The $\mathcal{D}$-maximal sets

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http://www.nd.edu/~cholak/papers/
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The Computably Enumerable Sets, $\mathcal{E}$

- W_e is the domain of the eth Turing machine.
- $(\{W_e : e \in \omega\}, \subseteq)$ are the c.e. (r.e.) sets under inclusion, $\mathcal{E}$.
- X is *automorphic* to Y, $X \approx Y$, iff there is an automorphism of $\mathcal{E}$ such that $\Phi(X) = Y$.

Main Question

Question (Completeness)

Which c.e. sets are automorphic to complete sets?

The Scott Rank of $\mathcal{E}$ is $\omega_1^{CK} + 1$

Theorem (Cholak, Downey, Harrington)

There is an c.e. set A such that the set

$$\mathcal{I}_A = \{i : A \text{ is automorphic to } W_i\}$$

is Σ_1^1-complete.

Main Open Question, Again

Look at the index set of all $\widehat{A}$ in the orbit of A with the hopes of finding some answers. The index set of such $\widehat{A}$ is in Σ^1_1.

Question (Completeness)

Which c.e. sets are automorphic to complete sets?

The orbits in the previous theorem all contain complete sets.

Question

Is the above question an arithmetical question?

Prompt Sets

Definition

A is *promptly simple* iff there is computable function p such that for all e, if W_e is infinite then there is a x and s with $x \in (W_{e,\text{at } s} \cap A_{p(s)})$.

Definition

A c.e. set A is *prompt* iff there is computable function p such that for all e, if W_e is infinite then there is a x and s with $x \in W_{e,\text{at } s}$ and $A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x$.

Theorem (Cholak, Downey, Stob)

All prompt simple sets are automorphic to a complete set.

Almost Prompt Sets

Definition

$X = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \cup \dots (W_{e_{2n-1}} - W_{e_{2n}})$ iff X is $2n$-c.e. and X is $2n+1$-c.e. iff $X = Y \cup W_e$, where Y is $2n$-c.e.

Definition

Let X_e^n be the eth n-c.e. set. A is *almost prompt* iff there is a computable nondecreasing function $p(s)$ such that for all e and n if $X_e^n = \overline{A}$ then $(\exists x)(\exists s)[x \in X_{e,s}^n$ and $x \in A_{p(s)}]$.

Lemma

Prompt implies almost prompt. So every Turing complete set is almost prompt.

Theorem (Harrington, Soare)

All almost prompt sets are automorphic to a complete set.

Tardy Sets

Definition

D is 2-*tardy* iff for every computable nondecreasing function $p(s)$ there is an e such that $X_e^2 = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^2$ then $x \notin D_{p(s)}]$.

Theorem (Harrington, Soare)

There are realizable & definable properties $Q(D)$ and $P(D,C)$ such that

- *$Q(D)$ implies that D is 2-tardy (so not Turing complete),*
- *if there is a C such that $P(D,C)$ and D is 2-tardy then $Q(D)$ (and D is high),*

n-Tardy Sets

Definition

D is *n-tardy* iff for every computable nondecreasing function $p(s)$ there is an e such that $X_e^n = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^n \text{ then } x \notin D_{p(s)}]$.

Theorem

There are realizable & definable properties $Q_n(D)$ such that $Q_n(D)$ implies that D is properly n-tardy (so not Turing complete).

Question

If D is not automorphic to a complete set must D satisfy some Q_n?

Degrees of n-Tardy sets

Splits of 2-tardys are 3-tardy but

Theorem

There is a 3-tardy that is not computable in any 2-tardy.

Question

Is there an $n+1$-tardy set that is not computed by any n-tardy set? Is there a very tardy sets which is not computed by any n-tardy?

Main Open Questions, Localized

Question ($\mathcal{D}$-Maximal Completeness)

Does the orbit of every $\mathcal{D}$-maximal set contain a complete set?

Question

Is the above question an arithmetical question?

Question

Why $\mathcal{D}$-maximal?

The Maximal Sets

Definition

M is *maximal* if for all W either $W \subseteq^* M$ or $M \cup W =^* \omega$.

Theorem (Friedberg)

Maximal sets exist.

Corollary

Let M be maximal. If $M \cup W =^* \omega$ then there is a computable set R_W such that $R_W \subseteq^* M$ and $W \cup R_W = \omega$. So $W = \overline{R}_W \sqcup (W \cap R_W)$.

Building Automorphisms

Theorem (Soare)

If A and $\hat{A}$ are both noncomputable then there is an isomorphism, Φ, between the c.e. subsets of A, $\mathcal{E}(A)$, to the c.e. subsets of $\hat{A}$ sending computable subsets of A to computable subsets of $\hat{A}$ and dually.

