# The Computably Enumerable Sets

## Open Questions

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# Main Goals

To understand orbits within the structure of c.e. (r.e.) sets under inclusion.

What are the good open interesting questions?

The story is rather complex.

# The Computably Enumerable Sets, $\mathcal{E}$

- $W_e$ is the domain of the $e$th Turing machine.
- $(\{W_e : e \in \omega\}, \subseteq)$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- These sets are the same as the $\Sigma^0_1$ sets, $\{x : (\mathbb{N}, +, \times, 0, 1) \vDash \varphi(x)\}$, where $\varphi$ is $\Sigma^0_1$.
- $W_{e,s}$ is the domain of the $e$th Turing machine at stage $s$.
- **Dynamic Properties**: How slow or fast can a c.e. set be enumerated with respect to another c.e. set? w.r.t. the standard enumeration of all c.e. sets?
- $0, 1, \cup$, and $\cap$ are definable from $\subseteq$ in $\mathcal{E}$.
- For safety all sets are c.e., infinite, and coinfinite unless otherwise noted.

# Automorphisms of $\mathcal{E}$

- $\Phi \in \text{Aut}(\mathcal{E})$ iff $\Phi(W)$ is a one to one, onto map taking c.e. sets to c.e. sets such that $W_e \subseteq W_{e'}$ iff $\Phi(W_e) \subseteq \Phi(W_{e'})$ and dually.
- For example, if $p$ is a computable permutation of $\mathbf{N}$ then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- If $p$ is a permutation of $\mathbf{N}$ such that $p(W)$ is r.e. and dually then $\Phi(W) = p(W)$ is an automorphism of $\mathcal{E}$.
- $\Phi \in \text{Aut}(\mathcal{E})$ iff there is a permutation, $p$, of $\mathbf{N}$ such that $\Phi(W_e) = \widehat{W}_{p(e)}$ and $W_e \subseteq W_{e'}$ iff $\widehat{W}_{p(e)} \subseteq \widehat{W}_{p(e')}$ and dually.
- $A$ is *automorphic* to $\hat{A}$ iff there is a $\Phi \in \text{Aut}(\mathcal{E})$ such that $\Phi(A) = \hat{A}$.

# Core Themes

- **Automorphisms** and **Orbits** of $\mathcal{E}$.
- **Definability**: If two sets are not in the same orbit there is some definable difference. So definability within a fixed structure; within arithmetic; (principal) types; elementary definability; within $\mathcal{L}_{\omega_1,\omega}$, within $\mathcal{L}_{\omega_1,\omega_1}$.
- **Dynamic Properties.**
- **Turing Complexity.**

# Main Open Questions

## Question (Completeness)

*Which c.e. sets are automorphic to complete sets?*

## Question (Cone Avoidance)

*Given an incomplete c.e. degree* $\mathbf{d}$ *and an incomplete c.e. set* $A$, *is there an* $\hat{A}$ *automorphic to* $A$ *such that* $\mathbf{d} \not\leq_T \hat{A}$?

# Almost Prompt Sets

## Definition

Let $X_e^n$ be the $e$th $n$-r.e. set. $A$ is *almost prompt* iff there is a computable nondecreasing function $p(s)$ such that for all $e$ and $n$ if $X_e^n = \overline{A}$ then $(\exists x)(\exists s)[x \in X_{e,s}^n$ and $x \in A_{p(s)}]$.

## Theorem (Maass, Shore, Stob)

*There is a definable property $P(A)$ which implies $A$ is prompt and furthermore for all prompt degree, $\mathbf{d}$, there is set $A$ such that $P(A)$ and $A \in \mathbf{d}$.*

## Theorem (Harrington and Soare)

*All almost prompt sets are automorphic to a complete set.*

# Tardy Sets

## Definition

$D$ is 2-*tardy* iff for every computable nondecreasing function $p(s)$ there is an $e$ such that $X_e^2 = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^2 \text{ then } x \notin D_{p(s)}]$.

## Definition

$D$ is *codable* iff for all $A$ there is an $\widehat{A}$ in the orbit of $A$ such that $D \leq_T \widehat{A}$.

## Theorem (Harrington and Soare)

*There are $\mathcal{E}$ definable properties $Q(D)$ and $P(D, C)$ such that*

- *$Q(D)$ implies that $D$ is 2-tardy,*
- *if there is an $C$ such that $P(D, C)$ and $D$ is 2-tardy then $Q(D)$ (and $D$ is high),*
- *$X$ is codable iff there is a $D$ such that $X \leq_T D$ and $Q(D)$.*

# Questions about Tardiness

## Question

*How do the following sets of degrees compare:*

- *the tardy degrees,*
- *for each $n$, $\{\mathbf{d} :$ there is a $n$-tardy $D$ such that $\mathbf{d} \leq_T D\}$,*
- *$\{\mathbf{d} :$ there is a 2-tardy $D$ such that $Q(D)$ and $\mathbf{d} \leq_T D\}$,*
- *$\{\mathbf{d} :$ there is a $A \in \mathbf{d}$ which is not automorphic to a complete set$\}$,*
- *the hemimaximal ($d$-simple) degrees.*

## Theorem (Harrington and Soare)

*There is a maximal 2-tardy set.*

## Question

*Is there a nonhigh (nonhemimaximal) 2-tardy set which is automorphic to a complete set?*

# Invariant Classes

## Definition

A class $\mathcal{D}$ of degrees is *invariant* if there is a class $\mathcal{S}$ of (c.e.) sets such that

1. $\mathbf{d} \in \mathcal{D}$ implies there is a $W$ in $\mathcal{S}$ and $\mathbf{d}$.
2. $W \in \mathcal{S}$ implies $\deg(W) \in \mathcal{D}$ and
3. $\mathcal{S}$ is closed under automorphic images (but need not be one orbit).

