RELATIVIZATION, CATEGORICITY, AND DIMENSION

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Abstract

by

Charles Frederick Dymphna McCoy

We examine questions in computable model theory by studying relativized counterparts. Specifically, our study centers on computable and higher-level categoricities and computable dimension.

A computable structure is Δ^0_α-*categorical* if every computable copy is isomorphic by a Δ^0_α isomorphism. Goncharov used a priority construction to give a syntactic characterization of Δ^0_1-categoricity, under added effectiveness hypotheses. Ash, Knight, Manasse, and Slaman, and independently Chisholm, showed that the syntactic property actually is equivalent to the stronger notion of *relativized* Δ^0_1-categoricity. Their proofs use forcing constructions.

A computable structure's *computable dimension* is the number of equivalence classes of computable copies under the relation of being computably isomorphic. Goncharov produced, for each finite n, a structure with computable dimension n; however, no "natural" characterization of dimension n had been discovered in any class of structures.

Could studying relative dimension yield a characterization, perhaps a syntactic one like that for Δ^0_1-categoricity? We show that, in fact, finite computable dimension does not relativize: any structure with relative finite computable dimension is relatively Δ^0_1-categorical. Our argument uses a forcing construction related to the one by Ash et al.

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Goncharov, Dzgoev, Remmel, and LaRoche characterized the Δ^0_1-categorical linear orderings and Boolean algebras. We examine Δ^0_2-categoricity in these structures. Unlike those who studied Δ^0_1-categoricity, we begin with the relativized notion; using the result by Ash et al, we give noneffective syntactic arguments to provide a complete description. We then classify, under added effectiveness hypotheses, the Δ^0_2-categorical Boolean algebras and linear orderings. Beginning with the relativized results proves useful in two ways. First, since relativized and unrelativized categoricities often coincide, the relativized results provide a probable statement of the unrelativized results, modulo extra hypotheses. Second, the proofs of the unrelativized results give a basic organization for the priority arguments and an idea of what extra hypotheses will make these arguments valid.

Next, we study Δ^0_3-categoricity. We provide insight into why a classification for the Δ^0_3-categorical linear orderings might be impossible, and we classify the *relatively* Δ^0_3-categorical Boolean algebras.

To my mom and my dad,

who challenge me and help me

to be more than I was “cut out” to be.

CONTENTS

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CHAPTER 1

INTRODUCTION

1.1 The basics of categoricity and dimension

We define the notions of categoricity and dimension relevant to our study, and we provide earlier results which motivate our questions or are used in our proofs.

Definition 1.1 *A computable structure* $\mathcal{A}$ *is* computably categorical *if for any computable copy* $\mathcal{B} \cong \mathcal{A}$ *there is a* Δ^0_1 *isomorphism* $\varphi_e : \mathcal{B} \cong \mathcal{A}$*.*

The relativized notion is defined as follows.

Definition 1.2 *A computable structure* $\mathcal{A}$ *is* relatively computably categorical *if for any copy* $\mathcal{B} \cong \mathcal{A}$ *there is a* $\Delta^0_1(\mathcal{D}(\mathcal{B}))$ *isomorphism* $\varphi_e^{\mathcal{D}(\mathcal{B})} : \mathcal{B} \cong \mathcal{A}$*.*

In Chapter 5, we will generalize this definition for structures $\mathcal{A}$ which are not computable and may not even have a computable copy.

We give some examples of computably categorical structures.

1. A linear order is computably categorical iff it has finitely many pairs of direct successors ([15], [24]).

2. A Boolean algebra is computably categorical iff it has finitely many atoms ([15], [25], [23]).

3. An Abelian p-group is computably categorical iff it can be written in one of the following two forms: $(\mathbb{Z}(p^\infty))^\alpha \oplus G_1$, where $\alpha \in \mathbb{N}$ or $\alpha = \infty$, and G_1 is finite; or $(\mathbb{Z}(p^\infty))^l \oplus G_1 \oplus (\mathbb{Z}_{p^k})^\infty$, where $k, l \in \mathbb{N}$ and G_1 is finite ([13]). (The notation is that of [16].)

Note that relative computable categoricity implies computable categoricity; we will discuss below why the converse is not true. In the above examples, however, the computably categorical structures are in fact all relatively computably categorical.

Definition 1.3 *For a computable ordinal α, a computable structure $\mathcal{A}$ is Δ^0_α-categorical if for any computable copy $\mathcal{B} \cong \mathcal{A}$ there is a Δ^0_α isomorphism $\varphi_e^{\Delta^0_\alpha} : \mathcal{B} \cong \mathcal{A}$.*

The relativized notion is defined as follows.

Definition 1.4 *A computable structure $\mathcal{A}$ is* relatively Δ^0_α-categorical *if for any copy $\mathcal{B} \cong \mathcal{A}$ there is a $\Delta^0_\alpha(\mathcal{D}(\mathcal{B}))$ isomorphism $\varphi_e^{\Delta^0_\alpha(\mathcal{D}(\mathcal{B}))} : \mathcal{B} \cong \mathcal{A}$.*

Goncharov generalized the notion of computable categoricity by defining computable dimension.

Definition 1.5 *A computable structure $\mathcal{A}$ has* computable dimension $m < \omega$ *if*

1. *there exist m computable copies $\mathcal{B}_1, \ldots, \mathcal{B}_m \cong \mathcal{A}$ with no two computably isomorphic; and*

2. *for any $m+1$ computable copies $\mathcal{B}_1, \ldots, \mathcal{B}_{m+1} \cong \mathcal{A}$, there are at least two which are computably isomorphic.*

(Therefore, a computable structure has dimension 1 iff it is computably categorical.)

Definition 1.6 *A computable structure $\mathcal{A}$ has* computable dimension ω *if there exists a sequence of computable copies $(\mathcal{B}_i)_{i<\omega}$ so that each $\mathcal{B}_i \cong \mathcal{A}$, but no two are computably isomorphic.*

In [14] Goncharov proved that for any finite $m \geq 2$ there is a computable structure of computable dimension m. In [15], Goncharov and Dzgoev defined a condition called *branching* which guarantees that a structure has computable dimension ω.

Goncharov and Ventsov defined the relativized notion of finite computable dimension.

Definition 1.7 *A computable structure $\mathcal{A}$ has* relative computable dimension $m < \omega$ *if*

1. *there exist m copies (with computable universes) $\mathcal{B}_1, \ldots, \mathcal{B}_m \cong \mathcal{A}$ with no two $\Delta^0_1((\mathcal{D}(\mathcal{B}_1) \oplus \ldots \oplus \mathcal{D}(\mathcal{B}_m))$ isomorphic; and*

2. *for any $m+1$ copies (with computable universes) $\mathcal{B}_1, \ldots, \mathcal{B}_{m+1} \cong \mathcal{A}$, there are at least two which are $\Delta^0_1((\mathcal{D}(\mathcal{B}_1) \oplus \ldots \oplus \mathcal{D}(\mathcal{B}_{m+1}))$ isomorphic.*

Goncharov proved a syntactic characterization of computable categoricity that provides the starting point for many of the results in this thesis. Before we can state this result precisely, we need to review some material on infinitary formulas.

Definition 1.8 *Fix a computable language $\mathcal{L}$. We define infinitary and computable infinitary formulas inductively for all computable ordinals α.*

1. *A Σ_0 or Π_0* formula *is a finitary, quantifier-free (q.f.) formula. We can assign these formulas Gödel numbers in the standard way.*

2. *An* infinitary q.f. formula *is constructed from Σ_0 formulas using only disjunction and conjunction.*

3. *A Σ_α* infinitary formula *is a countable disjunction of the form $\psi(\vec{x}) = \bigvee\!\!\!\bigvee_{i \in I} \exists \vec{u}_i \theta_i(\vec{x}, \vec{u}_i)$, where each θ_i is a Π_β formula for some $\beta < \alpha$. A Π_α* formula *is defined analogously, with universal quantifiers replacing existential quantifiers, and conjunctions replacing disjunctions.*

4. *A* computable Σ_α infinitary formula, *or* Σ^c_α *formula, is a* Σ_α *formula involving disjunctions over only computably enumerable sets of formulas. That is, a* Σ^c_α *formula* $\psi(\vec{x})$ *is of the form* $\psi(\vec{x}) = \bigvee\!\!\!\bigvee_{i\in I} \exists \vec{u}_i \theta_i(\vec{x}, \vec{u}_i)$, *where each* θ_i *is a* Π^c_β *formula for some* $\beta < \alpha$, *and the set of Gödel numbers for the* θ_i*'s is c.e. (We assume, by induction, that we can assign Gödel numbers to the computable formulas of lower complexity.)*

A more complete treatment of computable infinitary formulas is found in [1].

