THE COMPUTABLY ENUMERABLE SETS: RECENT RESULTS AND FUTURE DIRECTIONS

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ABSTRACT. We survey some of the recent results on the structure of the computably enumerable (c.e.) sets under inclusion. Our main interest is on collections of c.e. sets which are closed under automorphic images, such as the orbit of a c.e. set, and their (Turing) degree theoretic and dynamic properties. We take an algebraic viewpoint rather than the traditional dynamic viewpoint.

1. INTRODUCTION

One of the core themes in mathematical logic is the interplay between *definability* and *computability*[1]. We are going to focus on this interplay within the structure of the computably enumerable (c.e.) sets under inclusion. This structure is denoted $\mathcal{E}$. $\mathcal{E}$ is a natural structure as, for example, one might think of a c.e. set as the solution set to a diophantine equation or the theorems generated from a finite set of axioms. As auxiliary structures we will consider true arithmetic, $(\mathbb{N}, +, \times, 0, 1)$, and the c.e. Turing degrees.

Our primary interest is on collections of c.e. sets which are closed under automorphic images, such as the orbit of a c.e. set. When are these collections definable in $\mathcal{E}$? by an elementary formula? The idea that one can learn about a structure by studying the properties invariant under automorphisms goes back to 1872 and Klein's Erlangen Program.

We can also ask computability theoretic questions. How are these collections related degree theoretically? How are they related in terms

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[1] As suggested by Soare [23] we use "computability theory" rather than "recursion theory", "computable" rather than "recursive" and "computably enumerable" rather than "recursively enumerable".

of the jump? What is their complexity in the arithmetic and hyperarithmetic hierarchy as an index set? We consider these questions as a computability theoretic subtheme of our work.

Another subtheme to our work will be that of dynamic properties. A c.e. set is enumerated at some rate. How does this enumeration compare with the other enumerations? with the standard enumeration of all c.e. sets? As we will see this issue is interrelated with the above questions.

Our notation will follow Soare [22] unless otherwise noted. More or less this is a survey paper. But we make no claims that it is a complete picture of all the recent work in this area. For that one would have to find a paper or book which combines parts of Cholak [4], Soare [24], Cholak and Harrington [7] plus this paper. Unless otherwise noted the theorems and their proofs appear in one of Cholak and Harrington [10], Cholak and Harrington [9] or a paper with Harrington too young to have a working title.

2. The Structures $\mathcal{E}$ and $\mathcal{E}^*$

W_e is the domain of the eth Turing machine. Hence $\mathcal{E} = (\{W_e : e \in \omega\}, \subseteq)$. Since $0, 1, \cup$, and $\cap$ are definable from $\subseteq$ in $\mathcal{E}$, one might consider $\mathcal{E}$ as a lattice. $\mathcal{F}$ is the filter of finite sets (within $\mathcal{E}$). $\subseteq^*$ is inclusion modulo finite difference. $\mathcal{E}^*$ is $\mathcal{E}$ modulo $\mathcal{F}$. X^* is the equivalence class of X in $\mathcal{E}^*$.

We will consider the two structures $\mathcal{E}$ and $\mathcal{E}^*$ as interchangeable. This can be justified. A nontrivial orbit in $\mathcal{E}$ gives rise to an nontrivial orbit in $\mathcal{E}^*$. The following theorem shows the converse is true and hence when considered as collections of c.e. sets these structures have the same orbits.

Theorem 1 (Soare [21]). *If $\Phi \in Aut(\mathcal{E}^*)$, $\Phi(A^*) = \widehat{A}^*$ and A is infinite and coinfinite then there is a $\Psi \in Aut(\mathcal{E})$ such that $\Psi(A) = \widehat{A}$.*

Recall X is computable iff X and $\overline{X}$ are both c.e. sets. A set X is computable iff X and $\overline{X}$ are c.e. iff there is c.e. set Y such that $X \cup Y = \omega$ and $X \cap Y = \emptyset$. So being computable or not is definable in $\mathcal{E}$.

Being finite is definable in $\mathcal{E}$. A set X is finite iff every subset of X is computable. Therefore a definable collection of c.e. sets in $\mathcal{E}^*$ is a definable collection of c.e. sets in $\mathcal{E}$. If we have a definable collection of c.e. sets in $\mathcal{E}$, we can closed that collection under finite difference and still have a definable collection in $\mathcal{E}$ which is also definable in $\mathcal{E}^*$. So when considered as collections of c.e. sets closed under finite difference $\mathcal{E}$ and $\mathcal{E}^*$ have the same definable collections.

3. The Creative Sets

The simplest automorphism of $\mathcal{E}$ is a computable permutation of ω, a 1-reduction. The creative sets are 1-complete and hence the collection of creative sets is closed under computable permutations. Is this collection also closed under automorphic images? Surprisingly, in the early 80's, Harrington showed the answer was yes; the property of being creative is definable in $\mathcal{E}$.

Theorem 2 (Harrington, see Soare [22] Theorem XV.1.1)**.** *A computably enumerable set A is creative iff $(\exists C \supset A)(\forall B \subseteq C)(\exists R)[R$ is computable and $R \cap C$ is not computable and $R \cap A = R \cap B]$.*

Corollary 3. *The creative sets form an orbit.*

Based on these results one might ask "is it enough just to consider just computable permutations when dealing with collections of c.e. sets which are closed under automorphic images? In the early 80's, in unpublished work, Harrington showed the answer is no.