False Proof.

Let p be a computable 1-1 map from A to $\hat{A}$. Let $\Phi(W) = p(W)$. □

Orbits of Maximal Sets

Theorem (Soare)

The maximal sets form an orbit. This orbit includes a complete set.

Proof.

Take M and $\widehat{M}$ two maximal sets and apply Soare's Theorem to get Φ. Define if $W \subseteq^* M$ then let $\Psi(W) = \Phi(W)$. If $M \cup W =^* \omega$ then let $\Psi(W) = \overline{\Phi(R_W)} \sqcup \Phi(R_W \cap W)$. □

Friedberg Splits of Maximal Sets

Definition

$S_1 \sqcup S_2 = M$ is a Friedberg split of M, for all W, if $W \searrow M$ is infinite then $W \searrow S_i$ is also.

Lemma (Downey, Stob)

A nontrivial split of a maximal set (a hemimaximal set) is a Friedberg Split.

Theorem (Downey, Stob)

The Friedberg splits of the maximal sets form a definable orbit. This orbit includes a complete set.

Proof.

Let $M_1 \sqcup M_2 = M$. If $W \cup M =^* \omega$ then $W \searrow M$ is infinite.
$\Psi(W) = (\overline{\Phi_1(R_W \cap M_1)} \cap \overline{\Phi_2(R_W \cap M_2)}) \sqcup \Phi_1(R_W \cap W \cap M_1) \sqcup \Phi_1(R_W \cap W \cap M_2)$. $\square$

Herrmann Sets

Definition

A set H is *Herrmann* iff

- for all c.e. sets D disjoint from H there is a computable set R such that R is disjoint from H, $D \subset R$, and $R - D$ is infinite.
- for all c.e. sets B there is a c.e. set D disjoint from H such that either $B \subset H \cap B$ or $B \cup D \cup H = \omega$.

Theorem (Cholak, Downey, and Herrmann)

The Herrmann sets form a definable Δ^0_3 orbit containing a complete set different from the maximal and hemimaximal sets.

There is a Δ^0_3 list of disjoint computable sets R_i such that for all B disjoint from H there is an n with $B \subseteq \bigsqcup_{i \le n} R_i$.

Sets with A-Special Lists

Definition

A list of computably enumerable sets, $\mathcal{F} = \{F_i : i \in \omega\}$, is an *$A$-special list* iff $\mathcal{F}$ is a list of pairwise disjoint noncomputable sets, $F_0 = A$, and, for all e, there is an i such that $W_e \subseteq \bigsqcup_{l \leq i} F_l$ or $W_e \cup \bigsqcup_{l \leq i} F_l = \omega$.

Theorem (Cholak and Harrington)

The sets with A-special lists form a definable Δ^0_5 (but not Δ^0_3) orbit different from the maximal, hemimaximal, and Herrmann sets.

Question

Does this orbit contain a complete set?

$\mathcal{D}$-Maximal Sets

Definition (The sets disjoint from A)

$\mathcal{D}(A) = \{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}$ under inclusion. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$.

Definition

A is $\mathcal{D}$-*hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Σ^0_3 Boolean algebra. A is $\mathcal{D}$-*maximal* iff $\mathcal{E}_{\mathcal{D}(A)}$ is the trivial Boolean algebra iff for all c.e. sets B there is a c.e. set D disjoint from A such that either $B \subset A \cup D$ or $B \cup D \cup A = \omega$.

Lemma

Maximal sets, hemimaximal sets, Herrmann sets and sets with A-special lists are $\mathcal{D}$-maximal.

$\mathcal{D}$-hhsimple and Simple

Theorem (Maass 84)

If A is $\mathcal{D}$-hhsimple and simple (i.e., hhsimple) if $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$ then $A \approx \hat{A}$.

Theorem (Cholak, Harrington)

If A is hhsimple then $A \approx \hat{A}$ iff $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.

All such orbits contain complete sets.

Complexity Restrictions

Theorem (Cholak, Harrington)

If A is $\mathcal{D}$-hhsimple and A and $\hat{A}$ are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.

Does not provide an answer to the following:

Question ($\mathcal{D}$-maximal Completeness)

Which $\mathcal{D}$-maximal sets are automorphic to complete sets?

Question

Is the above question an arithmetical question?

Question

Can we classify the $\mathcal{D}$-maximal sets?

The Classification $\mathcal{D}$-maximal sets

Definition

For a $\mathcal{D}$-maximal set, a list of c.e. sets $\{X_i\}_{i\in\omega}$ *generates* $\mathcal{D}(A)$ iff for all D if D is disjoint from A then there is a n such that $D \subseteq^* \bigcup_{i\leq n} X_i$. (This list need not be computable.)