Definition 1.9 *Let* $\mathcal{A}$ *be an* $\mathcal{L}$*-structure for some computable language* $\mathcal{L}$*. A* formally Σ^0_α Scott family *for* $\mathcal{A}$ *is an ordered pair* $\langle \vec{c}, \Theta \rangle$ *where* $\vec{c}$ *is a tuple of distinct parameters; and* Θ *is a c.e. set of* Σ^c_α *formulas of* $\mathcal{L}$ *with parameters among* $\vec{c}$ *so that*

1. *for each tuple* $\vec{a} \in \mathcal{A}$, *there is* $\theta(\vec{x}) \in \Theta$ *with* $\mathcal{A} \models \theta(\vec{a})$*; and*

2. *for any* θ *in* Θ *and* $\vec{a}, \vec{a}\,' \in \mathcal{A}$, *if* $\mathcal{A} \models \theta(\vec{a})$ *and* $\mathcal{A} \models \theta(\vec{a}\,')$, *then* $\langle \mathcal{A}, \vec{a}, \vec{c} \rangle \cong \langle \mathcal{A}, \vec{a}\,', \vec{c} \rangle$.

For structures satisfying a stronger decidability condition, Goncharov provided a syntactic characterization of computable categoricity in [11].

Definition 1.10 *A structure is* 1-decidable *if its* $\exists$*-diagram is computable.*

Definition 1.11 *A structure is* 2-decidable *if its* $\forall\exists$*-diagram is computable.*

Theorem 1.12 (Goncharov) *Let* $\mathcal{A}$ *be 2-decidable. Then* $\mathcal{A}$ *is computably categorical iff it has a formally* Σ^0_1 *Scott family.*

Goncharov and others wondered, however, what relationship existed between the existence of a formally Σ^0_1 Scott family and computable categoricity under relaxed hypotheses. Was the above connection merely a by-product of the strong decidability conditions on $\mathcal{A}$? Ash, Knight, Manasse, and Slaman in [4], and independently, Chisholm in [6] proved that there is indeed a strong connection between computable categoricity and the existence of a formally Σ_1^0 Scott family.

Theorem 1.13 (Ash et al, Chisholm) *A computable structure $\mathcal{A}$ is relatively Δ_α^0-categorical iff it has a formally Σ_α^0 Scott family.*

In [12] Goncharov himself produced a structure that is computably categorical but has no formally Σ_1^0 Scott family, thus demonstrating that the relatively computably categorical structures constitute a proper subset of the computably categorical structures. In fact, Kudinov, using a characterization of computable categoricity in 1-decidable structures which he proved in [20], produced a 1-decidable structure that is computably categorical but has no no formally Σ_1^0 Scott family. Consequently, the extra decidability condition in Theorem 1.12 is as weak as possible.

1.2 A brief description of results

In Chapter 2 we prove that finite computable dimension does not relativize. In Chapter 3 we use Theorem 1.13 to characterize relativized Δ_2^0-categoricity in Boolean algebras and linear orderings. In Chapter 4 we prove the analogous unrelativized results. Finally, in Chapter 5 we discuss Δ_3^0-categoricity in these two classes of structures.

CHAPTER 2

COMPUTABLE DIMENSION n DOES NOT RELATIVIZE

Goncharov and Ventsov questioned whether computable dimension actually relativized to anything meaningful; if so, perhaps a syntactic characterization could be established to produce more "natural" examples of structures of dimension $n > 1$. During a talk in the summer of 1997, Knight stated the result that no computable structure has a finite relative computable dimension greater than 1. The main purpose of this chapter is to provide a proof of this result. Theorem 1.13 is critical to our proof: we demonstrate that any structure of finite relative computable dimension has a formally Σ_1^0 Scott family and thus has relative computable dimension 1.

The outline of this chapter closely follows that of [4]. In the first section we define products of structures and prove the definability of $\Delta_1^0((\mathcal{D}(\mathcal{B}_1) \oplus \ldots \oplus \mathcal{D}(\mathcal{B}_n))$ functions. In the next section we develop forcing machinery to define a finite number of generic copies of a structure. Finally, in the last section we establish our main result by using a forcing construction to produce a formally Σ_1^0 Scott family for a structure assumed to have finite relative computable dimension.

2.1 The Transition Lemma

To simplify certain definitions and arguments, we include the logical constants T (truth) and F (falsity) in addition to the usual logical symbols. For the rest of this chapter we fix a computable language $\mathcal{L}$ with only relation symbols and a computable

$\mathcal{L}$-structure $\mathcal{A}$ with universe $A = \mathbb{N}$. If $X \subset \mathbb{N}$ is computable, we write $\mathcal{L}(X)$ to denote the augmented language obtained by adding a constant symbol for each element of X.

Definition 2.1 *Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be $\mathcal{L}$-structures with respective pairwise disjoint universes $B_1, \ldots, B_n$. Let $\mathcal{L}'$ be the new language obtained from $\mathcal{L}$ by adding new unary predicate symbols $S_1, \ldots, S_n$. Then the* cardinal sum $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ *is an $\mathcal{L}'$-structure defined as follows (recall that $\mathcal{L}$ has only relation symbols):*

1. *the universe of $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ is $\bigcup_{i=1}^n B_i$;*
2. *for any relation symbol R in $\mathcal{L}$, $R^{\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n} = \bigcup_{i=1}^n R^{\mathcal{B}_i}$;*
3. *for $i = 1, \ldots, n$, $S_i^{\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n} = B_i$.*

The cardinal sum is one of a general kind of product structure considered by Feferman and Vaught in [10].

Note: Throughout the rest of the chapter we consider only cardinal sums where the universes $B_1, \ldots, B_n$ are computable, infinite and disjoint; and $\bigcup_{i=1}^n B_i = \mathbb{N}$.

For $\mathcal{L}$-structures $\mathcal{B}_1, \ldots, \mathcal{B}_n$, we notice the following about the satisfaction of *atomic* sentences γ of $\mathcal{L}(\mathbb{N})$:
If all the parameters of γ come from a single B_i, then $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \models \gamma$ iff $\mathcal{B}_i \models \gamma$.
If γ has parameters from $\mathcal{B}_i$ and $\mathcal{B}_j$, where $i \neq j$, then $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \not\models \gamma$.

We now give definitions and results analogous to some of those in the first two sections of [4].

Notation 1) For any tuples $\vec{u} = (u_1, \ldots, u_m)$ and $\vec{v} = (v_1, \ldots, v_m)$ and any function f, we write $f(\vec{u}) = \vec{v}$ if $f(u_i) = v_i$ for each $i \in \{1, \ldots, m\}$. (We think of the u_i's as distinct.)

2) When we write $\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)$, we mean the atomic diagram of $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ as an $\mathcal{L}$-structure, not an $\mathcal{L}'$-structure.

3) We denote the atomic sentence of $\mathcal{L}(\mathbb{N})$ with Gödel number x by θ_x.

Lemma 2.2 *(Transition Lemma): Let $B_1, \ldots, B_n$ have the properties stated after Definition 2.1, and let and $m > 0$. From an m-tuple $\vec{u}$ of distinct elements, an m-tuple $\vec{v}$, and $e \in \omega$ we may effectively obtain an index for a quantifier-free Σ_1^c sentence $\psi_{e,\vec{u},\vec{v}}$ of $\mathcal{L}(\mathbb{N})$ such that for all structures $\mathcal{B}_1, \ldots, \mathcal{B}_n$ with respective universes B_i we have $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}(\vec{u}) = \vec{v} \leftrightarrow \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \models \psi_{e,\vec{u},\vec{v}}$. Specifically, the sentence $\psi_{e,\vec{u},\vec{v}}$ has the form*

$$\bigvee\!\!\!\!\bigvee_{\sigma \in S} \gamma_{\sigma,1} \wedge \ldots \wedge \gamma_{\sigma,n}$$

where each $\gamma_{\sigma,i}$ is a finitary conjunction of atomic sentences and negations of atomic sentences from $\mathcal{L}(B_i)$.