Theorem 4 (Harrington, Unpublished)**.** *The creative sets are the only orbit of $\mathcal{E}$ which remains an orbit when we restrict the allowable automorphisms to computable permutations.*

4. Automorphisms of $\mathcal{E}^*$

Hence we need to further explore the automorphisms of $\mathcal{E}^*$. First we should note that there are many automorphisms of $\mathcal{E}^*$.

Theorem 5 (Lachlan, see Soare [22] Theorem XV.2.2)**.** *There are $2^{\aleph_0}$ automorphisms of $\mathcal{E}^*$.*

However this proof uses a cohesiveness argument which does not provide "useful" automorphisms. So we must find other ways of constructing automorphisms.

Given the sucess of the last section, one might ask can we continue using permutations as automorpisms?

Theorem 6 (Soare [21])**.** *Every automorphism Φ of $\mathcal{E}^*$ is induced by a permutation p of ω, i.e., $\Phi(W_e) =^* p(W_e)$.*

But the converse is not true. There is a permutation which takes the even numbers to K, the halting set. Hence this permutation takes a computable set to a non-computable set and cannot induce an automorphism of $\mathcal{E}^*$. So it is better to think of $\Phi(W_e) =^* W_{g(e)}$, for some function g. We say Φ is Δ^0_n iff g is Δ^0_n.

To make life notationally easier lets consider two copies of $\mathcal{E}$. One living in ω, the standard copy, and living in a hatted copy of ω, $\widehat{\omega}$. In the hatted copy, everything wears a hat.

So to build an automorphism of $\mathcal{E}^*$ it is enough to build g and h such that $\Phi(W_e) =^* \widehat{W}_{g(e)}$ and $\Phi^{-1}(\widehat{W}_e) = W_{h(e)}$. Lets try to construct g and h via a back and forth argument similar to how we might show all countable dense linear orders are isomorphic.

We start with W_0 and find a c.e. image, $\widehat{W}_{g(0)}$. We must ensure W_0 is finite iff $\widehat{W}_{g(0)}$ is finite, W_0 is cofinite iff $\widehat{W}_{g(0)}$ is cofinite, and W_0 is infinite iff $\widehat{W}_{g(0)}$ is infinite. Now given $\widehat{W}_0$ we will find a c.e. preimage, $W_{h(0)}$. But now we must find $W_{h(0)}$ such that $\widehat{W}_{g(0)} \cap \widehat{W}_0$ is infinite (finite or cofinite) iff $W_0 \cap W_{h(0)}$ is infinite (finite or cofinite) and similarly for $\widehat{W}_{g(0)} - \widehat{W}_0$ and $W_0 - W_{h(0)}$, $\widehat{W}_0 - \widehat{W}_{g(0)}$ and $W_{h(0)} - W_0$, and $\widehat{\omega} - (\widehat{W}_{g(0)} \cup \widehat{W}_0)$ and $\omega - (W_0 \cup W_{h(0)})$.

We can try and continue but things will get out of hand quickly. We need to know if a difference of c.e. (d.c.e.) sets is infinite (finite or cofinite). In general, this is a Π^0_3-complete question. As we add more sets, the number of d.c.e. sets we must consider grows exponentially. Furthermore we must somehow find whether these sets are infinite in some fashion which allows us to build all *computably enumerable* images and preimages.

The solution is to modify our back and forth argument and do it using a tree construction. The tree provides a good framework to organize all information needed. In computability theory, trees are normally Π^0_2 branching. That is, the true path is computable in $\mathbf{0}''$. In addition, we are allowed finite injury along the true path. This corresponds to asking Σ^0_3 questions.

A d.c.e. set is of the form $X - Y$ where X and Y are c.e. Computably enumerable sets come with an enumeration. We can use these enumerations to stagewise approximate $X - Y$. So $(X-Y)_s = X_s - Y_s$. Let $U = \cup_s (X_s - Y_s)$. The set U is c.e. and $X - Y \subseteq U$. It is Π^0_2 to see if U is infinite. If U is infinite we say that $X - Y$ is *well-visited*. If $X - Y$ is well-visited then infinitely many balls look like they will be in $X - Y$. Just because $X - Y$ is well-visited does not mean $X - Y$ is infinite. The set $X - Y$ is *well-resided* if infinitely many balls in $X_s - Y_s$ remain in $X_t - Y_t$ for all later stages t. We can assume $X - Y$ is infinite unless $X - Y$ is *not well-resided*. The set $X - Y$ is not well-resided if for almost all x and s there is a stage t such that if $x \in (X_s - Y_s)$ then $x \notin (X_t - Y_t)$. So whether a d.c.e. set is not well-resided is a Σ^0_3 question. Hence it is possible to use a tree to determine if a d.c.e. set is well-visited and/or not well-resided and from this information, using a tree, construct an automorphism.

As a result most previous constructions of automorphisms are tree constructions. At this point, but for the following observations, we are not going to discuss these constructions. We refer the reader to the papers mentioned below. We will bring up the issue of automorphisms later but in a different tone.

The art of the state in automorphism constructions has changed over the years but they all, in some form, use the idea of well-visited and not well-resided. As a result they all are very dynamic. They depend on dynamic properties rather than order theoretic properties of $\mathcal{E}$. Sometimes these theorems are labeled as *extension theorems*, as they extend an isomorphism between substructures of $\mathcal{E}^*$ to an automorphism of $\mathcal{E}^*$. Generally an extension uses *entry states*. The issues of extension theorems and entry sets will come up again in more detail.

