣

Proof of Claim 3. Suppose BLUE enumerates $\hat{x}$ such that $\hat{\nu}_2 = \nu(\alpha, \hat{x}, s+1)$, where

$\hat{x} \in \hat{Y}_{\alpha,s}$, $\nu_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}_\alpha$, and $s > v_\alpha$ (where v_α is defined as in Lemma 3.7.5 (iv)). Since $s > v_\alpha$, $\hat{x} \in \hat{R}_{\alpha,s} \cap \hat{R}_{\alpha,s+1}$. Since BLUE is the player acting, the enumeration must take place via Step $\hat{1}$, $\hat{3}$, or $\hat{5}$ applying to $\hat{x}$ and some $\gamma \supseteq \alpha$. If Step $\hat{1}$ applies, it will give $\hat{x}$ some γ-state $\hat{\nu}_3 = \hat{\nu}(\gamma, \hat{x}, s+1)$. By construction $\hat{\nu}_3 \in \widehat{\mathcal{M}}_\gamma$, so $\hat{\nu}_3 \restriction \alpha = \hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$. The same holds in Step $\hat{5}$, where for case 1, $\hat{x} \in \hat{Y}_{\gamma,s}$, and for case 2, $\hat{x} \in \hat{Y}_{\delta,s}$ for some $\delta^- = \gamma$. If the BLUE enumeration takes place in Step $\hat{3}$, γ is $\mathcal{M}$-inconsistent, so it must be that $\gamma \supsetneq \alpha$ since $\alpha \subset f$. Let $\hat{\nu}_3 = \nu(\gamma^-, \hat{x}, s+1)$. By construction condition $\hat{3}.4$, $\hat{\nu}_3 \in \widehat{\mathcal{M}}_{\gamma^-}$, so $\hat{\nu}_2 = \hat{\nu}_3 \restriction \alpha \in \widehat{\mathcal{M}}_\alpha$ by the definition of T. $\dashv$

Case 3. $\hat{e}_\alpha > \hat{e}_\beta$.

Holds by the dual proof to Case 2. □

Lemma 3.7.12. *$\alpha \subset f \Rightarrow \alpha$ is $\mathcal{R}$-consistent.*

Proof. Assume $\alpha \subset f$ is not $\mathcal{R}$-consistent, so $(\exists \nu_1 \in \mathcal{R}_\alpha)(\forall \nu_2 \in \mathcal{M}_\alpha)[\nu_1 \not<_R \nu_2]$. Choose such a ν_1. As in Lemma 3.7.10, $S_\alpha = R_\alpha$, $S_\alpha =^\diamond M$, and there is v_α such that for $s > v_\alpha$, no $x \in S_{\alpha,s}$ later leaves S_α. Lemma 3.7.11 gives that $\mathcal{M}_\alpha = \mathcal{E}_\alpha$. By definition, $\mathcal{R}_\alpha \subseteq \mathcal{M}_\alpha$, so $\nu_1 \in \mathcal{R}_\alpha \Rightarrow \nu_1 \in \mathcal{E}_\alpha$, giving

$$(\exists^{np} x)(\exists s > v_\alpha)[x \in S_{\alpha,s+1} - Y_{\alpha,s} \;\&\; \nu(\alpha, x, s) = \nu_1]. \tag{3.7.4}$$

For each such x and s, as in Lemma 3.7.10, neither Step 1 nor Step 2 can apply at any stage $t > s+1$. Step 3 cannot apply to $x \in S_{\alpha,t}$ because, by Lemma 3.7.10, α is $\mathcal{M}$-consistent. Step 5 cannot apply to x while $\nu(\alpha, x, t) = \nu_1$, because it requires $\nu_1 \in \mathcal{B}_\alpha$, which is disjoint from $\mathcal{R}_\alpha$. However, if $\nu(\alpha, x, t) = \nu_1$ for all $t \geq s$, then x witnesses that $F(\alpha^-, \nu_1)$ fails, and $\nu_1 \in \mathcal{R}_\alpha$ would force $\alpha \not\subset f$. Therefore at some stage $t > s$, the state of x must be changed so that $\nu(\alpha, x, t) = \nu_1$ but $\nu(\alpha, x, t+1) = \nu_2 \neq \nu_1$. The only step remaining which can do such a thing is Step 4, which will choose ν_2 such that $\nu_1 <_R \nu_2$. This must happen for all x satisfying (3.7.4), so choose ν_2 such that a nonprincipal collection of x are given state ν_2. Then $\nu_2 \in \mathcal{F}_\alpha$, so by Lemma 3.7.11, $\nu_2 \in \mathcal{M}_\alpha$, and α is $\mathcal{R}$-consistent. □

Lemma 3.7.13. *If* $\alpha \subset f$ *and* $\nu_1 \in \mathcal{B}_\alpha$*, then* $\{x : x \in Y_\alpha \;\&\; \nu(\alpha, x) = \nu_1\} =^* \emptyset$.