Proof: We must recall two very basic but important facts about computations. First, the program for the e^{th} Turing machine with oracle does *not* depend on the oracle we use in a particular computation. Second, any computation uses only a finite initial segment of the oracle. Therefore, $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}(\vec{u}) = \vec{v}$ iff there is a string $\sigma \in 2^{<\omega}$ with $\sigma \subset \mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)$ and $\varphi_e^{\sigma}(\vec{u}) = \vec{v}$. By the note above about the satisfaction of atomic sentences in a product structure, we note that the property "$\sigma \subset \mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)$" can be expressed by a conjunction of atomic sentences, where each atomic sentence has parameters from a single B_i.

More precisely, we let σ be an element of $2^{<\omega}$. For any $\mathcal{B}_1, \ldots, \mathcal{B}_n$, σ is an initial segment of $\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)$ iff for each $x \in dom\, \sigma$ we have the following:

1. if θ_x has parameters from $B_i, B_j, i \neq j$, then $\sigma(x) = 0$;

2. if θ_x has parameters from only one B_i, then $\sigma(x) = 0$ iff $\mathcal{B}_i \models \neg\theta_x$.

Define the c.e. set $S = \{\sigma : \varphi_e^\sigma(\vec{u}) = \vec{v}$ and $\forall x \in dom\,\sigma[(\sigma(x) = 1) \Rightarrow$ (all the parameters in θ_x come from a single B_i)$]\}$.

For each $i = 1, \ldots, n$ and each $\sigma \in S$, let

$R_0^{\sigma,i} = \{x\colon x \in dom\,\sigma[\sigma(x) = 0$, and θ_x involves only parameters from $B_i]\}$; and

$R_1^{\sigma,i} = \{x\colon x \in dom\,\sigma[\sigma(x) = 1$, and θ_x involves only parameters from $B_i]\}$.

If either $R_0^{\sigma,i}$ or $R_1^{\sigma,i}$ is nonempty, then we let $\gamma_{\sigma,i} := \bigwedge\!\!\!\bigwedge_{x \in R_0^{\sigma,i}} \neg\theta_x \wedge \bigwedge\!\!\!\bigwedge_{x \in R_1^{\sigma,i}} \theta_x$.

Otherwise, we let $\gamma_{\sigma,i} :=$ T.

Then for all $\mathcal{B}_1, \ldots, \mathcal{B}_n$, $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}(\vec{u}) = \vec{v}$ iff $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \models \bigvee\!\!\!\bigvee_{\sigma \in S} \gamma_{\sigma,1} \wedge \ldots \wedge \gamma_{\sigma,n}$.

QED

2.2 Forcing

Throughout the rest of this chapter, we fix $n > 1$ and *sets* $B_1, \ldots, B_n$ with the properties stated after Definition 2.1. Furthermore, we assume that any structure called $\mathcal{B}_i$ has universe B_i.

Notation We add function symbols $f_1, \ldots, f_n$ to $\mathcal{L}$ and denote the augmented language by $\mathcal{L}^\star$.

Definition 2.3 *For a q.f. sentence ψ of $\mathcal{L}(\mathbb{N})$ we define ψ' of $\mathcal{L}^\star(\mathbb{N})$ inductively:*

1. *if ψ is atomic and contains parameters from more than one B_i, then $\psi' = F$;*

2. *if ψ is atomic and contains parameters from only one B_i, then ψ' results from replacing the occurrence of each b_i in B_i with $f_i(b_i)$;*

3. *$(\neg\psi)' = \neg(\psi')$, $(\bigvee\!\!\!\bigvee \psi_i)' = \bigvee\!\!\!\bigvee(\psi_i')$, etc.*

Then the following is immediate.

Lemma 2.4 *Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be $\mathcal{L}$-structures with $g_i : \mathcal{B}_i \cong \mathcal{A}$. Extend each $g_i : B_i \to A$ to a total function $h_i : A \to A$. (We need h_i to be total to interpret the function* *symbol f_i. The precise way of extending g_i is irrelevant; for instance, we could just map everything in the complement of B_i to 0.) Let h_i interpret the function symbol f_i to obtain the $\mathcal{L}^\star$-structure $\langle \mathcal{A}, h_1, \dots, h_n \rangle$. Then for a q.f. sentence ψ of $\mathcal{L}(\mathbb{N})$, $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \models \psi$ iff $\langle \mathcal{A}, h_1, \dots, h_n \rangle \models \psi'$.*

The forcing construction we describe below will give us bijections $g_i : B_i \to A$ (Proposition 2.11). This bijection g_i immediately induces an $\mathcal{L}$-structure $\mathcal{B}_i \cong_{g_i} \mathcal{A}$. Lemma 2.4 guarantees that $\mathcal{L}^\star$ is a language in which we can talk about these induced structures.

Note: When we actually use the forcing concepts we now develop, the sentences of $\mathcal{L}^\star(\mathbb{N})$ we consider will be q.f. (finitary or infinitary) and have terms of the form a for $a \in A$ and $f_i(a_i)$ for $a_i \in B_i$. Therefore, these are the only sentences we consider here. That is, when we refer to a collection of $\mathcal{L}^\star(\mathbb{N})$ sentences, we automatically assume they are of this form.

Definition 2.5 *Let ψ be a sentence of $\mathcal{L}^\star(\mathbb{N})$ (with the properties described above). Let $p = \langle p^1, \dots, p^n \rangle$ be an n-tuple of finite partial 1-1 functions $p^i : B_i \to A$. (These tuples are called* forcing conditions.*) We write $q = \langle q^1, \dots, q^n \rangle \supseteq p$ iff $q^i \supseteq p^i$ for all i. We define the relation $p \Vdash \psi$ (read "p* forces *ψ") inductively:*

1. *If ψ is a finitary sentence with the above properties, then $p \Vdash \psi$ iff*

   (a) *for each $f_i(a_i)$ appearing in ψ, $p^i(a_i)$ is defined; and*

   (b) *$\mathcal{A} \models \hat{\psi}$ where $\hat{\psi}$ is obtained by substituting $p^i(a_i)$ for $f_i(a_i)$.*

2. *If ψ is not finitary, and $\psi = \bigwedge\!\!\!\bigwedge_n \psi_n$, $p \Vdash \psi$ iff $\forall n \forall q \supseteq p \exists r \supseteq q(r \Vdash \psi_n)$.*

3. *If ψ is not finitary, and $\psi = \bigvee\!\!\!\bigvee_n \psi_n$, $p \Vdash \psi$ iff $\exists n(p \Vdash \psi_n)$.*

4. *If ψ is not finitary, and $\psi = \neg\theta$, $p \Vdash \psi$ iff $\forall q \supseteq p(q \nVdash \theta)$.*

We establish some very simple, very important properties of forcing.

Proposition 2.6 *Let ψ be a sentence of $\mathcal{L}^\star(\mathbb{N})$ (with the properties described above) and p be a forcing condition. Then*

1. *p cannot force both ψ and $\neg\psi$;*
2. *if $p \Vdash \psi$ and $q \supset q$, then $q \Vdash \psi$;*
3. *there is $q \supset p$ with $q \Vdash \psi$ or $q \Vdash \neg\psi$.*

Proof: 1) Assume $p \Vdash \psi$. If ψ is finitary, then the definition of satisfaction precludes p from forcing $\neg\psi$. If ψ is infinitary, then since $p \subseteq p$, we cannot have $p \Vdash \neg\psi$.

2) Fix $q \supset p$. We argue by induction on the complexity of ψ that $p \Vdash \psi$ implies $q \Vdash \psi$.

Let $p \Vdash \psi$, a finitary sentence. Then the definition of forcing implies the result, because $q^i(a_i) = p^i(a_i)$ for each $f_i(a_i)$ appearing in ψ.

Let $p \Vdash \psi = \bigwedge\!\!\!\bigwedge_n \psi_n$, an infinitary conjunction. Suppose $n \in \mathbb{N}$ and $r \supseteq q$. Then $r \supseteq p$, so there is $s \supseteq r$ with $s \Vdash \psi_n$. Hence, $q \Vdash \psi$.

Let $p \Vdash \psi = \bigvee\!\!\!\bigvee_n \psi_n$, an infinitary disjunction. Since $p \Vdash \psi$, there is n with $p \Vdash \psi_n$. By induction hypothesis, $q \Vdash \psi_n$, so $q \Vdash \psi$.

Let $p \Vdash \psi = \neg\theta$, an infinitary negation. Suppose $r \supseteq q$. Then $r \supseteq p$, so $r \not\Vdash \theta$. Thus, $q \Vdash \psi$.

