Computability Theory

Domination, Measure, Randomness, and Reverse Mathematics

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The Game Plan

Basic Concepts

Computability

Cantor Space – Category and Measure

Domination

and Measure, again

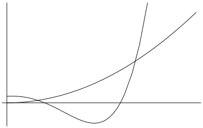
Randomness

Reverse Mathematics

(Given my contribution to this project please consider this as a survery talk not a research talk.)

Domination

Motivation and Definition



Definition

If $f(n) \geq g(n)$ for all but finitely many n, then f *dominates* g, $f \geq^* g$.

Primitive Recursion Functions

Ackermann's Function

Theorem

Ackermann's function dominates all primitive recursive functions.

Ackermann's function is computable. Hence the primitive recursive functions do not capture the informal notion of *computable*.

Computable Functions

Basic Definitions

Definition (Turing Machines)

- $\Phi^X_{e,s}(n) = \Phi_{e,s}(X, n) = \Phi^{\emptyset}_{e,s}(X, n)$ is the eth Turing machine with oracle X with input n run s stages. Determining if $\Phi^X_{e,s}(n)\downarrow$ is computable in X.The number and type of the inputs can vary.
- $\Phi^X_e(n)\downarrow$ if there is an stage s such that $\Phi^X_{e,s}(n)\downarrow$. Φ_e need not be total. But from now on we will assume we are dealing with total Φ.

Definition (Turing Reducibility)

- $f \leq_T X$ iff $f = \Phi^X_e$, for some e.
- $Y \leq_T X$ iff $\chi_Y \leq_T X$.
- $Y \equiv_T X$ iff $X \leq_T Y$ and $Y \leq_T X$. We write this as the Turing degree, $\mathbf{x}$.

Cantor Space
Category

We will work in Cantor Space, 2^ω.

Definition

- Let $\sigma \in 2^\omega$. $[\sigma]$ is a basic clopen (open and closed) class.
- A *open* class is a countable union of basic open classes. The complement of a open class is *closed*.
- The countable intersection of (basic) open classes is a G_δ class.
- The countable union of closed classes is a F_σ class.

Cantor Space is compact: if a closed class $\mathbb{X}$ can be covered by open classes, $\mathbb{X}$ can be covered by finitely open classes.

Cantor Space

Computability and Category

Definition

- $\mathbb{X} \subseteq 2^{\omega}$ is *computable* in Z iff there is a total Φ such that $X \in \mathbb{X}$ iff $\Phi^{Z}(X)\downarrow = 1$ and $X \notin \mathbb{X}$ iff $\Phi^{Z}(X)\downarrow = 0$.
- $\mathbb{X}$ is Π^{Z}_{1} iff $\mathbb{X} = \{X | \forall n (\Phi^{Z}(X, n)\downarrow = 1)\}$.
- $\mathbb{X}$ is Π^{Z}_{2} iff $\mathbb{X} = \{X | \forall n \exists m (\Phi^{Z}(X, n, m)\downarrow = 1)\}$.

Lemma

- *$\mathbb{X}$ is clopen iff $\mathbb{X}$ is computable. (Finite Use Principle)*
- *$\mathbb{X}$ is closed iff $\mathbb{X}$ is Π^{Z}_{1}, for some Z. If $Z = \emptyset$ then $\mathbb{X}$ is called a Π^{0}_{1}-class or effectively closed.*
- *A class $\mathbb{X}$ is G_{δ} iff $\mathbb{X}$ is Π^{Z}_{2}, for some Z. These classes are called boldface Π^{0}_{2} or $\mathbf{\Pi}^{0}_{2}$.*
- *Similarly a F_{σ} class is $\mathbf{\Sigma}^{0}_{2}$.*

Measure Theory on Cantor Space

Definition

- $\mu([\sigma]) = 2^{-|\sigma|}$. This determines the measure of G_δ classes and hence F_σ classes.
- A class $\mathbb{X}$ is measurable iff $\liminf \mu(\mathbb{G})$ exists and is equal to $\mu(\mathbb{X})$, where $\mathbb{G}$ is a G_δ class containing $\mathbb{X}$.

Definition

A Borel measure $\hat{\mu}$ is *regular* if every measurable (in terms of $\hat{\mu}$) class $\mathbb{P}$ there is a G_δ class $\mathbb{Q} \supseteq \mathbb{P}$ and an F_σ class $\mathbb{S} \subseteq \mathbb{P}$ such that $\hat{\mu}(\mathbb{S}) = \hat{\mu}(\mathbb{P}) = \hat{\mu}(\mathbb{Q})$.