Proof. Fix $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$. Let v_α be as in Lemma 3.7.5 (v), and let $x \in R_{\alpha,s}$ for some $s > v_\alpha$. Cofinitely many of the elements $x \in Y_\alpha$ will satisfy that hypothesis. Assume that for all $t \geq s$, $\gamma = \alpha(x, t)$ (some $\gamma \supseteq \alpha$) and $\nu_1 = \nu(\alpha, x, t)$. Since $\alpha \subset f$, by the (inductive) definition of $\mathcal{B}_\alpha$, $\nu_1 \in \mathcal{B}_\alpha \Rightarrow \nu_1' \in \mathcal{B}_\gamma$ for all $\nu_1' \in \mathcal{M}_\gamma$ such that $\nu_1' \restriction \alpha = \nu_1$. Note that x's γ-state must be some such ν_1'.

Case 1. γ is $\mathcal{R}$-consistent and $\mathcal{M}$-consistent.

Then the hypotheses of Step 5 case 1 remain satisfied, so at some stage $t+1 > s$, it applies with $\nu_1' = \nu(\gamma, x, t)$, $\nu_2' = \nu(\gamma, x, t+1)$, $\nu_1' <_B \nu_2'$, and $\nu_2' \in \mathcal{M}_\gamma - \mathcal{B}_\gamma$. Hence $\nu_2 = \nu_2' \restriction \alpha \in \mathcal{M}_\alpha - \mathcal{B}_\alpha$ and $\nu(\alpha, x, t+1) = \nu_2 >_B \nu_1$.

Case 2. Otherwise.

Then likewise, Step 5 case 2 applies to x and γ^-. □

Lemma 3.7.14. *The correspondence* $U_\alpha \leftrightarrow \widehat{U}_\alpha$ *and* $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, *defines an isomorphism from* $G^\diamondsuit$ *to* $\mathcal{E}^*$.

Proof. Choose $\alpha \subset f$. By Lemmas 3.7.10 and 3.7.12 α is consistent. Therefore every $\alpha \subset f$ has an extension in f and f is infinite. By Lemma 3.7.9 and its dual, $Y_\alpha =^\diamondsuit M$ and $\hat{Y}_\alpha =^* \omega$. Lemma 3.7.11 gives $\mathcal{F}_\alpha = \mathcal{M}_\alpha = \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha$, so the well-visited states on the M and $\widehat{\omega}$ sides coincide. Since for $\alpha \subset f$, $Y_\lambda - Y_\alpha =^\diamondsuit \emptyset$ $(\hat{Y}_\lambda - \hat{Y}_\alpha =^* \emptyset)$, Lemma 3.7.13 and its dual give $\mathcal{N}_\alpha = \widehat{\mathcal{N}}_\alpha$ (by the remarks preceding (3.4.12)), so the non-well-resided states also coincide. Therefore we have met the automorphism requirement stated and discussed in §3.2. □

CHAPTER 4

FURTHER RESULTS AND OPEN QUESTIONS

The final chapter addresses consequences of the isomorphism result between $G^\diamondsuit$ and $\mathcal{E}^*$ as well as open questions resulting from the study of G and $G^\diamondsuit$.

4.1 Transfer of Orbits from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$

Here we work in the setting of Π^0_1 classes. As discussed in §2.2, the transfer of orbits from $\mathcal{E}^*$ to $\mathcal{E}_\Pi$ is not completely successful. An orbit in $\mathcal{E}^*$ corresponds to an orbit in $G^\diamondsuit$, and an orbit in G corresponds to an orbit in $\mathcal{E}_\Pi$, but as Corollary 4.1.7 will show, the collection of Π^0_1 classes making up an orbit in $G^\diamondsuit$ is not necessarily an orbit in G. The invariance of the collection under automorphisms is preserved, but not transitivity; that is, there may be classes A and B in the collection such that no automorphism f of G takes A to B. The result draws on the idea of Cantor-Bendixson rank. Any Π^0_1 class from G with Cantor-Bendixson rank strictly less than ω_1^{CK} generates an orbit in $G^\diamondsuit$ which does not form an orbit in G.