3) Try to define $q \supset p$ to force ψ. If we cannot, and ψ is finitary, then we define $q = \langle q^1, \ldots, q^n \rangle$ so that $q^i(a_i)$ exists if $f_i(a_i)$ appears in ψ. Then $q \Vdash \neg\psi$. If we cannot, and ψ is infinitary, then $p \Vdash \neg\psi$ by definition. QED

The following definition of a fragment of sentences is not standard.

Definition 2.7 *A subset $\mathcal{F}$ of $(\mathcal{L}^\star(\mathbb{N}))_{\omega_1\omega}$ is called a* fragment *of q.f. sentences iff it has the following properties:*

1. *if* $\theta \in \mathcal{F}, \theta = \neg\psi$, *then* $\psi \in \mathcal{F}$;

2. *if* $\theta \in \mathcal{F}, \theta = \bigwedge\!\!\!\bigwedge_{n \in R} \psi_n$, *then* $\{\psi_n : n \in R\} \subset \mathcal{F}$, *and* $\bigvee\!\!\!\bigvee_{n \in R} \neg\psi_n \in \mathcal{F}$;

3. *if* $\theta \in \mathcal{F}, \theta = \bigvee\!\!\!\bigvee_{n \in R} \psi_n$, *then* $\{\psi_n : n \in R\} \subset \mathcal{F}$;

4. *for each* $i = 1, \dots, n$, $b \in B_i$, *and* $a \in \mathbb{N}$, $\bigvee\!\!\!\bigvee_{c \in \mathbb{N}} f_i(b) = c$, $\bigvee\!\!\!\bigvee_{d \in B_i} f_i(d) = a \in \mathcal{F}$.

Note: Any countable subset S of $(\mathcal{L}^\star(\mathbb{N}))_{\omega_1\omega}$ can be extended to a countable fragment $\mathcal{F}$ by a standard closing procedure.

Definition 2.8 *Let* $\mathcal{F}$ *be fragment of* $(\mathcal{L}^\star(\mathbb{N}))_{\omega_1\omega}$. *A sequence* $(p_m)_{m\in\omega}$ *of forcing conditions is* $\mathcal{F}$-complete *if for all* m, $p_m \subseteq p_{m+1}$, *and for each* ψ *in* $\mathcal{F}$, *there is* $m \in \omega$ *so that* $p_m \Vdash \psi$ *or* $p_m \Vdash \neg\psi$.

Proposition 2.9 *For any countable fragment* $\mathcal{F}$ *of* $(\mathcal{L}^\star(\mathbb{N}))_{\omega_1\omega}$, *there exists an* $\mathcal{F}$-*complete sequence of forcing conditions.*

Proof: Let the sentences of $\mathcal{F}$ be listed as $(\psi_m)_{m\in\omega}$. We let $p_0 = \langle\emptyset, \dots, \emptyset\rangle$. At each stage $m > 0$, we define $p_m \supseteq p_{m-1}$ so that $p_m \Vdash \psi_{m-1}$ or $p_m \Vdash \neg\psi_{m-1}$. We can do this by part 3 of Proposition 2.6. QED

Proposition 2.10 *For any countable fragment* $\mathcal{F}$ *and any forcing condition* p, *we can include* p *in an* $\mathcal{F}$-*complete forcing sequence.*

Proof: Let $p_0 = p$ and proceed as in the previous proof. QED

Proposition 2.11 *Let* $\mathcal{F}$ *be a fragment and* $(p_m)_{m\in\omega}$ *be an* $\mathcal{F}$-*complete sequence of forcing conditions. Then for each* $i = 1, \dots, n$, $g_i = \bigcup_{m\in\omega} p^i_m : B_i \to A$ *is a bijection.*

Proof: This follows directly from the definitions of forcing, fragments, and complete sequences. The formulas of the first type in property 4 of Definition 2.7 guarantee totality; those of the second type guarantee surjectivity. Since each p^i_m is injective, so is $\bigcup_{m\in\omega} p^i_m$. QED

For each $i = 1, \ldots, n$, let $\mathcal{B}_i$ be the structure on B_i induced by the bijection g_i. The product $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ obtained from a complete forcing sequence is called *generic*. We now establish the most important fact about forcing.

Lemma 2.12 *(Forcing Lemma) Let $\mathcal{F}$ be a fragment of sentences of the type described at the beginning of this section, and $(p_m)_{m\in\omega}$ be an $\mathcal{F}$-complete sequence. For each $i = 1, \ldots, n$ let h_i extend g_i as described in Lemma 2.4. Then for any ψ in $\mathcal{F}$*

$$\langle \mathcal{A}, h_1, \ldots, h_n \rangle \models \psi \text{ iff } \exists m (p_m \Vdash \psi).$$

Sketch of proof: The argument is standard, so we give only a brief outline. It proceeds by induction on the complexity of the sentences in our fragment. If ψ is finitary, then the very definition of forcing yields the desired conclusion. If ψ is a disjunction or negation, then the induction is relatively straightforward. If ψ is a conjunction, then the argument is a bit more complicated; the inclusion of the second set in property 2 of Definition 2.7 is crucial. QED

As we shall see in Section 4, we often use Lemma 2.4, which connects satisfaction in $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$ with satisfaction in $\langle \mathcal{A}, h_1, \ldots, h_n \rangle$, in conjunction with Lemma 2.12, which connects satisfaction in $\langle \mathcal{A}, h_1, \ldots, h_n \rangle$ with forcing.

2.3 Relative dimension

2.3.1 A different definition of formally Σ^0_1 Scott families suffices

We give a weaker definition of a formally Σ^0_1 Scott family than that appearing in the introduction.

Definition 2.13 *Let $\mathcal{A}$ be an $\mathcal{L}$-structure for some computable language $\mathcal{L}$. A* formally Σ^0_1 Scott family *for $\mathcal{A}$ is an ordered pair $\langle \vec{c}, \Theta \rangle$ where $\vec{c}$ is a tuple of distinct parameters; and Θ is a c.e. set of Σ^c_1 formulas of $\mathcal{L}$ with parameters among $\vec{c}$ so that*

1. *for each tuple $\vec{a} \in \mathcal{A}$ of distinct elements with those from $\vec{c}$ listed first, there is $\theta(\vec{x}) \in \Theta$ with $\mathcal{A} \models \theta(\vec{a})$; and*

2. *for any θ in Θ and $\vec{a}, \vec{a}\,' \in \mathcal{A}$, if $\mathcal{A} \models \theta(\vec{a})$ and $\mathcal{A} \models \theta(\vec{a}\,')$, then $\langle \mathcal{A}, \vec{a} \rangle \cong \langle \mathcal{A}, \vec{a}\,' \rangle$.*

The definition of a formally Σ^0_1 Scott family stated in Chapter 1 contains a stronger first and second condition:

1. for *each* tuple there must be a formula, not just those tuples of distinct elements with those from $\vec{c}$ listed first;

2. if $\vec{a}$ and $\vec{a}\,'$ satisfy the same formula, there must be an isomorphism between $\langle \mathcal{A}, \vec{a}, \vec{c} \rangle$ and $\langle \mathcal{A}, \vec{a}\,', \vec{c} \rangle$.

With some rather long but straightforward technical arguments, we can show that the two definitions are equivalent in the following sense: $\mathcal{A}$ has a formally Σ^0_1 Scott family according to one definition iff it has a formally Σ^0_1 Scott family according to the other definition. However, in this chapter, our sole purpose in constructing the formally Σ^0_1 Scott family is to invoke Theorem 1.13. Therefore, rather than argue for equivalence directly, we reprove Theorem 1.13 for our definition of a formally Σ^0_1 Scott family. Since we have altered the definition by weakening conditions, we need only check one direction of the result.

Theorem 1.13 *A computable structure $\mathcal{A}$ is relatively computably categorical iff it has a formally Σ^0_1 Scott family (as defined in Definition 2.13).*

Proof: ($\Leftarrow$) Assume that $\mathcal{A}$ has a formally Σ^0_1 Scott family $\langle \vec{c}, \Theta \rangle$, and let $\mathcal{B} \cong \mathcal{A}$. We must produce a $\Delta^0_1(\mathcal{D}(\mathcal{B}))$-isomorphism f between them.