The first automorphism construction is due to Soare [21]. A similar construction can be found in Soare [22], Section XV.4-6. Soare's original argument was not a tree construction. In the 80s, in unpublished work, Harrington placed the construction on a tree and proved a number of results, such as Theorems 2 and 4. The first published version of the tree method or, as it is also called, the Δ^0_3 method, for constructing automorphisms of $\mathcal{E}^*$ appeared in Cholak [2] and later in Cholak [3]. The current published state of the art appears in Harrington and Soare [15]. We will discuss some yet unpublished papers on automorphisms in later sections.

5. Some Previous Results

Before turning to our recent work we want to use some older results to highlight ours themes. The classical example is that of the maximal sets:

5.1. **Maximal Sets.**

Definition 7. M is *maximal* if for all c.e. sets W either $W \subseteq^* M$ or $M \cup W =^* \omega$.

Theorem 8 (Soare [21])**.** *If M and $\widehat{M}$ are maximal sets then there is a Δ^0_3 automorphism Φ of $\mathcal{E}$ such that $\Phi(M) = \widehat{M}$. Hence the maximal sets form an orbit.*

Note: in this case we say M is *automorphic* to $\widehat{M}$ and write $M \approx_{\Delta^0_3} \widehat{M}$.

Theorem 9 (Martin [19]). *A computably enumerable degree* $\mathbf{h}$ *is high iff there is a maximal set* M *such that* $M \in \mathbf{h}$.

The proof of the above theorem captures dynamic properties of the high degrees. Hence we have a collection of results about the maximal sets which involve definability, automorphisms and orbits, degree theoretic issues and some dynamic properties.

5.2. **Post's Program.** Post's Problem and the corresponding program has a great impact on the development of computability theory.

Post's Problem. *Find an incomplete, noncomputable computably enumerable degree.*

Post's Problem was solved by Friedberg [13] and Mučnik [20]. These solutions introduced the finite injury method and the use of priority and changed the face of computability theory. But these solutions did not solve the problem in the way that Post intended.

Post's Program. *Find a "thinness" property of a computably enumerable set which guarantees incompleteness.*

Various solutions have been posed. The first was Marčenkov [18] which ensures that a computably enumerable set is low$_2$. There have been other suggested solutions. For example, one by Ambos-Spies and Nies [1] which ensures that a set is cappable.

However it is unclear what Post meant by a "thinness property". It seems (at least to this author) that only Marčenkov [18] was truly offered as a solution to Post's Program given this ambiguity and has the most potential to satisfy Post.

One way to interpret Post's Program is find a definable property in $\mathcal{E}$ which ensures incompleteness. The following theorem implies that all the previous posed solutions are not solutions in this sense. None of these solutions are definable in $\mathcal{E}$.

Theorem 10 (Cholak [3], Harrington and Soare [15]). *Every computably enumerable set is automorphic to a high set.*

However Harrington and Soare were able to come up with a solution to Post's Program when considered this way.

Definition 11.

$$Q(A)$$
$$(\exists C)_{A \subset_m C}(\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C}[[B \cap (S - A) = D \cap (S - A)] \Rightarrow$$
$$(\exists T)[\overline{C} \subset T \wedge A \cap (S \cap T) = B \cap (S \cap T)]],$$

where $S \sqsubset T$ iff $(\exists \breve{S})[S \cap \breve{S} = \emptyset \wedge S \cup \breve{S} = T]$ and $A \subset_m C$ iff $C - A$ is not finite and $(\forall W)[\overline{C} \subseteq W \Rightarrow \overline{A} \subseteq W]$.

Theorem 12 (Harrington and Soare [14])**.** *$Q(A)$ implies A is incomplete and $Q(A)$ holds for some A.*

5.3. Automorphic to a Complete Set.

Post's Program and the possible solutions prompted the question which computably enumerable sets are automorphic to a complete set (in the same orbit as a complete set). For example, all maximal and creative sets are automorphic to a complete set. The best known answer involves dynamic properties. In order to avoid some complex notation, we present a predecessor to the best known result.

Definition 13. A set A is *promptly simple* iff there is a computable function p, and for all computably enumerable sets W, if W is infinite then $(\exists x)(\exists s)[x \in W_{\text{at } s} \cap A_{p(s)}]$.

Theorem 14 (Cholak et al. [6] (presumed by Harrington and Soare [15]))**.** *All promptly simple sets (almost prompt) are automorphic to a complete set.*

Question 15. *Which computably enumerable sets are automorphic to a complete set?*

Conjecture 16. *The index set of computably enumerable sets automorphic to a complete set is either Σ^1_1-complete (or reasonably low in the arithmetical hierarchy, say Δ^0_{10}).*

In the latter half of this paper we will present some evidence supporting this conjecture. We direct the reader to Section 10.

5.4. Degree Theoretic Control.

As we have seen we do not have a handle on controlling the degree of $\widehat{A}$. Harrington (see Harrington and Soare [15]) showed we can avoid a lower cone. Can we avoid an upper cone?

Question 17. *Let A and $\mathbf{d}$ be incomplete. Does there exist an $\widehat{A}$ such that $A \approx \widehat{A}$ and $\mathbf{d} \not\leq_T \widehat{A}$?*

Like the last question, this question can be shaped into an index set question.

5.5. Controlling the Double Jump.

Whereas we do not have control of the Turing degree of $\widehat{A}$ we do have more control over its double jump. This work leads to a large number of invariant degree classes. We will skip this area in interest of time. We direct the reader to

Cholak and Harrington [7] and Cholak and Harrington [8] for more information.

6. AUTOMORPHISMS, AGAIN

Theorems previously mentioned exhibited an automorphic vs. elementary definable difference dichotomy. For example, thirty years ago it was thought every orbit contained a complete set but we now know there is a definable property which implies incompleteness. Our goal is to make this dichotomy more explicit. For future results we need to know more about how to construct automorphisms. We will move towards an algebraic extension theorem.