Definition 4.1.1. *The* Cantor-Bendixson derivative *of a Π^0_1 class P is*

$$D(P) = P - \{f : f \text{ is isolated in } P\}.$$

We may iterate the derivative to get $D^2(P)$, $D^3(P)$, etc., with $D^\alpha(P) = \bigcap_{\beta<\alpha} D^\beta(P)$ for limit ordinals α. The Cantor-Bendixson rank *of P is the least ordinal α such that $D^\alpha(P) = D^{\alpha+1}(P)$. Let $CB(P)$ denote the Cantor-Bendixson rank of P.*

Definition 4.1.2. *The* computable ordinals *are the order types of computable well-orderings of* ω. *The least non-computable ordinal is* Church-Kleene ω_1, *or* ω_1^{CK}.

Theorem 4.1.3 (Kreisel [12], see [2]). *The set of Cantor-Bendixson ranks of* Π^0_1 *classes,* $\{\alpha : (\exists P)[CB(P) = \alpha]\}$, *is exactly the set of ordinals* $\{\alpha : \alpha \leq \omega_1^{CK}\}$.

Theorem 4.1.4. *Let* $A^\diamondsuit$ *be an element of* $G^\diamondsuit$. *The set* $\{CB(P) : P \in A^\diamondsuit\}$ *is closed upwards in the ordinals* $\leq \omega_1^{CK}$.

Proof. Let $A \in A^\diamondsuit$ such that $CB(A) = \alpha < \omega_1^{CK}$, and let $\beta > \alpha$ be a computable ordinal or ω_1^{CK}. Supposing $G = [N, 2^\omega]$, let $p \in 2^{<\omega}$ such that $[p] \cap N = \emptyset$. From Theorem 4.1.3, let P be a Π^0_1 class of rank β. Let $Q = (A \cap \overline{[p]}) \cup \{p^\frown f : f \in P\}$. Then $Q =^\diamondsuit A$ is a Π^0_1 class of rank β. □

Theorem 4.1.5. *Cantor-Bendixson rank is preserved under all automorphisms of* $\mathcal{E}_\Pi$ *or* G.

Proof. We demonstrate that Cantor-Bendixson rank is definable by $\mathcal{L}_{\omega_1\omega}$ formulas in the language of inclusion. We begin with some results on finite cardinality and isolated points. Note that the clopen classes are the same as the complemented classes, and so are definable. The empty set, firstly, is definable as $(\forall Q)[Q \subseteq P \rightarrow Q = P]$. A Π^0_1 class P is a singleton if for every clopen set C, either $C \cap P = P$ or $C \cap P = \emptyset$. Likewise, a point f is isolated in a Π^0_1 class P if there is some clopen C such that $C \cap P = \{f\}$. In that case we say C *isolates* f.

In a finite Π^0_1 class, every path is isolated. Therefore, the class P has cardinality k if there is a collection of k disjoint clopen sets C_i such that each C_i isolates a point of P and $\bigcup_{i=1}^k (C_i) \supseteq P$. The property of being finite is $\mathcal{L}_{\omega_1\omega}$ definable as $\bigvee\!\!\!\bigvee_k[P \text{ is of size } k]$.

Now we turn to Cantor-Bendixson rank. For P to have rank 0 it must be empty or perfect. A Π^0_1 class is perfect if it is nonempty and has no isolated points. Given that definition, one can show that a Π^0_1 class is perfect if its intersection with every clopen set is either infinite or empty. Therefore, the formula

$(\forall \text{ clopen } C)[P \cap C = \emptyset \ \vee \ P \cap C \text{ is infinite}]$ defines the perfect classes. The class P has rank one if it is finite or the union of a perfect class with a finite class.

Suppose we have defined a formula for all ordinals $\beta \leq \alpha$, that says P is of rank at least β. For a successor ordinal $\alpha + 1$, P has rank at least $\alpha + 1$ if it contains an infinite number of disjoint subclasses of rank α. Such P are defined by the $\mathcal{L}_{\omega_1\omega}$ formula $(\exists P_1, P_2, \ldots, P_n, \ldots) \bigwedge\!\!\!\bigwedge_{i,j}[P_i \subseteq P \ \& \ P_i \text{ is of rank at least } \alpha \ \& \ P_i \cap P_j = \emptyset]$. For a limit ordinal α, P has rank at least α if it has rank at least β for all $\beta < \alpha$. P has rank exactly α if it has rank at least α but not at least $\alpha + 1$.

