

On Ramsey's Theorem for Pairs

Peter A. Cholak, Carl G. Jockusch Jr., and Theodore A. Slaman

On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1-55, 2001.

www.nd.edu/~cholak

Ramsey's theorem

- $[X]^n = \{Y \subseteq X : |Y| = n\}$.
- A k -coloring C of $[X]^n$ is a function from $[X]^n$ into a set of size k .
- H is *homogeneous* for C if C is constant on $[H]^n$, i.e. all n -element subsets of H are assigned the same color by C .
- RT_k^n is the statement that every k -coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

Goals

- (Computability theory) Study the complexity (in terms of the arithmetical hierarchy or degrees) of infinite homogeneous sets for a coloring C relative to that of C . (For simplicity, assume that C is computable (recursive) and relativize.)
- (Reverse mathematics) Study the proof-theoretic strength of Ramsey's theorem (and its natural special cases) as a formal statement in second order arithmetic.

Language

Use the sorted language with the symbols: $=, \in, +, \times, 0, 1, <$;
Number variables: $n, m, x, y, z \dots$; Set Variables: $X, Y, Z \dots$

p is prime.

This just uses bounded quantification.

$$\forall \delta \exists \epsilon [|x - c| < \epsilon \Rightarrow |x^2 - c^2| < \delta]$$

This is an example of a Π_2^0 formula. The negation is Σ_2^0 . A formula which is logically equivalent (over our base theory) to both a Π_n^0 formula and a Σ_n^0 formula is Δ_n^0 .

Induction and Comprehension

$I\Sigma_n$ is the following statement: For every $\varphi(x)$, a Σ_n^0 formula, if $\varphi(0)$ and $\forall x[\varphi(x) \Rightarrow \varphi(x + 1)]$ then $\forall x[\varphi(x)]$.

Over our base theory, $I\Sigma_n^0$ and $II\Pi_n^0$ are equivalent. $I\Sigma_n$ is also equivalent to every Π_n^0 -definable set (Σ_n^0 set) has a least element.

Δ_1^0 comprehension is the statement: For every $\varphi(x)$, a Δ_1^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

For example, Δ_1^0 comprehension implies the set of all primes exists.

Arithmetic comprehension is the statement: For every $\varphi(x)$, a Δ_n^0 formula, there is an X such that $X = \{x : \varphi(x)\}$.

2nd-order arithmetic

We work over models of 2nd-order arithmetic.

The intended model: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), +, \times, 0, 1, <)$.

P^- is the theory of finite sets. The base theory, RCA_0 , is the logical closure of $P^- + I\Sigma_1^0$ and Δ_1^0 comprehension.

$$(\mathbb{N}, \{\text{all computable sets}\}, +, \times, 0, 1, <) \models RCA_0.$$

ACA_0 is RCA_0 plus arithmetic comprehension.

$$(\mathbb{N}, \{\text{all arithmetic sets}\}, +, \times, 0, 1, <) \models ACA_0.$$

Statements in 2nd-order arithmetic

Statement (WKL)

Every infinite tree of binary strings has an infinite branch.

$\forall T \exists P$ [if T is an infinite binary branching tree then P is an infinite path through T].

Statement (RT_k^n)

For every infinite set X and for every k -coloring of $[X]^n$ there is an infinite homogeneous set H .

These are Π_2^1 sentences: look at the set quantifiers ignore the (inside) number quantifiers.

Conservation

Definition

If T_1 and T_2 are theories and Γ is a set of sentences then T_2 is Γ -conservative over T_1 if $\forall \varphi [(\varphi \in \Gamma \wedge T_2 \vdash \varphi) \Rightarrow T_1 \vdash \varphi]$.

Theorem

ACA₀ is arithmetically conservative over PA.

Computability Theory

$A \leq_T B$ iff there is a computer which using an oracle for B can compute A . For all A , $\emptyset \leq_T A$.

A' (read A -jump) is all those programs e which using A as an oracle halt on input e . $A^{(n)}$ is the n^{th} jump of A . The jump operation is order preserving.

So for all A , $\emptyset^{(n)} \leq_T A^{(n)}$. A is low_n iff $A^{(n)} \leq_T \emptyset^{(n)}$.

Key idea: if A is low_n then sets which are Δ_{n+1}^0 in A are Δ_{n+1}^0 in \emptyset .

WKL

Theorem (Jockusch and Soare)

[The Low Basis Theorem] Every infinite computable tree of binary strings has a low path (working in the standard model).

Theorem (Harrington)

$RCA_0 + WKL$ is Π_1^1 -conservative over RCA_0 .

Ramsey's Theorem – Known Results

Theorem (Specker)

There is a computable 2-coloring of $[\mathbb{N}]^2$ with no infinite computable homogeneous set.

Corollary (Specker)

RT_2^2 is not provable in RCA_0 .

Theorem

1. (Jockusch) For each $n \geq 2$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that $0^{(n-2)} \leq_T A$ for each infinite homogeneous set A .
2. (Simpson) For each $n \geq 3$ and $k \geq 2$ (both n and k standard), the statements RT_k^n are equivalent to ACA_0 over RCA_0 .

Theorem (Seetapun)

For any computable 2-coloring C of $[\mathbb{N}]^2$ and any noncomputable sets C_0, C_1, \dots , there is an infinite homogeneous set X such that $(\forall i)[C_i \not\leq_T X]$.

Corollary (Seetapun)

RT_2^2 does not imply ACA_0 . Hence, over RCA_0 , RT_2^2 is strictly weaker than RT_2^3 .

Theorem (Hirst)

RT_2^2 is not Σ_3^0 -conservative over RCA_0 . RT_2^2 is stronger than RCA_0 . RT_2^2 proves $B\Sigma_2$.

$B\Sigma_2$ is strictly between $I\Sigma_1$ and $I\Sigma_2$.

Our Work – Computability Theory

Theorem

For any computable 2-coloring of $[\mathbb{N}]^2$, there is an infinite homogeneous set X which is low_2 , i.e. $X'' \leq_T 0''$.

Our work – Reverse mathematics

Theorem

$RCA_0 + RT_2^2$ is Π_1^1 -conservative over $RCA_0 + I\Sigma_2$.

Corollary

RT_2^2 does not imply PA over RCA_0 .

This improves Seetapun's result that RT_2^2 does not imply ACA_0 over RCA_0 .

Question

Is $RT_2^2 \Pi_2^0$ -conservative over RCA_0 ? In particular, does RT_2^2 prove the consistency of $P^- + I\Sigma_1$? Does RT_2^2 prove that Ackerman's function is total?