Definition 18. $\mathcal{L}^*(A)$ is $\{W \cup A : W$ a c.e. set$\}$ under $\subseteq$ modulo the ideal of finite sets ($\mathcal{F}$). (The outside of a set.)

Definition 19. $\mathcal{E}^*(A)$ is $\{W \cap A : W$ a c.e. set$\}$ under $\subseteq$ modulo $\mathcal{F}$. (The inside of a set and dual of Definition 18.)

If $A \approx \widehat{A}$ then $\mathcal{E}^*(A) \cong \mathcal{E}^*(\widehat{A})$ & $\mathcal{L}^*(A) \cong \mathcal{L}^*(\widehat{A})$. But what about the other direction? It turns out that it is always the case that $\mathcal{E}^*(A) \cong \mathcal{E}^*$; use a computable one-to-one function whose range is A to induce the desired isomorphism.

So the question becomes what conditions are necessary to ensure that $\mathcal{L}^*(A) \cong \mathcal{L}^*(\widehat{A})$ implies $A \approx \widehat{A}$. By Cholak [3] when $\mathcal{L}^*(A)$ is infinite, extra conditions are needed. We will discuss the case when $\mathcal{L}^*(A)$ is finite shortly. When these conditions hold we say that we have extended the isomorphism between $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ to an automorphism of $\mathcal{E}^*$.

6.1. The Structure $\mathcal{S}_{\mathcal{R}}(A)$.

The following is necessary for our algebraic extension theorem.

Definition 20. $\mathcal{S}(A) = \{B : \exists C(B \sqcup C = A)\}$. $\mathcal{R}(A) = \{R : R \subseteq A$ and R is computable$\}$. Let $\mathcal{S}_{\mathcal{R}}(A)$ be the quotient structure $\mathcal{S}(A)$ modulo $\mathcal{R}(A)$.

$\mathcal{S}(A)$ is the splits of A and $\mathcal{S}(A)$ forms a Boolean algebra. $\mathcal{R}(A)$ is the computable subsets of A and is an ideal of $\mathcal{S}(A)$.

Theorem 21 (Nies, see Cholak and Harrington [10])**.** *$\mathcal{S}_{\mathcal{R}}(A)$ is a Σ^0_3 atomless Boolean algebra.*

We are interested in extendible substructures, $\mathcal{B}$, of $\mathcal{S}_{\mathcal{R}}(A)$.

Definition 22. A subalgebra, $\mathcal{B}$, of $\mathcal{S}_{\mathcal{R}}(A)$ is an *extendible* subalgebra of $\mathcal{S}_{\mathcal{R}}(A)$ if there is a Δ^0_3 set B, an effective listing of splits of A, $\{S_i\}_{i<\omega}$ and $\{\breve{S}_i\}_{i<\omega}$ such that for all i, $S_i \sqcup \breve{S}_i = A$ and $\{S_i\}_{i\in B}$ generates $\mathcal{B}$.

We are interested in a particular extendible subalgebra. Given a computably enumerable set W, A breaks into two disjoint computably enumerable pieces, $W \searrow A$ (the integers which enter W first and then A) and $A \backslash W$ (the integers which enter A first and then maybe enter W). The sets of the form $W \searrow A$ are called *entry sets*. The integers which are in some state ν as they enter A form a computably enumerable set W. In this case $W = W \searrow A$. ν is an entry state iff $W \searrow A$ is infinite. Now $W \searrow A \sqcup A \backslash W = A$ and hence the entry sets form splits of A.

Definition 23. $\mathcal{E}_A$ is the Boolean algebra generated by the entry sets.

Lemma 24. *$\mathcal{E}_A$ is an* extendible *subalgebra of $\mathcal{S}_{\mathcal{R}}(A)$.*

7. The Algebraic Extension Theorems

The statement of the following theorem is an algebraic version of an extension theorem. While the statement is algebraic, the proof of this theorem is not algebraic. This is the one place (in this paper) we have to get our hands dirty and use the automorphism method mentioned earlier. The theorem is equivalent to Soare's New Extension Theorem (see Soare [24]) and the extension theorem stated in Soare [22].

Definition 25. Let $\mathcal{B}$ be an extendible algebra witnessed by B, $\{S_i\}_{i<\omega}$ and $\{\breve{S}_i\}_{i<\omega}$. Similarly for $\widehat{\mathcal{B}}$. $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are *extendibly Δ^0_3 isomorphic* via a Δ^0_3 function Θ if the map S_i goes to $\widehat{S}_{\Theta(i)}$, for $i \in B$, and $\widehat{S}_i$ goes to $S_{\Theta^{-1}(i)}$, for $i \in \widehat{B}$, induces an isomorphism between $\mathcal{B}$ and $\widehat{\mathcal{B}}$.

Theorem 26. *Let $\mathcal{B} \subseteq \mathcal{S}_{\mathcal{R}}(A)$ and $\widehat{\mathcal{B}} \subseteq \mathcal{S}_{\mathcal{R}}(\widehat{A})$ be two extendible Boolean algebras which are extendibly Δ^0_3 isomorphic via Θ (and both A and $\widehat{A}$ are not finite). Then there is a Φ such that Φ is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\widehat{A})$ and Φ extends Θ (Φ and Θ agree modulo $\mathcal{R}(A)$ on $\widehat{\mathcal{B}}$).*

This gives us an isomorphism Φ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\widehat{A})$ extending a given Θ between $\mathcal{B}$ and $\widehat{\mathcal{B}}$. (We will come back to Θ later.) How do we extend this into an automorphism Λ such that $\Lambda(A) = \widehat{A}$?