Automorphisms of both $\mathcal{E}_\Pi$ and G must preserve definability by $\mathcal{L}_{\omega_1\omega}$ formulas in the language of inclusion, so they must preserve Cantor-Bendixson rank. □

Corollary 4.1.6. *For any equivalence class $A^\diamondsuit$ in $G^\diamondsuit$ which contains a Π^0_1 class of Cantor-Bendixson rank $< \omega_1^{CK}$, there are classes $A, B \in A^\diamondsuit$ which are not automorphic in G.*

Proof. By Theorems 4.1.3 and 4.1.4, $X = \{CB(P) : P \in A^\diamondsuit\}$ is a subset of the ordinals $\leq \omega_1^{CK}$ which is closed upward. Therefore for any computable ordinal $\alpha \in X$ there is an ordinal $\beta > \alpha$ such that $\beta \in X$ also. Thus as long as there is some element of $A^\diamondsuit$ with computable ordinal rank, there exist elements of $A^\diamondsuit$ with different Cantor-Bendixson rank. Let A, B be two such elements. Since automorphisms of G must preserve Cantor-Bendixson rank, A and B cannot be automorphic in G. □

Corollary 4.1.7. *Any orbit $\mathrm{Orb}(A^\diamondsuit)$ of $G^\diamondsuit$ generated by a Π^0_1 class A such that $CB(A) < \omega_1^{CK}$ corresponds to an invariant class in G which is not an orbit.*

4.2 Creative Sets and Creative Ideals

Corollary 4.1.7 leaves open the possibility of an orbit of Π^0_1 classes which are all of rank ω_1^{CK}. Let $\mathscr{C}$ be the collection of all ideals A which satisfy the following formula, where quantifiers range over G.

$$\begin{aligned}(\exists C \supset A)(\forall B \subseteq C)(\exists R)[&R \text{ complemented } \& \ R \cap B = R \cap A \\ &\& \ (\forall X =^{\diamondsuit} R \cap C)[X \text{ noncomplemented}]]\end{aligned} \tag{4.2.1}$$

The formula (4.2.1) for $\mathscr{C}$ is a direct translation to G of Harrington's definition of creativity in $\mathcal{E}$ (see [14] XV.1.1). André Nies has made the following claim.

Claim 4.2.1 (Nies). *The collection of ideals $\mathscr{C}$ is nonempty and forms an effective orbit in G.*

Let $\mathscr{C}'$ be the collection of ideals satisfying the condition below, obtained by pushing Harrington's definition from $\mathcal{E}$ to G via $\mathcal{E}^*$ and $G^{\diamondsuit}$, and seemingly weaker than Nies' condition (4.2.1).

$$\begin{aligned}(\exists C \supset^{\diamondsuit} A)(\forall B \subseteq^{\diamondsuit} C)(\exists R)[&R \text{ complemented } \& \ R \cap A =^{\diamondsuit} R \cap B \\ &\& \ (\forall X =^{\diamondsuit} R \cap C)[X \text{ noncomplemented}]]\end{aligned} \tag{4.2.2}$$

Proposition 4.2.2. *$\mathscr{C} = \mathscr{C}'$.*

Proof. ($\mathscr{C} \subseteq \mathscr{C}'$): Suppose ideals A and C satisfy Nies' formula (4.2.1). Since $R \cap A = R \cap B \Rightarrow R \cap A =^{\diamondsuit} R \cap B$, it is clear that for any $B \subseteq C$, the bracketed part of (4.2.2) holds. In order to show A and C satisfy (4.2.2) we must show that any ideal $B \subseteq^{\diamondsuit} C$ (with $B \subseteq C$ not necessarily true) has a corresponding R making the bracketed part of (4.2.2) true.