Lemma

- *$\{\emptyset\}$ generates $\mathcal{D}(A)$ iff A is maximal.*
- *For any computable set R, $\{R\}$ generates $\mathcal{D}(A)$ iff A is maximal on $\overline{R}$.*
- *For any noncomputable c.e. set W, $\{W\}$ generates $\mathcal{D}(A)$ iff A is hemimaximal.*
- *In all other cases, the list of generators is infinite.*

Disjoint Computable Generators

Lemma

- *If infinitely many computable sets are used to (partially) generate $\mathcal{D}(A)$ we can assume that (partial) list is pairwise disjoint.*
- *If an infinite list of computable sets generates $\mathcal{D}(A)$ then A is Herrmann.*

Disjoint Noncomputable Generators

Lemma

Assume an infinite pairwise disjoint list generates $\mathcal{D}(A)$. Then either

- *A is Herrmann*
- *$\mathcal{D}(A)$ is generated by a infinite pairwise disjoint list where exactly one of the sets is noncomputable and A is hemi-Herrmann.*
- *A has a A-special list.*

Almost Disjoint Computable Generators

Lemma

Assume W and $\{R_i\}_{i\in\omega}$ generates $\mathcal{D}(A)$ where $\{R_i\}_{i\in\omega}$ is a list of pairwise disjoint computable sets and W is noncomputable set such that for all i, $W \cap R_i \neq^* \emptyset$. Then A is a nontrivial split of hhsimple set H and $\mathcal{E}_{\mathcal{D}(H)}$ must be infinite.

Theorem (Herrmann and Kummer)

For every possible infinite Σ^0_3 Boolean Algebra $\mathcal{B}$ there is a hhsimple set H and a split A of H such that A is $\mathcal{D}$-maximal and $\mathcal{E}_{\mathcal{D}(H)} \approx \mathcal{B}$.

Theorem

The class of $\mathcal{D}$-maximal splits of hhsimple sets breaks up into infinitely many orbits.

Question

Do these orbits contain complete sets?

Nested Noncomputable Generators

Definition

H is *r-maximal* iff no computable set splits $\overline{H}$ into two infinite sets. H is *atomless* iff H has no maximal superset.

Theorem

- *There are D-maximal splits of atomless r-maximal sets.*
- *The class of D-maximal splits of atomless r-maximal sets breaks into infinitely many orbits. (Uses a similar result about atomless r-maximal sets by Cholak and Nies.)*
- *A is a D-maximal split of atomless r-maximal set iff there is a list of nested noncomputable generators $\{X_i\}_{i \in \omega}$ for $\mathcal{D}(A)$, i.e. for all i then $X_i \subset X_{i+1}$.*

Question

Do these orbits contain complete sets?

Anti-Friedberg Splits

Definition

A_0 is an *anti-Friedberg* split of A iff there is an A_1 such that $A_0 \sqcup A_1 = A$ and, for all W, either $W - A_0$ is c.e. or $W \cup A =^* \omega$.

Lemma

All $\mathcal{D}$-maximal splits of maximal, hhsimple, Herrmann and atomless r-maximal sets are anti-Friedberg. All anti-Friedberg splits are $\mathcal{D}$-maximal.

Question

Do the anti-Friedberg splits of a set (orbit) form an orbit?

Disjoint Noncomputable Generators and Disjoint Computable Generators

Theorem

- *There are $\mathcal{D}$-maximal sets that are generated by pairwise disjoint noncomputable sets $\{W_i\}_{i\in\omega}$ and pairwise disjoint computable sets $\{R_i\}_{i\in\omega}$ such that for all $i \geq j$, $W_i \cap \mathcal{R}_j \neq^* \emptyset$.*
- *The class of such $\mathcal{D}$-maximal sets breaks into infinitely many orbits.*

Question

Do these orbits contain complete sets?

Nested Noncomputable Generators and Disjoint Computable Generators

Theorem

- *There are $\mathcal{D}$-maximal sets which are generated by noncomputable sets $\{W_i\}_{i\in\omega}$ and pairwise disjoint computable sets $\{R_i\}_{i\in\omega}$ such that for all $i \geq j$, $W_i \cap \mathcal{R}_j \neq^* \emptyset$ and $W_i \cap \overline{R_i} \subset^* W_{i+1}$.*
- *The class of such $\mathcal{D}$-maximal sets breaks into infinitely many orbits.*

Theorem (Classification)

That is all the possible generating sets of $\mathcal{D}$-maximal sets.

Main Open Questions, Localized

Question ($\mathcal{D}$-Maximal Completeness)

Does the orbit of every $\mathcal{D}$-maximal set contain an complete set?

Question

Is the above question an arithmetical question?