Lets assume that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via Ψ. (Otherwise A and $\widehat{A}$ cannot be in the same orbit.) Now $W = (W - A) \sqcup (W \cap A)$. What about letting $\Lambda(W) = (\Psi(W \cup A) - \widehat{A}) \sqcup \Phi(W \cap A)$. Λ is clearly order preserving.

But why is $\Lambda(W)$ and $\Lambda^{-1}(W)$ computably enumerable? For this we need the notion of supports.

7.1. Supports.

Definition 27. $S \in \mathcal{S}(A)$ *supports* X iff $S \subseteq X$ and $(X - A) \sqcup S$ is a computably enumerable set.

Lemma 28. *$W \searrow A$ supports W.*

Proof. $W = (W - A) \sqcup (W \searrow A) \sqcup (A \searrow W)$ and $(W - A) \sqcup (W \searrow A) = W \backslash A$ is a computably enumerable set. ☐

Definition 29. An extendible subalgebra $\mathcal{B}$ *supports* $\mathcal{L} \subseteq \mathcal{L}^*(A)$ if for all $W \in \mathcal{L}$ there is an $S \in \mathcal{B}$ such that S supports W.

Lemma 30. *$\mathcal{E}_A$ supports $\mathcal{L}^*(A)$.*

Definition 31. Assume that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via Ψ, $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are isomorphic via Θ, $\mathcal{B}$ supports $\mathcal{L}$, and $\widehat{\mathcal{B}}$ supports $\widehat{\mathcal{L}}$. Then the isomorphisms Ψ and Θ *preserve* the supports of $\mathcal{L}$ and $\widehat{\mathcal{L}}$ if $W \in \mathcal{L}$, $B \in \mathcal{B}$, and B supports W then $(\Psi(W \cup A) - \widehat{A}) \sqcup \Theta(B)$ is a computably enumerable set and if $\widehat{W} \in \widehat{\mathcal{L}}$, $\widehat{B} \in \widehat{\mathcal{B}}$, and $\widehat{B}$ supports $\widehat{W}$ then $(\Psi^{-1}(\widehat{W} \cup \widehat{A}) - A) \sqcup \Theta^{-1}(\widehat{B})$ is a computably enumerable set. Notice we do not worry about inclusion for the image (the first clause of the definition of support). For shorthand if $\mathcal{L} = \mathcal{L}^*(A)$ and $\widehat{\mathcal{L}} = \mathcal{L}^*(\widehat{A})$ we just say isomorphisms Ψ and Θ *preserve supports.*

7.2. A Stronger Extension Theorem.

Theorem 32. *Assume that*

(1) *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via Ψ;*
(2) *$\mathcal{B}$ and $\widehat{\mathcal{B}}$ are extendible algebras which are extendibly Δ^0_3 isomorphic via Θ;*
(3) *$\mathcal{B}$ supports $\mathcal{L}^*(A)$;*
(4) *$\widehat{\mathcal{B}}$ supports $\mathcal{L}^*(\widehat{A})$;*
(5) *Ψ and Θ preserve supports.*

Then $\Lambda(W) = (\Psi(W \cup A) - \widehat{A}) \sqcup \Phi(W \cap A)$ is an automorphism of $\mathcal{E}$ such that $\Lambda(A) = \widehat{A}$, $\Lambda \restriction \mathcal{L}^*(A) = \Psi$, and $\Lambda \restriction \mathcal{E}^*(A)$ is Δ^0_3. (Φ is from Theorem 26.)

This theorem follows from Theorem 26 by purely algebraic means.

8. Using the Algebraic Extension Theorems

8.1. **Splits, again.** Assume that A and $\widehat{A}$ are automorphic via Ψ. Hence $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via Ψ. So the structures $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ are isomorphic via isomorphism induced by Ψ. What is surpising is this isomorphism is Δ^0_3

Theorem 33. *$\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ are Δ^0_3-isomorphic structures via an isomorphism Θ* induced *by Ψ.*

Note $\mathcal{S}_{\mathcal{R}}(A)$ and $\mathcal{S}_{\mathcal{R}}(\widehat{A})$ are Σ^0_3 atomless Boolean algebras so we know they are Δ^0_4 isomorphic. So this theorem is an improvement of one quantifier or jump. The proof heavily uses coding, is difficult, and very technical.

8.2. **Automorphisms to Automorphisms.** Lets assume that A and $\widehat{A}$ are in the same orbit and see what the above theorems tell us.

Theorem 34. *If A and $\widehat{A}$ are automorphic via Ψ then*

1. *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via Ψ;*
2. *there is an extendible $\mathcal{B}$ supporting $\mathcal{L}^*(A)$*
3. *there is an extendible $\widehat{\mathcal{B}}$ supporting $\mathcal{L}^*(\widehat{A})$;*
4. *$\mathcal{B}$ and $\widehat{\mathcal{B}}$ are extendibly Δ^0_3 isomorphic via Θ;*
5. *the isomorphisms Ψ and Θ preserve supports.*

We are given Θ via Theorem 33. It turns out that $\Theta(\mathcal{E}_A)$ is an extendible algebra which is extendible Δ^0_3 isomorphic to $\mathcal{E}_A$ via a Θ' induced by Θ and similarly for $\Theta^{-1}(\mathcal{E}_A)$. It is not hard to show that the join of extendible algebras is extendible. So $\mathcal{B} = \mathcal{E}_A \oplus \Theta^{-1}(\mathcal{E}_{\widehat{A}})$ and $\widehat{\mathcal{B}} = \Theta(\mathcal{E}_A) \oplus \mathcal{E}_{\widehat{A}}$ are extendible. It is straightforward to show they extendibly isomorphic via an Θ' induced via Θ. We will just drop Θ' in favor of Θ.