Theorem

μ is regular.

(Uniformly) Almost Everywhere Dominating

Definition (Dobrinen and Simpson)

- A Turing degree **a** is *almost everywhere* (*a.e.*) *dominating* if for almost all Z for all $g \leq_T Z$ there is function f of degree **a** which dominates g.
- A Turing degree **a** is *uniformly almost everywhere* (*u.a.e.*) *dominating* if there is function f of degree **a** such that

$$\mu\left(\{Z \in 2^\omega : (\forall g)[g \leq_T Z \Rightarrow g \leq^* f]\}\right) = 1.$$

We also call such a function f *uniformly a.e. dominating*.

Lemma

U.a.e. dominating implies a.e dominating.

Existence

Theorem (Martin)

An uniformly a.e. dominating function f dominates all computable functions and hence must be high. I.e. $f' \equiv_T \mathbf{0}''$, where $f' = \{e | \Phi_e^f(e)\downarrow\}$.

Theorem (Kurtz)

There is a uniformly a.e. dominating function of degree $\mathbf{0}'$.

Theorem (Cholak, Greenberg, Miller)

There is an incomplete (c.e.) uniformly a.e. dominating degree.

Positive Measure Dominating

Definition (Kjos-Hanssen)

- $\text{Tot}(\Phi) = \{X | \forall n \exists s \Phi_s(X, n)\downarrow$ is total$\}$
- $\Phi < \mathbf{a}$ iff either $\mu(\text{Tot}(\Phi)) = 0$ or there is an $f \leq_T \mathbf{a}$ such that
$$\mu(\{X \in \text{Tot}(\Phi) | f \geq \Phi(X)\}) > 0.$$
- If, for all Φ, $\Phi < \mathbf{a}$ then $\mathbf{a}$ is *positive measure (p.m.) dominating*.

Lemma

U.a.e. dominating implies a.e dominating implies p.m. dominating.

Theorem (Binns, Kjos-Hanssen, Miller, Soloman)

Converse holds. P.m. dominating implies a.e dominating which implies u.a.e. dominating.

U.a.e. Domination and Measure

Theorem (Dobrinen and Simpson)

A Turing degree **a** *is u.a.e. dominating iff for every* Π^0_2 *class* $\mathbb{Q} \subseteq 2^\omega$ *there is a* $\Sigma^{\mathbf{a}}_2$ *class* $\mathbb{S} \subseteq \mathbb{Q}$ *such that* $\mu(\mathbb{S}) = \mu(\mathbb{Q})$*.*

($\Rightarrow$)
$$\mathbb{Q} = \{X \mid \forall n(\Phi_e(X,n)\downarrow)\},$$

for some e. Let Ψ be such that $\Psi^X(n)$ is the least s where $\Phi_{e,s}(X,n)\downarrow$. f dominates Ψ^X for almost all $X \in \mathbb{Q}$.

$$\mathbb{S} = \{X : \exists k \forall n(\Phi_{e,f(n)+k}(X,n)\downarrow)\}.$$

P.m. Domination and Measure

Theorem (Kjos-Hanssen after Dobrinen and Simpson)

A Turing degree $\mathbf{a}$ *is p.m. dominating iff* $Tot(\Phi)$ *has a* $\Pi^{\mathbf{a}}_1$ *subclass,* $\mathbb{S}$*, of positive measure.*

$(\Leftarrow)$ By compactness, $\{\Phi(X,n) | X \in \mathbb{S}\}$ is finite for all n. Therefore $\{\langle n,m\rangle : \forall X(X \in \mathbb{S} \to \Phi(X,n) < m)\}$ is $\Sigma^{\mathbf{a}}_1$. Hence by $\Sigma^{\mathbf{a}}_1$ uniformization there is a function $f \leq \mathbf{a}$ such that $\forall n \forall X(X \in \mathbb{S} \to \Phi(X,n) < f(n))$.

Goal Check I: We have related domination and measure. Now lets add randomness to this mixture.

1-Random Reals

Want to miss all "effectively null classes".

Definition (Martin-Löf)

- A *Martin-Löf test* (*relative to* X) is a computable (in X) collection of Σ^0_1 open classes $\{\mathbb{U}_e\}$ with $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- R misses a test, $\{\mathbb{U}_e\}$, iff $R \notin \bigcap_e \mathbb{U}_e$.
- R is 1-*random* (*relative to* X) iff R misses all Martin-Löf tests (relative to X).