Let the top element of G be M. Since $B \subseteq^{\diamondsuit} C$, there is some $n \in M$ such that $B \vee \langle n \rangle \subseteq C \vee \langle n \rangle$. Consider $B' = B \cap \overline{\langle n \rangle}$. Since $B \cap \overline{C} \subseteq \langle n \rangle$, $B' \subseteq C$, and therefore there is an R which satisfies the bracketed part of Nies' formula (4.2.1); in particular, $R \cap A = R \cap B'$. The same R satisfies the bracketed part of (4.2.2), because $R \cap B =^{\diamondsuit} (R \cap B) \vee \langle n \rangle = (R \cap B') \vee \langle n \rangle =^{\diamondsuit} R \cap A$.

($\mathscr{C}' \subseteq \mathscr{C}$): Now suppose ideal A satisfies the $\mathscr{C}'$ formula (4.2.2). Then there is some witness C to that satisfaction such that for some $m \in M$, $A \vee \langle m \rangle \subseteq C \vee \langle m \rangle$. Let $C' = C \vee \langle m \rangle$; C' will be our witness that A satisfies the $\mathscr{C}$ formula (4.2.1).

To show that, let $B \subseteq C'$. By the definition of C', we know $B \subseteq^{\diamondsuit} C$, so for A, B, C, there is an R such that the bracketed part of (4.2.2) holds. We need R' such that the complementation and noncomplementation properties of R are retained, but $R' \cap A = R' \cap B$. We perform a similar procedure to the one by which we obtained C'. Since $R \cap A =^{\diamondsuit} R \cap B$ by hypothesis, let $n \in M$ be such that $(R \cap A) \vee \langle n \rangle = (R \cap B) \vee \langle n \rangle$. Let $R' = R \cap \overline{\langle n \rangle}$. Then $R' \cap A = R' \cap B$ immediately, and we need only check the complementation properties. If R is complemented by X, then one may check that R' is complemented by $X \vee \langle n \rangle$. For noncomplementation, it is sufficient to show that $R' \cap C' =^{\diamondsuit} R \cap C$, since $=^{\diamondsuit}$ is an equivalence relation. First note that $R' \cap C' = R \cap \overline{\langle n \rangle} \cap C'$, so $\langle n \rangle$ witnesses $R' \cap C' =^{\diamondsuit} R \cap C'$. Next, we expand $R \cap C'$ into $R \cap (C \vee \langle m \rangle) = (R \cap C) \vee (R \cap \langle m \rangle)$. Since $R \cap \langle m \rangle \subseteq \langle m \rangle$, $\langle m \rangle$ witnesses $R \cap C' =^{\diamondsuit} R \cap C$, so by transitivity, $R' \cap C' =^{\diamondsuit} R \cap C$. Therefore, for $A \in \mathscr{C}'$, C' witnesses that A satisfies (4.2.1) and is thus in $\mathscr{C}$. $\square$

Since $\mathscr{C}'$ was pushed up from $G^{\diamondsuit}$ to G, in order for it to form an orbit, all of its members must correspond to Π^0_1 classes of Cantor-Bendixson rank ω_1^{CK}. (From here on we will refer to the rank of an ideal, meaning the rank of its corresponding Π^0_1 class.) Thus, give Claim 4.2.1 and Proposition 4.2.2 together, we obtain the fact that all "creative ideals" must have Cantor-Bendixson rank ω_1^{CK}.

In the c.e. sets, creativity is equivalent to 1-completeness, and thus all creative sets are Turing-complete (the significance of this is discussed in §4.4), but the degree properties of creative *ideals* are unknown. However, we may make a definition of 1-reducibility in the Π^0_1 classes and consider the relationship between 1-reducibility and Cantor-Bendixson rank.

To define 1-reducibility for Π^0_1 classes, we must consider functions from 2^ω into itself. However, we are not equipped to take infinite input and produce infinite output, so we must actually define the function to be from $2^{<\omega}$ to itself.

Definition 4.2.3. *A 1:1 function f from 2^ω into itself is* given by *a total 1:1 order-preserving map Φ from $2^{<\omega}$ into itself if for all finite binary strings σ, τ, the following holds.*

$$\Phi(\sigma) = \tau \iff (\forall X \in 2^\omega)[\sigma \subset X \ \rightarrow \ \tau \subset f(X)].$$

Definition 4.2.4. *Let P, Q be Π^0_1 classes. P is* 1-reducible *to Q, denoted $P \leq_1 Q$, if there is a computable 1:1 function f from 2^ω into itself such that $f(P) \subseteq Q$ and $f(\overline{P}) \subseteq \overline{Q}$. The class Q is* 1-complete *if for all Π^0_1 classes P, $P \leq_1 Q$.*

