Computability Theory

Domination, Measure, Randomness, and Reverse Mathematics

Peter Cholak

University of Notre Dame
Department of Mathematics

Peter.Cholak.1@nd.edu

http://www.nd.edu/~cholak/papers/nyc2.pdf

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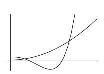
The Game Plan

Domination
Motivation and Definition



Definition

- Let $f, g\colon \omega \to \omega$. The function f *majorizes* g, $f \geq g$, if $f(n) \geq g(n)$ for all $n \in \omega$.
- If $f(n) \geq g(n)$ for all but finitely many n, then f *dominates* g, $f \geq^* g$.

Primitive Recursion Functions
Ackermann's Function

Theorem

Ackermann's function dominates all primitive recursive functions.

Ackermann's function is computable. Hence the primitive recursive functions do not capture the informal notion of *computable.*

Computable Functions

Basic Definitions

Definition (Turing Machines)

- $\Phi^X_{e,s}(n) = \Phi_{e,s}(X,n) = \Phi^{\emptyset}_{e,s}(X,n)$ is the eth Turing machine with oracle X with input n run s stages. Determining if $\Phi^X_{e,s}(n)\downarrow$ is computable in X and is Δ^X_1. The number and type of the inputs can vary.
- $\Phi^X_e(n)\downarrow$ if there is an stage s such that $\Phi^X_{e,s}(n)\downarrow$. Φ_e need not be total.

Definition (Turing Reducibility)

- $f \leq_T X$ iff $f = \Phi^X_e$, for some e.
- $Y \leq_T X$ iff $\chi_Y \leq_T X$.
- $Y \equiv_T X$ iff $X \leq_T Y$ and $Y \leq_T X$. We write this as the Turing degree, $\mathbf{x}$.

Post's Program

Highness and dominant functions

Definition (Dominant)

f is *dominant* iff f dominates every computable function.

Statement (Post's Program)

To find a "sparseness" property of a c.e. set which ensure incompleteness.

Theorem (Tennenbaum and Martin)

If M is a maximal set (the only partial computable sets in the complement of M are finite) then p_M is dominant.(If $\overline{M} = \{a_0 < a_1 \ldots\}$ then $p_M(n) = a_m$.)

Theorem (Martin)

- *Every high c.e. degree contains a maximal set.*
- *There is a dominant function of degree $\mathbf{a}$ iff $\mathbf{0}'' \leq_T \mathbf{a}'$, i.e. $\mathbf{a}$ is high. ($A' = \{e|\Phi^A_e(e)\downarrow\}$).*

Cantor Space

Category

We will work in Cantor Space, 2^ω.

Definition

- Let $\sigma \in 2^\omega$. $[\sigma]$ is a basic clopen (open and closed) class.
- A *open* class is a countable union of basic open classes. The complement of a open class is *closed*.
- The countable intersection of (basic) open classes is a G_δ class.
- The countable union of closed classes is a F_σ class.

Cantor Space is compact: if a closed class $\mathcal{X}$ can be covered by open classes, $\mathcal{X}$ can be covered by finitely open classes.

Cantor Space

Computability and Category

Definition

- $\mathcal{X} \subseteq 2^\omega$ is *computable* in Z iff there is a total Φ such that $X \in \mathcal{X}$ iff $\Phi^Z(X)\downarrow = 1$. (Φ total here and below.)
- $\mathcal{X}$ is Π^Z_1 iff $\mathcal{X} = \{X|\forall n(\Phi^Z(X,n)\downarrow = 1)\}$.
- $\mathcal{X}$ is Π^Z_2 iff $\mathcal{X} = \{X|\forall n \exists m(\Phi^Z(X,n,m)\downarrow = 1)\}$.

Lemma

- *$\mathcal{X}$ is clopen iff $\mathcal{X}$ is computable.*
- *$\mathcal{X}$ is closed iff $\mathcal{X}$ is Π^Z_1, for some Z. If $Z = \emptyset$ then $\mathcal{X}$ is called a Π^0_1-class or effectively closed.*
- *A class $\mathcal{X}$ is G_δ iff $\mathcal{X}$ is Π^Z_2, for some Z. These classes are called boldface Π^0_2 or $\boldsymbol{\Pi}^0_2$.*
- *Similarly a F_σ class is $\boldsymbol{\Sigma}^0_2$.*

Measure Theory on Cantor Space

Definition

- $\mu([\sigma]) = 2^{-|\sigma|}$. This determines the measure of G_δ classes and hence F_σ classes.
- A class $\mathscr{X}$ is measurable iff $\liminf \mu(\mathscr{G})$ exists and is equal to $\mu(\mathscr{X})$, where $\mathscr{G}$ is a G_δ class containing $\mathscr{X}$.