Applying Theorem 32 to the above theorem results in the following:

Theorem 35. *If A and $\widehat{A}$ are automorphic via Ψ then they are automorphic via Λ where $\Lambda \restriction \mathcal{L}^*(A) = \Psi$ and $\Lambda \restriction \mathcal{E}^*(A)$ is Δ^0_3.*

In all the previously known techniques for constructing automorphisms Ψ it is *always* the case that $\Psi \restriction \mathcal{E}^*(A)$ is Δ^0_3. One wondered if this was a limitation in our techniques or the automorphisms themselves.

8.3. **Preserving the Computable Subsets.** Our current goal is to show that there is an isomorphism Λ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\widehat{A})$ which preserve the computable sets. Hence $\Lambda(R)$ is computable iff R is computable. This is more subtle than it appears. For example, if we will find a computable permutation p taking A to $\widehat{A}$ then defining $\Lambda(W) = p(W)$ does the desired job. But p only exists if A and $\widehat{A}$ have the same 1-degree.

Recall $\mathcal{R}(A)$ is the computable subset of A. Let $\overline{\mathcal{R}}(A) = \{\overline{R} : R \in \mathcal{R}(A)\}$.

Theorem 36. *Assume that*

1. $\mathcal{B}$ *and* $\widehat{\mathcal{B}}$ *are extendible algebras which are extendibly* Δ^0_3 *isomorphic via* Θ*;*
2. *for all* $R \in \mathcal{R}(A)$ *there is an* S *such that* $S \in \mathcal{B} \cap \mathcal{R}(A)$ *and* $R \subseteq S$*;*
3. *For all* $\widehat{R} \in \mathcal{R}(\widehat{A})$ *there is an* $\widehat{S}$ *such that* $\widehat{S} \in \widehat{\mathcal{B}} \cap \mathcal{R}(\widehat{A})$ *and* $\widehat{R} \subseteq \widehat{S}$.
4. Θ *preserves the computable subsets.*

Then there is a Λ *such that* Λ *is a* Δ^0_3 *isomorphism between* $\mathcal{E}^*(A) \cup \overline{\mathcal{R}}(A)$ *and* $\mathcal{E}^*(\widehat{A}) \cup \overline{\mathcal{R}}(\widehat{A})$. Λ *preserves the computable subsets.*

Item 2 says that the computable subset of $\mathcal{B}$ supports the complements of the computable subsets of A. Item 3 is the dual of Item 2. Like Theorem 32, this theorem follows from Theorem 26 by purely algebraic means. The hypotheses of Theorem 36 are easy to meet.

Lemma 37. *Let* A *and* $\widehat{A}$ *be two noncomputable computably enumerable sets. Then there are extendible Boolean algebras* $\mathcal{B}$ *and* $\widehat{\mathcal{B}}$ *and a* Δ^0_3 Θ *such that*

1. $\mathcal{B}$ *and* $\widehat{\mathcal{B}}$ *are extendibly isomorphic via* Θ*;*
2. *for all* $R \in \mathcal{R}(A)$ *there is an* S *such that* $S \in \mathcal{B} \cap \mathcal{R}(A)$ *and* $R \subseteq S$*;*
3. *for all* $\widehat{R} \in \mathcal{R}(\widehat{A})$ *there is an* $\widehat{S}$ *such that* $\widehat{S} \in \widehat{\mathcal{B}} \cap \mathcal{R}(\widehat{A})$ *and* $\widehat{R} \subseteq \widehat{S}$*;*
4. Θ *preserves the computable subsets.*

Corollary 38. *If* A *and* $\widehat{A}$ *are two noncomputable computably enumerable sets then there is a* Λ *such that* Λ *is a* Δ^0_3 *isomorphism between* $\mathcal{E}^*(A) \cup \overline{\mathcal{R}}(A)$ *and* $\mathcal{E}^*(\widehat{A}) \cup \overline{\mathcal{R}}(\widehat{A})$. Λ *preserves the computable subsets.*

Corollary 38 is not a new result. It was known to Herrmann but unpublished. It follows from Theorem 26 which is equivalent to almost all the previous extension theorems and Lemma 37 which is equivalent to Soare's Order-Preserving Enumeration Theorem (Soare [21] see XV.5.1 of [22]).

8.4. The Maximal Sets, again.

Theorem 39. *Assume that* M *and* $\widehat{M}$ *are maximal. Then an isomorphism* Λ *between* $\mathcal{E}^*(M) \cup \overline{\mathcal{R}}(M)$ *and* $\mathcal{E}^*(\widehat{M}) \cup \overline{\mathcal{R}}(\widehat{M})$ *(as given by Corollary 38) induces an automorphism* Ψ *taking* M *to* $\widehat{M}$.