Proposition 4.2.5. *A computable 1:1 function f from 2^ω into itself induces an endomorphism of $\mathcal{E}_\Pi$ which does not decrease Cantor-Bendixson rank.*

Proof. The induced map f' must automatically respect inclusion, since it is pointwise defined. The underlying map Φ on $2^{<\omega}$ giving f is also computable. It is easily checked that Φ will take a computable tree T to a computable tree S such that $f([T]) = [S]$, so f' takes Π^0_1 classes to Π^0_1 classes. Therefore f is an endomorphism, and so must preserve any property which is existentially $\mathcal{L}_{\omega_1\omega}$ definable. This includes the property of having Cantor-Bendixson rank at least α for some computable ordinal α. Therefore for a Π^0_1 class P, $f'(P)$ will have Cantor-Bendixson rank no less than $CB(P)$. $\square$

Corollary 4.2.6. *For Π^0_1 classes P and Q, $P \leq_1 Q \ \Rightarrow \ CB(P) \leq CB(Q)$, and therefore a 1-complete Π^0_1 class must have Cantor-Bendixson rank ω_1^{CK}.*

Since the creative ideals form an orbit and are thus all of rank ω_1^{CK}, Corollary 4.2.6 supports the following conjecture.

Conjecture 4.2.7. *Creativity is equivalent to 1-completeness in G, as it is in $\mathcal{E}$.*

4.3 Degrees of Ideals and $\diamondsuit$-Equivalence Classes

The correspondence between ideals of $2^{<\omega}$ and ideals of Q preserves degree, so when we show facts about degrees we may use either setting, as convenient. Recall the notation that $\boldsymbol{a}$ is the Turing degree of a.

There is an ideal of every Turing degree. For the set W, let I_W be the ideal of $2^{<\omega}$ generated by the set

$$\{0^{n+1}1 : n \in W\}.$$

I_W is clearly computable from W, and from I_W we can compute its root set, which gives W. Thus $I_W \equiv_T W$.

Theorem 4.3.1. *The set* $\{\boldsymbol{A} : A \in A^{\diamondsuit}\}$ *is closed upward in the c.e. degrees.*

Proof. Without loss of generality, we work in $2^{<\omega}$ with $M = 2^{<\omega} - \{0^n\}_{n\in\omega}$. Given $A \subseteq M$ and a c.e. degree $\boldsymbol{B} > \boldsymbol{A}$, choose a representative ideal B of degree $\boldsymbol{B}$. Let $\widetilde{A}$ be A with the interval $[01]$ replaced by a copy of B; that is, let $01^\frown\tau \in \widetilde{A}$ iff $\tau \in B$. It is clear that $\widetilde{A} =^{\diamondsuit} A$ and $\widetilde{\boldsymbol{A}} \geq \boldsymbol{B}$. In fact, since B computes both A and B, $\widetilde{\boldsymbol{A}} = \boldsymbol{B}$. □

Definition 4.3.2. $\boldsymbol{A}^{\diamondsuit} = \min\{\boldsymbol{A} : A \in A^{\diamondsuit}\}$, *if this minimum exists.*

This prompts the question of whether degree is a well-defined concept in $G^{\diamondsuit}$, or if there are equivalence classes for which the minimum does not exist. Definition 4.3.2 is analogous to the definition of degree of an isomorphism class of models in computable model theory (for a survey, see Knight [11]), and there are certainly isomorphism classes of models without degree (Richter [13]). We hope to use the techniques from computable model theory to prove the following conjecture.

Conjecture 4.3.3. *There exists an element of* $G^{\diamondsuit}$ *with no degree.*

The *jump degree* of an equivalence class $A^\diamondsuit \in G^\diamondsuit$ is the minimum degree in $\{\boldsymbol{A'} : A \in A^\diamondsuit\}$. An affirmative answer to Conjecture 4.3.3 would leave open the following question.

Question 4.3.4. *Is jump degree a well-defined concept in $G^\diamondsuit$?*

4.4 Degree Invariant Classes and Translation

We turn now to degree-theoretic concerns of G and individual ideals in $G^\diamondsuit$ (as opposed to equivalence classes). For our purposes there are two kinds of invariance, *set invariance* and *degree invariance*, the latter defined below. Set invariance is the only kind of invariance we have been discussing thus far, where a collection of c.e. sets is invariant if it is closed under automorphisms of $\mathcal{E}$. We will also use the term "set invariance" for collections of ideals closed under automorphisms of G or $I(Q)$.