Definition

A Borel measure $\hat{\mu}$ is *regular* if every measurable (in terms of $\hat{\mu}$) class $\mathscr{P}$ there is a G_δ class $\mathscr{Q} \supseteq \mathscr{P}$ and an F_σ class $\mathscr{S} \subseteq \mathscr{P}$ such that $\hat{\mu}(\mathscr{S}) = \hat{\mu}(\mathscr{P}) = \hat{\mu}(\mathscr{Q})$.

Theorem

μ is regular.

(Uniformly) Almost Everywhere Dominating

Definition (Dobrinen and Simpson)

- A Turing degree $\mathbf{a}$ is *almost everywhere* (*a.e.*) *dominating* if for almost all Z for all $g \leq_T Z$ there is function f of degree $\mathbf{a}$ which dominates g.
- A Turing degree $\mathbf{a}$ is *uniformly almost everywhere* (*u.a.e.*) *dominating* if there is function f of degree $\mathbf{a}$ such that

$$\mu\left(\{Z \in 2^\omega : (\forall g)[g \leq_T Z \Rightarrow g \leq^* f]\}\right) = 1.$$

  We also call such a function f *uniformly a.e. dominating.*

Lemma

U.a.e. dominating implies a.e dominating. An u.a.e. dominating function is dominant.

Positive Measure Dominating

Definition (Kjos-Hanssen)

- $\text{Tot}(\Phi) = \{X \mid \forall n \exists s \Phi_s(X, n)\downarrow \text{ is total}\}$
- $\Phi < \mathbf{a}$ iff either $\mu(\text{Tot}(\Phi)) = 0$ or there is an $f \leq_T \mathbf{a}$ such that

$$\mu(\{X \in \text{Tot}(\Phi) \mid f \geq \Phi(X)\}) > 0.$$

- If, for all Φ, $\Phi < \mathbf{a}$ then $\mathbf{a}$ is *positive measure (p.m.) dominating.*

Lemma

U.a.e. dominating implies a.e dominating implies p.m. dominating.

Question (Main Open Question)

Does p.m. dominating imply a.e dominating? Does a.e dominating imply u.a.e. dominating?

U.a.e. Domination and Measure

Theorem (Dobrinen and Simpson)

A Turing degree $\mathbf{a}$ is u.a.e. dominating iff for every Π^0_2 class $\mathscr{Q} \subseteq 2^\omega$ there is a $\Sigma^{\mathbf{a}}_2$ class $\mathscr{S} \subseteq \mathscr{Q}$ such that $\mu(\mathscr{S}) = \mu(\mathscr{Q})$.

$(\Rightarrow)$
$$\mathscr{Q} = \{X \mid \forall n(\Phi_e(X, n)\downarrow)\},$$

for some e. Let Ψ be such that $\Psi^X(n)$ is the least s where $\Phi_{e,s}(X, n)\downarrow$. f dominates Ψ^X for almost all $X \in \mathscr{Q}$.

$$\mathscr{S} = \{X : \exists k \forall n(\Phi_{e,f(n)+k}(X, n)\downarrow)\}.$$

P.m. Domination and Measure

Theorem (Kjos-Hanssen after Dobrinen and Simpson)
A Turing degree $\mathbf{a}$ *is p.m. dominating iff* $Tot(\Phi)$ *has a* $\Pi_1^{\mathbf{a}}$ *subclass,* $\mathcal{S}$*, of positive measure.*

($\Leftarrow$) By compactness, $\{\Phi(X, n) | X \in \mathcal{S}\}$ is finite for all n. Therefore $\{\langle n, m\rangle : \forall X(X \in \mathcal{S} \to \Phi(X, n) < m)\}$ is $\Sigma_1^{\mathbf{a}}$. Hence by $\Sigma_1^{\mathbf{a}}$ uniformization there is a function $f \le \mathbf{a}$ such that $\forall n \forall X(X \in \mathcal{S} \to \Phi(X, n) < f(n))$.

Goal Check I: We have related domination and measure. Now lets add randomness to this mixture.

1-Random Reals

Want to miss all "effectively null classes".

Definition (Martin-Löf)

- A *Martin-Löf test* (*relative to* X) is a computable (in X) collection of Σ_1^0 open classes $\{\mathcal{U}_e\}$ with $\mu(\mathcal{U}_e) \le 2^{-e}$.
- R misses a test, $\{\mathcal{U}_e\}$, iff $R \notin \bigcap_e \mathcal{U}_e$.
- R is 1-*random* (*relative to* X) iff R misses all Martin-Löf tests (relative to X).