Proof. If $W \subseteq^* M$ let $\Psi(W) = \Lambda(W)$. If $\overline{M} \subseteq W$ then $M \cup W = \omega$ and there is a computable set $R \subset M$ such that $\overline{R} \subseteq W$ which implies

$W = \overline{R} \sqcup (W \cap R)$. In this case, let $\Psi(W) = \overline{\Lambda(R)} \sqcup \Lambda(W \cap R)$. Determining which case applies can be done computable in $\mathbf{0}''$. $\square$

Claim 40. *All of the known definable orbits have a similar algebraic proof using Corollary 38. However, in many cases, dealing with the sets outside or disjoint from A takes more than an oracle for $\mathbf{0}''$.*

One particular orbit we should mention is that of the hemimaximal sets, nontrivial splits of maximal sets. Downey and Stob [12] first showed that the hemimaximal sets formed an orbit by using the standard methods known at that time. But Herrmann claimed to have found an algebraic proof. Herrmann never published this proof. The first algebraic proof of this result can be found in Cholak et al. [5]. Herrmann's work and the work in [5] can be seen as a forerunner to the current work.

9. New Orbits

First we need to generalize the structure $\mathcal{L}^*(A)$.

Definition 41 (The sets disjoint from A)**.** Let $\mathcal{D}(A) = \{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}$. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$. A is $\mathcal{D}$-*hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Boolean algebra. A is $\mathcal{D}$-*maximal* iff $\mathcal{E}_{\mathcal{D}(A)}$ is the two element Boolean algebra.

Lemma 42. *If A is simple then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{L}^*(A)$.*

The lemma follows since if A is simple the only computably enumerable subsets of it's complement are finite. Except for the creative sets, up to this year all known orbits were orbits of $\mathcal{D}$-hhsimple sets. For example, the hemimaximal sets are $\mathcal{D}$-maximal.

9.1. A "Definable" Δ^0_5 Orbit which is Not a Δ^0_3 Orbit.

Definition 43. F is A-special if $F = A$ or $F - A$ is computably enumerable and for all computably enumerable V if $V \cap A = \emptyset$ then $V - F$ is computably enumerable.

If F is A-special then $F \in \mathcal{D}(A)$.

Definition 44. A computably enumerable set A is *pseudo hemi $\mathcal{D}$-maximal*: iff

1. $\forall W$ if $(W \cap A = \emptyset)$ then $\exists F$ such that F is A-special and $W \subseteq F$.
2. $\forall F$ if F is A-special then $\exists F_1$ such that F_1 is A-special, $F \cap F_1 = \emptyset$, and F_1 is not computable.

(3) $\forall W \exists F$ such that F is A-special and either $W \subseteq^* F \cup A$ or $W \cup F \cup A =^* \omega$.

Using Item 2, we can find a disjoint sequence of A-special sets, $\{F_i\}$, such that $A = F_0$ and for all e either $W_e \subseteq \bigsqcup_{i \leq e} F_i$ or $W_e \cup \bigsqcup_{i \leq e} F_i =^* \omega$. It turns out that the A-special sets are also pseudo hemi $\mathcal{D}$-maximal. Hence the set $\bigsqcup_{i \in \omega} F_i$ behaves like a maximal set (it is not c.e.) and the F_is behave like splits of $\bigsqcup_{i \in \omega} F_i$. Item 3 implies that A is $\mathcal{D}$-maximal. This explains why these sets are called pseudo hemi $\mathcal{D}$-maximal. Clearly being pseudo hemi $\mathcal{D}$-maximal is definable.

Theorem 45. *The pseudo hemi $\mathcal{D}$-maximal sets form a definable Δ^0_5 orbit. Furthermore there are two pseudo hemi $\mathcal{D}$-maximal sets which are not in the same Δ^0_3-orbit.*

Let A and $\widehat{A}$ be two different pseudo hemi $\mathcal{D}$-maximal sets. We can use Corollary 38 to build the desired automorphism Ψ like we did for the maximal sets. Apply Corollary 38 to F_i and the corresponding $\widehat{F_i}$ to get Λ_i. Now if $W_e \subseteq \bigsqcup_{i \leq e} F_i$ then let $\Psi(W_e) = \bigsqcup \Lambda_i(W_e \cap F_i)$. Otherwise $W_e \cup \bigsqcup_{i \leq e} F_i =^* \omega$. In which case there is a computable subset R of $\bigsqcup_{i \leq e} F_i$ such that $W_e \cup R =^* \omega$. So $W_e = \overline{R} \cup \bigsqcup_{i \leq e}(W_e \cap R \cap F_i)$. Now $R \cap F_i$ is computable. So $\Psi(R) = \bigsqcup_{i \leq e}(\Lambda_i(R \cap F_i)$ is computable. Let $W_e = \overline{\Psi(R)} \cup \bigsqcup_{i \leq e}(\Lambda(W_e \cap R \cap F_i))$.

The reason the automorphism is Δ^0_5 is that we need $\mathbf{0^{(4)}}$ to find the sequence $\{F_i\}$. To show that it is not an orbit under Δ^0_3 we build two pseudo hemi $\mathcal{D}$-maximal sets, A and $\widehat{A}$, one such that there is Δ^0_3 sequence of disjoint A-special sets like above and one such that there is no Δ^0_3 sequence of disjoint A-special sets. While these sets are in the same orbit they cannot be in the same Δ^0_3 orbit.

9.2. **Some Restrictions.** Note the complexity in the above orbit arises because of the sequence of A-special sets. One might wonder if this is necessary. Using Theorem 33 we can show:

Theorem 46. *If A is $\mathcal{D}$-hhsimple and A and $\widehat{A}$ are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\widehat{A})}$.*

Hence the complexity in the orbit must come from how A interacts with the sets in $\mathcal{D}(A)$ at least in the case that A is $\mathcal{D}$-hhsimple. Using a different coding we can improve this to all sets A.

Theorem 47. *If A and $\widehat{A}$ are automorphic then $\mathcal{E}_{\mathcal{D}(A)}$ and $\mathcal{E}_{\mathcal{D}(\widehat{A})}$ are Δ^0_6-isomorphic.*

These theorems plus some work of Maass [17] allow us to classify the complexity of the orbits of several different sets.