Definition 4.4.1. *A collection of degrees $\mathcal{C}$ is* invariant *in $\mathcal{E}$ if there is a collection of sets $\mathcal{S}$ such that*

(i) *For every degree $\boldsymbol{d} \in \mathcal{C}$, there is a set $X \in \mathcal{S}$ of degree $\boldsymbol{d}$,*

(ii) *If $X \in \mathcal{S}$ has degree $\boldsymbol{d}$, then $\boldsymbol{d} \in \mathcal{C}$, and*

(iii) *$\mathcal{S}$ is closed under automorphisms of $\mathcal{E}$ ($\mathcal{S}$ is set invariant).*

To define degree invariance in G, everywhere in the definition above replace $\mathcal{E}$ with G and "set" with "ideal."

The isomorphism between $\mathcal{E}^*$ and $G^\diamondsuit$ gives a translation of $\mathcal{S}$ which is invariant in $G^\diamondsuit$, but there is no guarantee the translation corresponds to $\mathcal{C}$. That is, set invariance is retained in moving from $\mathcal{E}^*$ to $G^\diamondsuit$, but degree invariance may not be. Even if Conjecture 4.3.3 is false and every equivalence class $A^\diamondsuit$ has degree, the isomorphism construction does not ensure that a c.e. set W has the same degree as the equivalence class of W's image.

If we could prove degree invariance results in $G^{\diamondsuit}$ we would have results for G. A set invariant class in $G^{\diamondsuit}$ may break up into many orbits in G, but since invariance is all that is needed, degree invariance in $G^{\diamondsuit}$ pushes to G. However, $\diamondsuit$-equivalence classes are difficult to work with, and degree is not necessarily even well-defined in $G^{\diamondsuit}$, so it appears unproductive to attempt to prove degree invariance results in $G^{\diamondsuit}$.

The remaining tactic is to work directly in G. One approach is to begin with a degree-invariant class in $\mathcal{E}$ where the corresponding class of sets $\mathcal{S}$ is neatly definable, such as the non-low$_2$ degrees and the atomless sets. Using concepts from $G^{\diamondsuit}$, we may translate the definition of $\mathcal{S}$ to G and attempt to re-prove the appropriate theorems.

To expand on that example, let $\mathcal{C} = \{\boldsymbol{d} : \boldsymbol{d}'' \neq \mathbf{0}''\}$, the collection of non-low$_2$ Turing degrees. The invariance of $\mathcal{C}$ may be shown using the collection of atomless sets, $\mathcal{S} = \{A : A \subset_\infty \omega \ \ \& \ \ (\forall B)[B \text{ maximal} \rightarrow A \not\subset^* B]\}$. The translation of $\mathcal{S}$ to $G^{\diamondsuit}$ is $\{A : A \subset^{\diamondsuit} M \ \ \& \ \ (\forall B)[B \text{ maximal} \rightarrow A \not\subset^{\diamondsuit} B]\}$, where B is maximal in $G^{\diamondsuit}$ if $X \supseteq^{\diamondsuit} B$ implies $X =^{\diamondsuit} B$ or $X =^{\diamondsuit} M$. The following two statements or their equivalents must be proved in $G^{\diamondsuit}$ (in $\mathcal{E}$ they are due to Shoenfield and Lachlan, respectively; see [14] XI.4.1, 5.1):

1. For every non-low$_2$ c.e. degree $\boldsymbol{d}$, there is a non-coprincipal c.e. ideal A of degree $\boldsymbol{d}$ such that A is not $\diamondsuit$-contained in any maximal ideal.

2. If A is a non-coprincipal c.e. ideal which is low$_2$, then A is $\diamondsuit$-contained in a maximal ideal.

In the above approach, each degree invariant class must be translated individually. However, Cholak and Harrington [8] have proved a more general result.

Theorem 4.4.2 ([8] 8.5). *Let* $\mathcal{C} = \{\boldsymbol{a} : \boldsymbol{a}$ *is the Turing degree of a* Σ^0_3 *set* $J \geq_T \mathbf{0}''\}$*. Let* $\mathcal{D} \subseteq \mathcal{C}$ *such that* $\mathcal{D}$ *is upward closed in* $\mathcal{C}$*. Then there is an* $\mathcal{L}(A)$ *property* $\varphi_D(A)$ *such that*

$$(\forall \text{ c.e. } F)[\boldsymbol{F}'' \in \mathcal{D} \Leftrightarrow (\exists A)[\varphi_D(A) \text{ and } A \equiv_T F]].$$

As a corollary this shows degree invariance of the non-low$_n$ and high$_n$ degrees for $n \geq 2$. We have the following conjecture.