Theorem (Martin-Löf, Solovay, Levin, Chaitin, Kolmogorov)
The definition of 1*-randomness is very robust.*

Low for 1-Random

Definition

- A is *low for random* iff the class of 1-randoms is the class of 1-randoms relative to A.
- A is *low for random over* Z iff the class of 1-randoms relative to Z is the class of 1-randoms relative to A and Z (or equivalently $A \oplus Z$).

Theorem (Downey, Hirschfeldt, *Nies*, Solovay, Stephan, Terwijn)
The class of A *such that* A *is low for random is a nontrivial robust class. Furthermore for all such* A*,* $A' \le \emptyset'$.

Lowness and Domination

Theorem (Kjos-Hanssen)
A *is low for random iff every* Π_1^A *class of positive measure has a* Π_1^0 *subclass of positive measure.*

Theorem (Kjos-Hanssen)
$0'$ *is low for random over* X *iff* $Tot(\Phi)$ *has a* Π_1^X *subclass,* $\mathcal{S}$*, of positive measure iff* X *is of p.m. dominating degree.*

Some of this work is joint with Binns, Lerman and Solomon and also depends on work by Nies and Stephan.

Goal Check II: The above theorem tells us that p.m. dominating degrees exist but does not provide a direct construction. It also says nothing about u.a.e. dominating degrees.

Nontrivial Existence

Theorem (Kurtz)

Any $\mathbf{a} \geq_T \mathbf{0}'$ *is uniformly a.e. dominating.*

Theorem (Cholak, Greenberg, Miller)

There is an incomplete (c.e.) uniformly a.e. dominating degree.

- A proof via priority argument. Gives an c.e. degree. But does not seem to mix with cone avoidance (open question).
- A proof via flexible forcing construction. The resulting generic f can avoid any PA degree or any DNR-degree (or c.e. degree) and can avoid the cone above any Turing degree $\mathbf{a}$.

Reverse Mathematics

Statement (G_δ-REG)

For every G_δ class $\mathcal{Q} \subseteq 2^\omega$ there is a F_σ class $\mathcal{F} \subseteq \mathcal{Q}$ such that $\mu(\mathcal{F}) = \mu(\mathcal{Q})$.

Theorem (Dobrinen and Simpson)

ACA_0 *implies* G_δ-REG.

Theorem (Kjos-Hanssen)

$\mathsf{RCA}_0 + G_\delta$-REG *does not imply* ACA_0.

G_δ-REG and traditional systems

G_δ-REG seems to be "orthogonal" to the traditional systems.

Theorem (Referee of Dobrinen and Simpson)

WKL_0 *does not imply* G_δ-REG.

Theorem (Cholak, Greenberg, Miller)

$\mathsf{RCA}_0 + G_\delta$-REG *does not imply* DNR_0.

Theorem (Cholak, Greenberg, Miller)

$\mathsf{WKL}_0 + G_\delta$-REG *does not imply* ACA_0*;* $\mathsf{WWKL}_0 + G_\delta$-REG *does not imply* WKL_0.

Question

Does $\mathsf{DNR}_0 + G_\delta$-REG *imply* WWKL_0*?*

Universal Functional and Conventions

Lemma

There is a partial computable functional $\Phi\colon 2^\omega \to \omega^\omega$ *such that if , for almost all* Z*, either* $\Phi(Z)$ *is not total or* $\Phi(Z) \leq^* f$ *then* f *is uniformly a.e. dominating.*

Definition

- $D_n = \{Z \in 2^\omega : \Phi(Z;n)\downarrow\}$.
- $D_n[s] = \{Z \in 2^\omega : \Phi(Z;n)[s]\downarrow\}$. So, by convention, $\bigcup D_n[s] = D_n$, $D_{n+1} \subseteq D_n$, and $\mathrm{Tot}(\Phi) = \cap_n D_n$.
- For $g \in \omega^{\leq\omega}$, let $D_{[n,m)}[g]$ be

  $$\{Z \in 2^\omega : (\forall k \in [n,m))\Phi(Z;k)\downarrow\ [g(k)]\}.$$

  Hence if $Z \in D_{[n,m)}[g]$, then g majorizes $\Phi(Z)$ on the interval $[n,m)$.

The Domination Requirements

Goal: Define f s.t. $\mu(\text{Tot}(\Phi) - D_{[N_i,\infty)}[f]) \leq \beta_i$, for all i. Must have that $N_i \geq i$ exists and $\lim_i \beta_i = 0$. We will focus on one i.

We want to define $f(s)$ s.t. $\mu(D_s - D_s[f(s)]) \leq \beta'_s$, where $s \geq N_i$.