First there is some $\vec{d}$ with $\langle \mathcal{A}, \vec{c}\rangle \cong \langle \mathcal{B}, \vec{d}\rangle$. Let $f(\vec{c}) = \vec{d}$. Let $a_1 \in A - \{\vec{c}\}$. There is $\theta_1(\vec{x}, y) \in \Theta$ so that $\mathcal{A} \models \theta_1(\vec{c}, a_1)$. So there is $b_1 \in \mathcal{B} - \{\vec{d}\}$ with $\mathcal{B} \models \theta_1(\vec{d}, b_1)$. Define $f(a_1) = b_1$. We argue, in fact, that $\langle \mathcal{A}, \vec{c}, a_1\rangle \cong \langle \mathcal{B}, \vec{d}, b_1\rangle$. Of course, for some a_1', we have $\langle \mathcal{A}, \vec{c}, a_1'\rangle \cong \langle \mathcal{B}, \vec{d}, b_1\rangle$. Then $\mathcal{A} \models \theta_1(\vec{c}, a_1')$, so $\langle \mathcal{A}, \vec{c}, a_1\rangle \cong \langle \mathcal{A}, \vec{c}, a_1'\rangle \cong \langle \mathcal{B}, \vec{d}, b_1\rangle$.

Let $b_2 \in \mathcal{B} - \{\vec{d}, b_1\}$. There is $\theta_2(\vec{x}, y_1, y_2) \in \Theta$ so that $\mathcal{B} \models \theta_2(\vec{d}, b_1, b_2)$. Thus there is $a_2 \in \mathcal{A} - \{\vec{c}, a_1\}$ so that $\mathcal{A} \models \theta_2(\vec{c}, a_1, a_2)$. Define $f(a_2) = b_2$. We can again argue that $\langle \mathcal{A}, \vec{c}, a_1, a_2\rangle \cong \langle \mathcal{B}, \vec{d}, b_1, b_2\rangle$. For some a_2', $\langle \mathcal{A}, \vec{c}, a_1, a_2'\rangle \cong \langle \mathcal{B}, \vec{d}, b_1, b_2\rangle$. Then $\mathcal{A} \models \theta_2(\vec{c}, a_1, a_2')$, so $\langle \mathcal{A}, \vec{c}, a_1, a_2\rangle \cong \langle \mathcal{A}, \vec{c}, a_1, a_2'\rangle \cong \langle \mathcal{B}, \vec{d}, b_1, b_2\rangle$.

We continue this back-and-forth argument to obtain the desired isomorphism.

QED

2.3.2 An important lemma

Lemma 2.14 *For each $e \in \omega$ and $i, j \in \{1, \ldots, n\}$, there is a sentence $\rho_{e,i,j}$ of $\mathcal{L}^\star(\mathbb{N})$ so that for $\mathcal{B}_1 \cong_{g_1} \mathcal{A}, \ldots, \mathcal{B}_n \cong_{g_n} \mathcal{A}$, $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)} : \mathcal{B}_i \cong \mathcal{B}_j$ iff $\langle \mathcal{A}, h_1, \ldots, h_n\rangle \models \rho_{e,i,j}$, where each h_i is defined from g_i as usual. (Note that in viewing $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}$ as an isomorphism from $\mathcal{B}_i$ onto $\mathcal{B}_j$, we are concerned only with its behavior on the* computable *subdomain B_i.)*

Proof: By the Transition Lemma, we know that for all $e, \vec{x}, \vec{y}$, there is a sentence $\psi_{e,\vec{x},\vec{y}}$ such that, regardless of the structure of $\mathcal{B}_1, \ldots, \mathcal{B}_n$, $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}(\vec{x}) = \vec{y}$ iff $\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n \models \psi_{e,\vec{x},\vec{y}}$. Therefore, by Lemma 2.4 we conclude that for $\mathcal{B}_1, \ldots, \mathcal{B}_n \cong \mathcal{A}$, $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}(\vec{x}) = \vec{y}$ iff $\langle \mathcal{A}, h_1, \ldots, h_n\rangle \models \psi'_{e,\vec{x},\vec{y}}$. Consequently, the sentence

$\bigwedge\!\!\!\bigwedge_{x \in B_i} \bigvee\!\!\!\bigvee_{y \in B_j} \psi'_{e,x,y}$ expresses the totality of $\varphi_e^{\mathcal{D}(\mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n)}$; $\bigwedge\!\!\!\bigwedge_{y \in B_j} \bigvee\!\!\!\bigvee_{x \in B_i} \psi'_{e,x,y}$, surjectivity;

$\bigwedge\!\!\!\bigwedge_{x_1, x_2 \in B_i, x_1 \neq x_2} \bigwedge\!\!\!\bigwedge_{y \in B_j} \neg\psi'_{e,x_1,y} \vee \neg\psi'_{e,x_2,y}$, injectivity; and

$\bigwedge\!\!\!\bigwedge_{R \in \mathcal{L}, \vec{x} \in B_i, \vec{y} \in B_j \text{ of } R\text{'s arity}} (\neg\psi'_{e,\vec{x},\vec{y}} \vee [R(f_i(\vec{x})) \leftrightarrow R(f_j(\vec{y}))])$, preservation of relations. The conjunction of these four sentences is $\rho_{e,i,j}$. QED

2.3.3 Finite computable dimension does not relativize

Theorem 2.15 *If $\mathcal{A}$ has finite relative computable dimension, then it has a formally Σ_1^0 Scott family, and therefore has dimension 1 by Theorem 1.13.*

Proof: Assume that $\mathcal{A}$ has relative computable dimension $< n$. We will build generic copies $\mathcal{B}_1, \ldots, \mathcal{B}_n \cong \mathcal{A}$ by forcing conditions, as described in the previous section. Our fragment $\mathcal{F}$ should contain all q.f. Σ_1^c sentences; and $\neg\rho_{e,i,j}$ for each $e \in \omega$ and each $i, j \in \{1, \ldots, n\}$ with $i < j$. If we construct an $\mathcal{F}$-complete sequence $(p_m)_{m\in\omega}$, then the fact that $\mathcal{A}$ has relative computable dimension $< n$ and Lemma 2.14 imply that for some e and $i < j$, $\langle \mathcal{A}, h_1, \ldots, h_n\rangle \models \rho_{e,i,j}$. Therefore, one direction of the Forcing Lemma implies that for some m, $p_m \Vdash \rho_{e,i,j}$. The other direction implies that no matter how we would have extended p_m to an $\mathcal{F}$-complete sequence, we still would have $\langle \mathcal{A}, h_1, \ldots, h_n\rangle \models \rho_{e,i,j}$. Therefore, *any generic copies* $\mathcal{B}_1, \ldots, \mathcal{B}_n$ constructed by a complete forcing sequence extending p_m would have the property that $\varphi_e^{\mathcal{D}(\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_n)} : \mathcal{B}_i \cong \mathcal{B}_j$. We are now in a position to construct the formally Σ_1^0 Scott family.

For the rest of this discussion, we let p denote p_m. (Therefore, any subsequent use of the character m has nothing to do with this index.) Let $p = \langle p^1, \ldots, p^n\rangle$; $\vec{b}_k = dom(p^k)$; and $\vec{a}_k = p^k(\vec{b}_k)$. Fix

1. $\vec{t} = \vec{t}_1\vec{t}_2$ an m-tuple of distinct elements of A with $\vec{t}_1$ an l-tuple from $\vec{a}_j$, and $\vec{t}_2$ an $(m-l)$-tuple disjoint from $\vec{a}_j$;

2. $\vec{u} = \vec{u}_1\vec{u}_2$ an m-tuple of distinct elements of B_i with $\vec{u}_1$ an o-tuple from $\vec{b}_i$ and $\vec{u}_2$ an $(m-o)$-tuple disjoint from $\vec{b}_i$;

3. $\vec{v} = \vec{v}_1\vec{v}_2$ an m-tuple of distinct elements of B_j with $p^j(\vec{v}_1) = \vec{t}_1$ and $\vec{v}_2$ an $(m-l)$-tuple disjoint from $\vec{b}_j$.

Fix $\vec{x}$, an m-tuple of distinct variables. We will construct a formula $\theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{x})$ of $\mathcal{L}$ with the following property:

for m-tuples $\vec{a}$ of distinct elements of $\mathcal{A}$, $\mathcal{A} \models \theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{a})$ iff $\exists q = \langle q^1, \ldots, q^n \rangle \supseteq p$ $[q^i(\vec{u}) = \vec{a}, q^j(\vec{v}) = \vec{t}$, and $q \Vdash \psi'_{e,\vec{u},\vec{v}}]$.