## Corollary

*The high degrees are invariant by a single orbit.*

## Corollary

*The prompt degrees are invariant by a single orbit.*

# More Invariant degree classes

## Theorem (Shoenfield 76)

*The nonlow$_2$ degrees are invariant.*

## Theorem (Harrington and Soare)

- *The nonlow degrees are not invariant.*
- *There is properly low$_2$ degree $\mathbf{d}$ such that if $A \in \mathbf{d}$ then $A$ is automorphic to a low set.*
- *There is a low$_2$ set which is not automorphic to a low set.*

## Theorem (Cholak and Harrington 02)

*For $n \geq 2$, nonlow$_n$ and high$_n$ degrees are invariant.*

# More Single Orbit Invariant Classes?

## Theorem (Downey and Harrington – No fat orbit)

*There is a property $S(A)$, a prompt low degree $\mathbf{d_1}$, a prompt high$_2$ degree $\mathbf{d_2}$ greater than $\mathbf{d_1}$, and tardy high$_2$ degree $\mathbf{e}$ such that for all $E \leq_T \mathbf{e}$, $\neg S(E)$ and if $\mathbf{d_1} \leq_T D \leq_T \mathbf{d_2}$ then $S(D)$.*

# Orbits Containing Prompt (Tardy) High Sets

## Question

*Let $A$ be incomplete. If the orbit of $A$ contains a set of high prompt degree must the orbit of $A$ contain a set from all high prompt degrees?*

## Question

*If the orbit of $A$ contains a set of high tardy degree must the orbit of $A$ contain a set from all high tardy degrees?*

A positive answer to both questions would answer the cone avoidance question. But not the completeness question.

## Question

*For every degree $\mathbf{a}$ is there a set $A \in \mathbf{a}$ whose orbit contains every high degree?*

# On the Complexity of the Orbits

Look at the index set of all $\widehat{A}$ in the orbit of $A$ with the hopes of finding some answers. The index set of such $\widehat{A}$ is in $\Sigma^1_1$.

## Theorem (Cholak and Harrington)

*If $A$ is hhsimple then $\{e : W_e$ is automorphic to $A\}$ is $\Sigma^0_5$.*

## Theorem (Cholak and Harrington)

*If $A$ is simple then $\{e : W_e$ is automorphic to $A\}$ is $\Sigma^0_8$.*

## Theorem

*If $A$ and $\widehat{A}$ are promptly simple and automorphic then they are $\Delta^0_3$ automorphic.*

## Question

*Is every low$_2$ simple set automorphic to a complete set? At least this is an arithmetic question.*

# $\Sigma^1_1$-completeness

## Theorem (Cholak, Downey, and Harrington)

*Let $T_i$ be a computable listing of all computable infinitely branching trees. For each $i$ there is an $A_{T_i}$ such that, for all $j, k$, the orbits of $A_{T_j}$ and $A_{T_k}$ are the same iff $T_j$ and $T_k$ are isomorphic.*

## Corollary

*There is an $A$ such that whether $\hat{A}$ is automorphic to $A$ is a $\Sigma^1_1$-question.*

$T$ is an invariant which determines the orbit of $A_T$. Only works for some $A$; those in the orbit of some $A_T$.

## Question

*Can we find an "invariant" (in terms of trees?) which works for all $A$?*

# Main Open Questions, Again

## Question (Completeness)

*Which c.e. sets are automorphic to complete sets?*

## Question (Cone Avoidance)

*Given an incomplete c.e. degree* $\mathbf{d}$ *and an incomplete c.e. set* $A$*, is there an* $\hat{A}$ *automorphic to* $A$ *such that* $\mathbf{d} \not\leq_T \hat{A}$*?*

## Question

*Are these arithmetical questions?*

# $d$-Simple Sets

## Definition (Lerman and Soare)

A coinfinite set $A$ is *$d$-simple* if for all $X$ there is a $Y \subseteq X$ such that

1. $X \cap \overline{A} = Y \cap \overline{A}$ and
2. $(\forall Z)[Z - X \text{ infinite } \rightarrow (Z - Y) \cap A \neq \emptyset]$.

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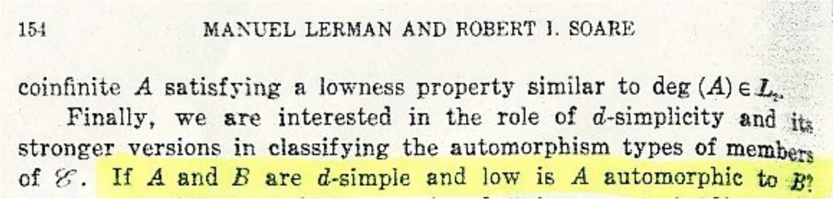
d-SIMPLE SETS, SMALL SETS, AND DEGREE CLASSES

MANUEL LERMAN AND ROBERT I. SOARE

# Questions about $d$-simple sets

## Question



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coinfinite $A$ satisfying a lowness property similar to deg $(A) \in L_2$.
Finally, we are interested in the role of $d$-simplicity and its stronger versions in classifying the automorphism types of members of $\mathcal{E}$. If $A$ and $B$ are $d$-simple and low is $A$ automorphic to $B$?

## Question

*Are all prompt (low$_2$) $d$-simple sets automorphic?*

## Question (Harrington and Soare)

*Is every $d$-simple (low$_2$) set automorphic to a complete set?*