Now $\text{Tot}(\Phi) - D_{[N_i,\infty)}[f] \subseteq \bigcup_{s \geq N_i}(D_s - D_s[f(s)])$.

So must pick the β's s.t. $\sum_{s \geq m_i} \beta'_s \leq \beta_i$.

All error is disjoint from previous error.

Kurtz's Result

Assume that we have inductively defined f for all $n' < n$ s.t. $\mu(D_{n'} - D_{n'}[f(n')]) \leq \beta'_{n'}$.

$\mu(D_n - D_n[t]) > \beta'_n$ iff there is a $t' > t$ and finitely incomparable many τ_i s.t. $\tau_i \in (D_n[t'] - D_n[t])$ and $\mu(\bigcup[\tau_i]) \geq \beta'_n$. This is Σ_1.

Hence using $\mathbf{0}'$ we can find a t s.t. $\mu(D_n - D_n[t]) \leq \beta'_n$. Let $f(n) = t$.

Towards a Priority Construction

Assume that we allow $f = lim_s f_s$ where $f_s \in \omega^\omega$ and for all s, s', n if $s \leq s'$ then $f_s(n) \leq f_{s'}(n)$. Then f is of c.e. degree.

Let $f_0(n) = n$. If there is ever a stage t s.t. $\mu(D_n[t] - D_n[f(n)]) > \beta'_n$ let $f_t(n) = t$.

But how is it possible to restrain some initial segment of f_s which might involve finitely many n?

Need a better way to measure $\text{Tot}(\Phi)$.

Measuring $\text{Tot}(\Phi)$ to $\geq q$

$\mu(\text{Tot}(\Phi)) \geq q$ iff there is an M s.t.

$$(\forall n \geq M)[\mu(\text{dom}\, D_n) \geq q].$$

Fix $0 < \epsilon < 1$. We can code M and K into the tree and ask

$$(\forall n \geq M)[\mu(\text{dom}\, D_n) \geq K\epsilon].$$

This can be done by a Π_2 branching tree construction. Hence we can measure $\text{Tot}(\Phi)$ to within $\epsilon/2$.

Incompleteness Requirements

If we restraint f up to r we must ensure

$$\mu\left(\mathrm{Tot}(\Phi) \setminus D_{[N,r)}[f]\right) \leq \epsilon.$$

Asume that we know the least K s.t. $\mathrm{Tot}(\Phi) \geq K\epsilon$. Pick a witness x and wait for a stage s and g s.t.

- $f_s \upharpoonright N \subseteq g$,
- $(\forall n \in [N, |g|))\ f_s(n) < g(n)$,
- $\varphi^g_{e,s}(x)\downarrow = 0$, and
- $\mu(D_{[N,|g|)}[g]) \geq K\epsilon$.

More or less straightforward to put everything into a tree construction. Assume all the nodes of length $2e$ are working on the φ_e. If one is successful, stop. Everything is set up to allow ONE restraint of the form:

$$\mu\left(\mathrm{Tot}(\Phi) \setminus D_{[N,r)}[f]\right) \leq \epsilon.$$

Hence the current c.e. construction does not mix with cone avoiding.

Forcing Conditions

$\mathbf{p} = \langle f, \epsilon \rangle$ is a condition, where $f \in \omega^{<\omega}$ and ϵ is a promise we will dominate $\mathrm{Tot}(\Phi)$ from $|f|$ onwards within ϵ.

$\langle g, \delta \rangle$ *extends* $\langle f, \epsilon \rangle$ if $f \subset g$, $\delta \leq \epsilon$ and, if $f \neq g$, then

$$\mu\left(\mathrm{Tot}(\Phi) \setminus D_{[|f|,|g|)}[g]\right) + \delta < \epsilon.$$

Lemma

If G is sufficiently generic then f^G is uniformly a.e. dominating.

Lemma (Key Lemma)

Let $\mathbf{p} \in \mathbb{P}$. Then there is a c.e. set

$$S \subset \{f^{\mathbf{q}} : \mathbf{q} \leq \mathbf{p}\}$$

and a $\mathbf{p}^* \leq \mathbf{p}$ such that $\{\mathbf{q} \leq \mathbf{p}^* : f^{\mathbf{q}} \in S\}$ is dense below $\mathbf{p}^*$.

Avoiding

Lemma (Cone Avoiding)

If $G \subset \mathbb{P}$ is generic over a non-computable A, then $\Psi(f^G) \neq A$.

Lemma (Avoiding PA degrees)

If $G \subset \mathbb{P}$ is generic, then f^G does not have PA-degree.

Lemma (Avoiding DNR degrees)

If $G \subset \mathbb{P}$ is generic, then f^G does not have DNR-degree.

THANKS!