We could include such a q in an $\mathcal{F}$-complete forcing sequence. Then q^i would be a piece of an isomorphism $g_i : \mathcal{B}_i \cong \mathcal{A}$; q^j would be a piece of an isomorphism $g_j : \mathcal{B}_j \cong \mathcal{A}$; and $q \Vdash \rho_{e,i,j}$, since p does. Thus, such a q would "force" the following diagram to hold. (Note that when we include q in our forcing sequence, we have defined only a finite part of each g_k, and thus only a finite part of the structure of each $\mathcal{B}_k$. Nevertheless, the diagram necessarily holds no matter how we complete the construction after q.)

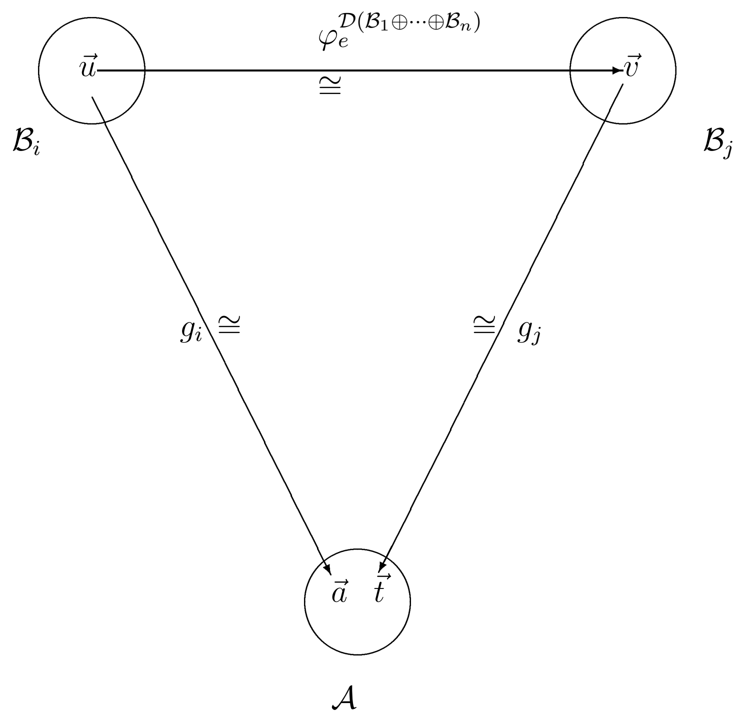

Figure 4.3

By the Transition Lemma, we know that $\psi_{e,\vec{u},\vec{v}}$ is of the form

$\bigvee_{\sigma\in S}\gamma_{\sigma,1}(\vec{b}_1,\vec{b}'_{\sigma,1})\wedge\ldots\wedge\gamma_{\sigma,i}(\vec{b}_i,\vec{u}_2,\vec{b}'_{\sigma,i})\wedge\ldots\wedge\gamma_{\sigma,j}(\vec{b}_j,\vec{v}_2,\vec{b}'_{\sigma,j})\wedge\ldots\wedge\gamma_{\sigma,n}(\vec{b}_n,\vec{b}'_{\sigma,n})$ where

1. $\vec{b}'_{\sigma,k}$ is a tuple of distinct elements of B_k distinct from $\vec{b}_k$ (and from $\vec{u}_2$ when $k=i$, $\vec{v}_2$ when $k=j$); and

2. each $\gamma_{\sigma,k}$ is a conjunction of atomic sentences and negations of atomic sentences of $\mathcal{L}(B_k)$.

The Transition Lemma, Lemma 2.4, and the Forcing Lemma imply that for any generic structures defined by an $\mathcal{F}$-complete sequence, $\varphi_e^{\mathcal{D}(\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_n)}(\vec{u})=\vec{v}$ iff $\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_n\models\psi_{e,\vec{u},\vec{v}}$ iff $\langle\mathcal{A},h_1,\ldots,h_n\rangle\models\psi'_{e,\vec{u},\vec{v}}$ iff for some q in our sequence, $q\Vdash\psi'_{e,\vec{u},\vec{v}}$. Therefore, we consider all $\sigma\in S$ where $\mathcal{A}\models\{\exists\vec{z}_{\sigma,1}[\vec{z}_{\sigma,1}$ distinct and disjoint from $\vec{a}_1$ and $\gamma_{\sigma,1}(\vec{a}_1,\vec{z}_{\sigma,1})]\wedge\ldots\wedge\exists\vec{z}_{\sigma,j}[\vec{z}_{\sigma,j}$ distinct and disjoint from $\vec{a}_j$, $\vec{t}_2$ and $\gamma_{\sigma,j}(\vec{a}_j,\vec{t}_2,\vec{z}_{\sigma,j})]\wedge\ldots\wedge\exists\vec{z}_{\sigma,n}[\vec{z}_{\sigma,n}$ distinct and disjoint from $\vec{a}_n$ and $\gamma_{\sigma,n}(\vec{a}_n,\vec{z}_{\sigma,n})]\}$. Let $S'=\{\sigma\in S:\sigma$ has the above property$\}$. Since $\mathcal{A}$ is computable and S is c.e., S' is c.e.

For an m-tuple $\vec{x}$ of variables, let $\vec{x}_1$ be its first o variables, and $\vec{x}_2$ be the next $(m-o)$ variables. Consider $\theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i,\vec{x})=\bigvee_{\sigma\in S'}[(\vec{x}$ distinct$)\wedge(\vec{x}_1=p^i(\vec{u}_1))\wedge(\vec{x}_2$ disjoint from $\vec{a}_i)\wedge(\exists\vec{y}_\sigma(\vec{y}_\sigma$ distinct and disjoint from $\vec{a}_i$, $\vec{x}_2$ and $\gamma_{\sigma,i}(\vec{a}_i,\vec{x}_2,\vec{y}_\sigma))]$.

No longer view $\vec{t},\vec{u},\vec{v}$ as fixed.
Let $T=\{\langle\vec{t},\vec{u},\vec{v}\rangle$: $\vec{t}\in A,\vec{u}\in B_i,\vec{v}\in B_j$ are as described after the first paragraph of this proof for some $m,l,o\in\mathbb{N}$ with $m\geq l,o,\ \}$.
Let $\Theta=\{\theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i,\vec{x}):\langle\vec{t},\vec{u},\vec{v}\rangle\in T\}$.
Finally, let our proposed Scott family be $\langle\vec{a}_i,\Theta\rangle$.

Claim 1: For each tuple $\vec{a}\in A$ of distinct elements with those of $\vec{a}_i$ (if any) appearing first, there is $\theta_{\vec{t},\vec{u},\vec{v}}\in\Theta$ such that $\mathcal{A}\models\theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i,\vec{a})$.
Sketch of proof: For each $a\in\vec{a}$, $\mathcal{F}$ contains the sentence $\bigvee_{x\in B_i}f_i(x)=a$, and we define the sequence of forcing conditions to be $\mathcal{F}$-complete. Thus, for some $q=$

$\langle q^1, \ldots, q^n \rangle \supseteq p$ and some $\vec{u}$, $q^i(\vec{u}) = \vec{a}$. Since $p \Vdash \rho_{e,i,j}$, we may assume, without loss of generality, that $q \Vdash \psi'_{e,\vec{u},\vec{v}}$ for some $\vec{v}$. Finally, since for each $v \in \vec{v}$, $\mathcal{F}$ contains the sentence $\bigvee\!\!\!\bigvee_{x \in \mathbb{N}} f_j(v) = x$, we may also assume that $q^j(\vec{v}) = \vec{t}$ for some $\vec{t}$.

Claim 2: If for some $\langle \vec{t}, \vec{u}, \vec{v} \rangle \in T$, $\mathcal{A} \models \theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{a})$ and $\mathcal{A} \models \theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{a}\,')$, then $\langle \mathcal{A}, \vec{a} \rangle \cong \langle \mathcal{A}, \vec{a}\,' \rangle$.

Sketch of proof: Fix $\langle \vec{t}, \vec{u}, \vec{v} \rangle \in T$. If $\mathcal{A} \models \theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{a})$, then we can extend p to a forcing condition q with $q^i(\vec{u}) = \vec{a}$, $q \Vdash \psi'_{e,\vec{u},\vec{v}}$, and $q^j(\vec{v}) = \vec{t}$. Furthermore, since $q \supseteq p$ and $p \Vdash \rho_{e,i,j}$, $q \Vdash \rho_{e,i,j}$ by Proposition 2.6 (ii). By Proposition 2.9, we can include q in an $\mathcal{F}$-complete sequence. Hence, we can induce structures $\mathcal{B}_i$ and $\mathcal{B}_j$ with $\langle \mathcal{A}, \vec{a} \rangle \cong \langle \mathcal{B}_i, \vec{u} \rangle \cong \langle \mathcal{B}_j, \vec{v} \rangle \cong \langle \mathcal{A}, \vec{t} \rangle$. If $\mathcal{A} \models \theta_{\vec{t},\vec{u},\vec{v}}(\vec{a}_i, \vec{a}\,')$ as well, then $\langle \mathcal{A}, \vec{a}\,' \rangle \cong \langle \mathcal{A}, \vec{t} \rangle$, so $\langle \mathcal{A}, \vec{a} \rangle \cong \langle \mathcal{A}, \vec{a}\,' \rangle$.