Theorem 48. *If A is $\mathcal{D}$-hhsimple and simple (i.e., hhsimple) then $A \approx \widehat{A}$ iff $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{L}^*(\widehat{A})$. (The "only if" is by Maass [17]).*

Theorem 49. *If A is simple then $A \approx \widehat{A}$ iff $A \approx_{\Delta^0_6} \widehat{A}$.*

Theorem 50. *If A and $\widehat{A}$ are both promptly simple then $A \approx \widehat{A}$ iff $A \approx_{\Delta^0_3} \widehat{A}$.*

9.3. **The r-maximal sets.** Lets reflect on what our results say about the orbit of r-maximal sets.

Theorem 51. *Let A and $\widehat{A}$ be r-maximal. Then $\Lambda(A) = \widehat{A}$ iff*

1. *$\mathcal{L}^*(A)$ and $\mathcal{L}^*(\widehat{A})$ are isomorphic via a Φ which is Δ^0_6.*
2. *There is a Θ, extendible $\mathcal{B}$ and $\widehat{\mathcal{B}}$ such that $\mathcal{B}$ supports $\mathcal{L}^*(A)$, $\widehat{\mathcal{B}}$ supports $\mathcal{L}^*(\widehat{A})$, $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are extendibly Δ^0_3 isomorphic via Θ, Θ is Δ^0_3, and Φ and Θ preserve supports.*

Every item but the last one in Item 2 are always true; see Theorem 32 and the first few sentences after Theorem 34. Hence the Φ is the issue. One approach to algebraically building Φ was more or less suggested in Cholak and Nies [11].

10. Slaman-Woodin Conjecture

In 1990, Slaman and Woodin make the following conjecture.

The Slaman-Woodin Conjecture. *The set $\{\langle i, j\rangle : W_i \approx W_j\}$ is Σ^1_1-complete.*

Clearly this set is in Σ^1_1. It is a natural question to ask whether this set is complete for this class. Slaman and Woodin were led to this conjecture by a series of results.

Theorem 52 (Remmel/Folklore)**.** *Let $\mathcal{B}_i$ be a listing of computable Boolean algebras. The set $\{\langle i, j\rangle : \mathcal{B}_i \cong \mathcal{B}_j\}$ is Σ^1_1-complete.*

Theorem 53 (Lachlan [16])**.** *There is a computably enumerable set H_i such that $\mathcal{L}^*(H_i) \cong \mathcal{B}_i$.*

Corollary 54 (Slaman and Woodin)**.** *The set $\{\langle i, j\rangle : \mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)\}$ is Σ^1_1-complete.*

This led to the Slaman-Woodin Conjecture. The idea was to replace "$\mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)$" with "$H_i \approx H_j$". By Theorem 48, this is impossible.

In 1995, the conjecture was shown to be true. However the proof is very difficult and most likely will never be properly written up.

Theorem 55 (Cholak, Downey, and Harrington)**.** *The set* $\{\langle i, j\rangle : W_i \approx W_j\}$ *is* Σ^1_1*-complete.*

There is, however, a slightly stronger result with some very interesting corollaries.

Theorem 56. *There is a (computably enumerable) set* A *such that the index set* $\{i : W_i \approx A\}$ *is* Σ^1_1*-complete.*

Corollary 57.

1. *Not all orbits of* $\mathcal{E}$ *are elementarily definable.*
2. $\mathcal{E}$ *is not a prime model.*
3. *There is no arithmetic description of all orbits of* $\mathcal{E}$*.*
4. *Scott rank of* $\mathcal{E}$ *is* $\omega_1^{CK} + 1$*.*
5. *(corollary of the proof) For all* $\alpha \geq 10$*, there is a properly* Δ^0_α *orbit.*

To prove this theorem we encode into the orbit of A the isomorphisms of a computably branching tree. Given a (certain) Δ^0_3 list of parameters, $\mathcal{B}_i \subset \mathcal{S}_{\mathcal{R}}(A)$, we can define a definable invariant, $(\mathcal{N}(A), \prec)$. If $A \approx \widehat{A}$ then $(\mathcal{N}(A), \prec) \cong (\mathcal{N}(\widehat{A}), \prec)$.

Modulo some unmentioned (here) definable properties, if, for all i, there is an (certain) extendible $\widehat{\mathcal{B}}_i$ and a Δ^0_3 isomorphism between $\mathcal{B}_i$ and $\widehat{\mathcal{B}}_i$ and $(\mathcal{N}(A), \prec) \cong (\mathcal{N}(\widehat{A}), \prec)$ then $A \approx \widehat{A}$. The proof is similar to the proof that pseudo hemi $\mathcal{D}$-maximal sets form an orbit.

Now given a computably branching tree $(T, \prec)$ we can construct an A (and the $\mathcal{B}_i$) such that $(\mathcal{N}(A), \prec) \cong (T, \prec)$. Depending on T, $\{\widehat{T} : \widehat{T} \cong T\}$ ranges from properly Δ^0_α to Σ^1_1-complete and so does the index set of the orbit of A.

10.1. **Last Question.**

$(\mathcal{N}(A), \prec)$ is a definable (with the parameters $\mathcal{B}_i$s) invariant which determines the orbit of A. But it only works for some A, those with the correct $\mathcal{B}_i$s.

Question 58. *Can we define the invariant* $(\mathcal{N}(A), \prec)$ *(or something similar) without using any parameters?*

A positive answer would allow us to describe all orbits in terms of a tree.

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