Theorem (Martin-Löf, Solovay, Levin, Chaitin, Kolmogorov)

The definition of 1*-randomness is very robust.*

Low for 1-Random

Definition

- A is *low for* 1*-random* iff the class of 1-randoms is the class of 1-randoms relative to A.
- A is *low for* 1*-random over* Z iff the class of 1-randoms relative to Z is the class of 1-randoms relative to A and Z (or equivalently $A \oplus Z$) iff $A \leq_{LR} Z$.

Theorem (Downey, Hirschfeldt, *Nies*, Solovay, Stephan, Terwijn)

The class of A such that A is low for 1-random is a nontrivial robust class. Furthermore for all such A, $A' \leq \emptyset'$.

Lowness and Domination

Theorem (Binns, Kjos-Hanssen, Lerman, Solomon)

If B is a.e. domainating then $\mathbf{0}' \leq_{LR} B$.

Theorem (Kjos-Hanssen)

$A \leq_{LR} \mathbf{0}'$ iff every Π^0_2 class of positive measure has a Π^A_1 subclass of positive measure iff A is postive measure dominating.

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

If $A \leq_T B'$ and $A \leq_{LR} B$ then every Σ^A_2 class has a Σ^B_2 subclass of the same measure.

Question (Aside)

What is needed for this theorem in reverse math?

Lowness and Domination, II

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

B is a.e. domainating iff $\mathbf{0}' \leq_{LR} B$.

Proof.

($\Leftarrow$) Let P be any Π^0_2 class. P is a Σ^0_3 class, so it has a $\Sigma^{0'}_2$ subclass Q of the same measure (Kurtz, 1981). But, by the above theorem, Q has a Σ^B_2 class of the same measure. □

Corollary

U.a.e. domainating iff a.e domainating iff p.m. domainating.

Weakly 2-random

- A *Martin-Löf test* (*relative to* X) is a computable (in X) collection of Σ_1^0 open classes $\{\mathbb{U}_e\}$ with $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- We will remove the restriction that $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- Consider a computable (in X) collection of Σ_1^0 open classes $\{\mathbb{U}_e\}$ with $lim_e \mu(\mathbb{U}_e) = 0$.
- $\bigcap_e \mathbb{U}_e$ is a measure zero Π_2^0 class (G_δ).
- R is *weakly* 2*-random* (*relative to* X) iff R misses all measure zero Π_2^0 (Π_2^X) classes.

Theorem (Kurtz)

Weakly 2 *random implies* 1*-random but converse does not hold.*

Low for weakly 2-random

- A is *low for weakly 2-random* if every weak 2-random is weak 2-random over A.
- A is *low for weak 2-tests* iff every Π_2^A nullclass is covered by a Π_2^0 nullclass.

Lemma

Low for weak 2-tests implies low for weak 2-ramdon. (Shortly we will see the converse also holds)

Theorem (Downey, Nies, Weber, Yu)

- *If A is low for weakly 2-random then A is low for random.*
- *There is a noncomputable c.e. A is that is low for weak 2-tests.*

Low for random and weakly 2-random

Theorem (Binns, Kjos-Hanssen, Nies, Miller, Solomon)

If A is low for random then A is low for weak 2-tests.

Corollary

A is low for random iff A is low for weak 2-tests iff A is low for weakly 2-randoms.

Reverse Mathematics

Statement (G_δ-REG)

For every G_δ class $\mathbb{Q} \subseteq 2^\omega$ there is a F_σ class $\mathbb{S} \subseteq \mathbb{Q}$ such that $\mu(\mathbb{S}) = \mu(\mathbb{Q})$.

Theorem (Dobrinen and Simpson)

ACA_0 *implies* G_δ-REG.

Theorem (Kjos-Hanssen)

$\mathrm{RCA}_0 + G_\delta$-REG *does not imply* ACA_0.

G_δ-REG and traditional systems

G_δ-REG seems to be “orthogonal" to the traditional systems.

Theorem (Referee of Dobrinen and Simpson)

WKL_0 *does not imply* G_δ-REG*.*

Theorem (Cholak, Greenberg, Miller)

$\mathrm{RCA}_0 + G_\delta$-REG *does not imply* DNR_0*.*

Theorem (Cholak, Greenberg, Miller)

$\mathrm{WKL}_0 + G_\delta$-REG *does not imply* ACA_0*;* $\mathrm{WWKL}_0 + G_\delta$-REG *does not imply* WKL_0*.*

Question

Does $\mathrm{DNR}_0 + G_\delta$-REG *imply* WWKL_0*?*