Therefore, we indeed have defined a formally Σ^0_1 Scott family for $\mathcal{A}$. QED

CHAPTER 3

RELATIVIZED Δ_2^0-CATEGORICITY IN BOOLEAN ALGEBRAS AND LINEAR ORDERINGS

3.1 Introduction

Recall that Goncharov and others characterized computable categoricity in many classes of structures, among them Boolean algebras and linear orderings.

Theorem 3.1 *(Goncharov, La Roche, Remmel) A Boolean algebra is computably categorical iff it has finitely many atoms.*

Theorem 3.2 *(Goncharov-Dzgoev, Remmel) A linear ordering is computably categorical iff it has finitely many pairs of direct successors.*

In fact, using Theorem 1.13, we can easily show that these theorems remain true if "computably categorical" is replaced with "relatively computably categorical." In computable model theory there have been several other instances where a priority argument was used to prove a result, and later a forcing argument was used to prove the analogous relativized result. In many cases, the forcing argument often is actually simpler than the priority construction. In this thesis, we examine the notions of Δ_2^0-categoricity and relativized Δ_2^0-categoricity in Boolean algebras and linear orderings by first studying the *relativized* notion. We shall see not only that here the arguments for the relativized notion are again easier than those for the unrelativized notion, but also that the former provide a helpful guide for the latter.

By Theorem 1.13, in order to examine relativized Δ^0_2-categoricity, we must understand the satisfaction of Σ_2 formulas in the structures we examine. In [1], Ash and Knight described back-and-forth relations that give us the necessary tools to characterize the satisfaction of Σ_2 formulas in Boolean algebras and linear orders quite easily.

Definition 3.3 *Let $\mathcal{A}$ be any structure, and $\vec{a}$, $\vec{b}$ tuples of the same length from $\mathcal{A}$. Then $\vec{a} \leq_1 \vec{b}$ iff the Σ_1 formulas true of $\vec{b}$ are true of $\vec{a}$ iff the Π_1 formulas true of $\vec{a}$ are true of $\vec{b}$.*

Proposition 3.4 *Let $\mathcal{A}$ be a Boolean algebra and $\vec{a}$, $\vec{b}$ be tuples from $\mathcal{A}$ so that the Σ_0 formulas true of $\vec{a}$ and $\vec{b}$ are the same. Then $\vec{a} \leq_1 \vec{b}$ iff each atom of the finite algebra determined by $\vec{a}$ is infinite or at least as large as the corresponding atom of $\vec{b}$.*

Proposition 3.5 *Let $\mathcal{A}$ be a linear ordering and $\vec{a}$, $\vec{b}$ be tuples from $\mathcal{A}$ so that the ordering of $\vec{a}$ is the same as that of $\vec{b}$. Then $\vec{a} \leq_1 \vec{b}$ iff each interval in $\mathcal{A}$ determined by $\vec{a}$ is infinite or at least as large as the corresponding interval determined by $\vec{b}$.*

3.2 The relativized results

In order to state our result for Boolean algebras, we must recall the definition of a 1-atom.

Definition 3.6 *Let $\sim$ be the equivalence relation in a Boolean algebra $\mathcal{A}$ where $a \sim b$ iff $a \cap \overline{b}$ and $b \cap \overline{a}$ are both either empty or a union of finitely many atoms of $\mathcal{A}$. The set of equivalence classes $\mathcal{A}/\sim$ is again a Boolean algebra. An element $a \in \mathcal{A}$ is a* 1-atom *of $\mathcal{A}$ iff the equivalence class $[a]$ is an atom of $\mathcal{A}/\sim$.*

Theorem 3.7 *A Boolean algebra $\mathcal{A}$ is relatively Δ^0_2-categorical iff it can be expressed as a finite direct sum $c_1 \vee \cdots \vee c_n$, where each c_i is either atomless, an atom, or a 1-atom.*

Proof: ($\Leftarrow$) If the summands are all either atomless or atoms, then it is Δ^0_1-categorical by Theorem 3.1. If $\mathcal{A}$ has some 1-atoms, then absorb all of the atoms into one of the 1-atoms, so we write $\mathcal{A} = c_1 \vee \cdots \vee c_n$, where c_1 is atomless, and $c_2, \ldots, c_n$ are 1-atoms. Consider the collection of parameters $\vec{c} = c_1, \ldots, c_n$; with these we will construct a c.e. Scott family of Σ^c_2 formulas.

Let $\vec{a} = a_1, \ldots, a_j \in \mathcal{A}$, and let $b_1, \ldots, b_{2^j}$ the atoms in the *formal* finite subalgebra generated by $\vec{a}$. (Some of the b_i's might equal 0.) For each b_i, we construct the following formulas:

1. $\theta^{b_i}_1(y_i, c_1)$ is "$y_i \cap c_1 = 0$" if $b_i \cap c_1 = 0$; "$y_i \cap c_1 = c_1$" if $b_i \cap c_1 = c_1$; or "$y_i \cap c_1 \neq 0 \wedge y_i \cap c_1 \neq c_1$" if $b_i \cap c_1$ is a proper subset of c_1, respectively.

2. For each $k \in \{2, \ldots, n\}$, exactly one of $b_i \cap c_k$, $\overline{b_i} \cap c_k$ is empty or a finite join of atoms (of $\mathcal{A}$), precisely because each c_k is a 1-atom. The formula $\gamma_m(y) = \exists z_1 \cdots z_m[z_1 \neq 0 \wedge \cdots \wedge z_m \neq 0 \wedge z_1 \cup \cdots \cup z_m = y \wedge \forall w(\neg(0 < w < z_1) \wedge \cdots \wedge \neg(0 < w < z_m))]$ expresses that y is a join of m atoms. For each $k \in \{2, \ldots, n\}$, let $n_{b_i,k}$ be the size of $y_i \cap c_k$ or $\overline{y_i} \cap c_k$, and let $\theta^{b_i}_1(y_i, c_k)$ be $\gamma_{n_{b_i,k}}(y_i \cap c_k)$ if $|b_i \cap c_k| = n_{b_i,k}$; or $\gamma_{n_{b_i,k}}(\overline{y_i} \cap c_k)$ if $|\overline{b_i} \cap c_k| = n_{b_i,k}$.

For a tuple of variables $\vec{x} = x_1, \ldots, x_j$, let $\vec{y} = y_1, \ldots, y_{2^j}$ be the terms in the formal finite subalgebra determined by $\vec{x}$. Let $\psi_{\vec{a}}(\vec{x}, \vec{c}) = \bigwedge\!\!\!\bigwedge_{i \in \{1, \ldots, 2^j\}} \theta^{b_i}_1 \wedge \cdots \wedge \theta^{b_i}_n$. Of course, $\mathcal{A} \models \psi_{\vec{a}}(\vec{a}, \vec{c})$. Furthermore, if for some $\vec{a}\,'$, $\mathcal{A} \models \psi_{\vec{a}}(\vec{a}\,', \vec{c})$, then we immediately have $(\mathcal{A}, \vec{a}, \vec{c}) \cong (\mathcal{A}, \vec{a}\,', \vec{c})$. Consequently, $\{\psi_{\vec{a}} | \vec{a} \in \mathcal{A}\}$ is a c.e. Scott family of Σ^c_2 formulas.

($\Rightarrow$) Assume that $\mathcal{A}$ is not of the described form, and let $\vec{c}$ be parameters in a putative formally Σ^0_2 Scott family. Consider the finite subalgebra determined by $\vec{c}$. One of the atoms a of this algebra must be such that

1. a is not a 1-atom; and

2. a contains infinitely many atoms (of $\mathcal{A}$).

Why must this be true? Otherwise, each atom of the subalgebra determined by $\vec{c}$ is either a 1-atom or is a finite join of atoms and an atomless algebra. Therefore, contrary to hypothesis, $\mathcal{A}$ can be written as a finite join of 1-atoms, atoms, and an atomless algebra.