Conjecture 4.4.3. *The non-low*$_n$ *and high*$_n$ *classes of degrees,* $n \geq 2$*, may be shown invariant in* G *by use of the proper translation of Theorem 4.4.2.*

A translation of the double jump definability result would leave very few open questions about degree invariance, namely the following.

Question 4.4.4. *Are the high degrees invariant in* G*?*

The high degrees were shown to correspond to the maximal sets by Martin ([14] XI.1.5, 2.3). Maximality is definable, so the maximal sets form an invariant class, and thus the high degrees are invariant in $\mathcal{E}$.

Question 4.4.5. *Is Turing-completeness invariant in* G*?*

In $\mathcal{E}$, the creative sets are the 1-complete sets, and 1-completeness implies Turing-completeness. Harrington's lattice-theoretic definition of the creative sets ([14] XV.1.1) shows that the creative sets are an invariant class, and so form an orbit of Turing-complete sets, answering the $\mathcal{E}$ version of Question 4.4.5 affirmatively.

As discussed in the end of §4.1, André Nies has a translation to G of Harrington's definition of creativity. The ideals satisfying that translation also form an invariant class. We do not know if they are 1-complete, although we conjecture they are (Conjecture 4.2.7). As in $\mathcal{E}$, 1-completeness implies Turing-completeness, so if Conjecture 4.2.7 holds, the answer to Question 4.4.5 is "yes".

4.5 G and Π^0_1 Classes

After we work out the open questions of the $\mathcal{E}^*$ to $\mathcal{E}_\Pi$ translation, the next question to ask is if we can work in the other direction. In particular, the *array noncomputable degrees*, introduced by Downey, Jockusch, and Stob [9], are an invariant class in $\mathcal{E}_\Pi$, as shown in Cholak, Coles, Downey, and Herrmann [7]. Is there a "reverse translation" by which we may show they are invariant in $\mathcal{E}$? In $\mathcal{E}_\Pi$, the invariance is shown via perfect thin classes, and in fact many interesting results in $\mathcal{E}_\Pi$ revolve around thin and minimal classes.

Definition 4.5.1. *An infinite* Π^0_1 *class* P *is* thin *if every subclass of* P *is relatively clopen in* P*. That is, for any* $Q \subseteq P$*, there is some principal* Π^0_1 *class* C *such that* $Q = P \cap C$*.*

Definition 4.5.2. *An infinite* Π^0_1 *class* P *is* minimal *if every subclass of* P *is finite or cofinite in* P*.*

A minimal class may be visualized as a tree with exactly one non-isolated path, off of which an infinite number of isolated paths branch. Notice that minimal classes are also thin, so results proved assuming thinness hold for minimal classes. The following proposition says that every thin member of G is trivial.

Proposition 4.5.3. *Suppose* $G = [N, 2^\omega]$ *contains a thin* Π^0_1 *class* P*. Then* $P =^\diamond N$*.*

Proof. Since $P \in G$, $N \subseteq P$. Therefore N is relatively clopen in P; that is, there is some principal C such that $N = P \cap C$. But then $P = (P \cap C) \cup (P \cap \overline{C}) = N \cup (P \cap \overline{C})$. The complement of C is also principal, so $N \cup (P \cap \overline{C}) =^\diamond N$ and $P =^\diamond N$. $\square$

A *perfect thin* class is a thin class where every extendible node has at least two infinite paths through it. A perfect tree may be visualized as the complete tree, $2^{<\omega}$, after it has been "stretched out" to possibly add more nodes in between branching points. By Proposition 4.5.3, the perfect thin classes are at best trivial members of any copy of G. In fact they cannot be members of any $G \subset \mathcal{E}_\Pi$ with singleton least element, because in a thin class a computable path must be isolated, and there are no isolated paths in a perfect class. The results we use to move from G to $\mathcal{E}_\Pi$ consider only copies of G which are maximal (with singleton least element, from the $\mathcal{E}_\Pi$ perspective), and so new techniques will have to be developed to prove degree invariance of the array non-computable degrees in $\mathcal{E}$.

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