Let a be the disjoint union of a_1, a_2, where a_1 is an infinite Boolean algebra and a_2 contains infinitely many atoms. We claim that for any Σ_2 formula $\gamma(x, \vec{c})$ satisfied by a_1, there is a_1' composed of only finitely many atoms of $\mathcal{A}$ so that a_1' also satisfies γ. Since obviously $(\mathcal{A}, a_1, \vec{c}) \not\cong (\mathcal{A}, a_1', \vec{c})$, $\mathcal{A}$ cannot have the supposed Scott family.

We can assume, without loss of generality, that the formula γ is of the form $\exists \vec{u} \delta(x, \vec{u}, \vec{c})$, where δ is the conjunction of *all* finitary Π_1 formulas satisfied by $a_1, \vec{c}$, and some $\vec{u}$. Consequently, we must show that, given a tuple $\vec{u}$, there are $a_1', \vec{u}\,'$ so that *every* Π_1 formula satisfied by $a_1, \vec{u}, \vec{c}$ is satisfied by $a_1', \vec{u}\,', \vec{c}$, or that $(a_1, \vec{u}, \vec{c}) \leq_1 (a_1', \vec{u}\,', \vec{c})$. By Proposition 3.4 we must show that there are $a_1', \vec{u}\,'$ so that each atom of the finite algebra determined by $a_1, \vec{u}, \vec{c}$ is infinite or at least as big as the corresponding atom determined by $a_1', \vec{u}\,', \vec{c}$.

Consider the finite subalgebra determined by $a_1, \vec{u}, \vec{c}$; let its atoms be $z_1, \ldots, z_m$. For each z_i with $z_i \cap a = 0$, let $z_i' = z_i$. In this subalgebra, the element a_1 is divided into $z_{j_1}, \ldots, z_{j_s}$ and a_2 into $z_{k_1}, \ldots, z_{k_t}$. One of the z_k's must be infinite; without loss of generality, let it be z_{k_1}. In a_2 find atoms (of $\mathcal{A}$) $z_{j_1}', \ldots, z_{j_s}', z_{k_2}', \ldots, z_{k_t}'$; and let $z_{k_1}' = a - [\bigcup_{p \in \{1,\ldots,s\}} z_{j_p}' \cup \bigcup_{q \in \{2,\ldots,t\}} z_{k_q}']$. Define $a_1', \vec{u}\,'$, and $\vec{c}\,'$ in the subalgebra generated by the z' elements in the same way that $a_1, \vec{u}$, and $\vec{c}$ are generated by the z elements. For instance, $a_1' = \bigcup_{p \in \{1,\ldots,s\}} z_{j_p}'$, so a_1' is a finite join of atoms of $\mathcal{A}$. Note that $\vec{c}\,' = \vec{c}$, since $\bigcup_{p \in \{1,\ldots,s\}} z_{j_p}' \cup \bigcup_{q \in \{1,\ldots,t\}} z_{k_q}' = \bigcup_{p \in \{1,\ldots,s\}} z_{j_p} \cup \bigcup_{q \in \{1,\ldots,t\}} z_{k_q} = a$, and for each z_i with $z_i \cap a = 0$, we let $z_i' = z_i$. Notice that for each $i = 1, \ldots, m$, z_i

is a Boolean algebra of size at least as large as z_i', and that a_1' is finite. Thus we have proven our claim. QED

Throughout the rest of the chapter, we will assume that a linear ordering $\mathcal{A}$ has a greatest and a least element; this assumption makes the statements of certain results simpler. However, it should be clear that a linear ordering without a greatest or least element is (relatively) Δ_2^0-categorical iff the same ordering with a greatest and least element "attached" is. Hence, our assumption is merely one of convenience.

Theorem 3.8 *Let $\mathcal{A} = (A, <_A)$ be a sum of finitely many intervals, each of type n, ω, ω^*, $\mathbb{Z}$, or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and infimum. Then $\mathcal{A}$ is relatively Δ_2^0-categorical.*

Proof: We use Theorem 1.13. For each interval of type ω or ω^*, name the "0"; for interval of type $\mathbb{Z}$, pick a single element and name it; for each interval of type $n \cdot \eta$, name the supremum and infimum; finally, name all of the remaining elements, of which there are a finite number. These will be the parameters appearing in our Scott family.

Let $\vec{a} = a_1, \ldots, a_k \in A$. We explicitly construct a formula $\gamma_{\vec{a}}(x_1, \ldots, x_k, \vec{c})$ such that $\mathcal{A} \models \gamma_{\vec{a}}(a_1, \ldots, a_k, \vec{c})$ and if $\mathcal{A} \models \gamma_{\vec{a}}(b_1, \ldots, b_k, \vec{c})$, then
$(\mathcal{A}, a_1, \ldots, a_k, \vec{c}) \cong (\mathcal{A}, b_1, \ldots, b_k, \vec{c})$.

First, express the ordering of $\vec{a}, \vec{c}$ in a formula $\theta(\vec{x}, \vec{c})$. We may assume without loss of generality that $a_1, \ldots, a_k$ are listed in increasing order. Next, let j_1 be the least number and j_2 the greatest number so that $1 \leq j_1 \leq j_2$, $a_{j_1}, \ldots, a_{j_2}$ all fall in the same interval of $\mathcal{A}$ of the types described above, and neither a_{j_1} nor a_{j_2} is an element of $\vec{c}$. We describe how to find a formula $\psi(x_{j_1}, \ldots, x_{j_2}, \vec{c})$ which characterizes $a_{j_1}, \ldots, a_{j_2}$ up to isomorphism. The formula $\gamma_{\vec{a}}$ will be a conjunction of such formulas.

If $a_{j_1}, \ldots a_{j_2}$ lie in the same interval of type ω with $c =$ "0", then $a_{j_1} =$ "p_1," $\ldots, a_{j_2} =$ "$p_{j_2 - j_1 + 1}$" in this copy of ω. Therefore, a_{j_1} satisfies the formula $\psi_1(x_1, c) =$

$\exists y_1 \cdots y_{p_1-1}[c < y_1 < \cdots < y_{p_1-1} < x_1 \wedge \forall z(\neg(c < z < y_1) \wedge \ldots \neg(y_{p_1-1} < z < x_1))]$. We have similar formulas $\psi_2(x_2, c), \ldots, \psi_{j_2-j_1+1}(x_{j_2-j_1+1}, c)$. Let $\psi(x_1, \ldots, x_{j_2-j_1+1}, c)$ be the conjunction.

If $a_{j_1}, \ldots, a_{j_2}$ lie in an interval of type ω^* or $\mathbb{Z}$, we similarly obtain a formula $\psi(x_1, \ldots, x_{j_2-j_1+1}, c)$, where c is the named element of the interval.

Assume $a_{j_1}, \ldots, a_{j_2}$ lie in the same interval of type $n \cdot \eta$ with $a_{j_1}, \ldots, a_{m_1}$ in the same n interval of $n \cdot \eta$; $a_{m_1+1}, \ldots, a_{m_2}$ in the same n interval that is different from a_1's; $\ldots$; $a_{m_t+1}, \ldots, a_{j_2}$ in the same n interval that is different from the t previous n intervals. For instance, consider the case where $n = 5$, $m_1 = j_1 + 1$, $a_{j_1} =$ "1," and $a_{m_1} =$ "3." Then a_{j_1}, a_{m_1} satisfy the formula $\psi_1(x_1, x_2) = \exists y_0 y_1 y_2[(y_0 < x_1 < y_1 < x_2 < y_2) \wedge \forall z(\neg(y_0 < z < x_1) \wedge \cdots \neg(x_2 < z < y_2))]$. We have similar formulas $\psi_2(x_{m_1+1}, \ldots, x_{m_2}), \ldots, \psi_{t+1}(x_{m_t+1}, \ldots, x_{j_2})$. Furthermore, a_{m_1} and a_{m_1+1} satisfy the formula $\rho_1(x_{m_1}, x_{m_1+1}) = \exists y_0 \cdots y_n(x_{m_1} < y_0 < \cdots < y_n < x_{m_1+1})$. Similarly, there are formulas $\rho_2, \ldots, \rho_t$. Let $\psi(x_1, \ldots, x_j)$ be the conjunction of all of $\psi_1, \ldots, \psi_{t+1}$, $\rho_1, \ldots, \rho_t$.

By design, $\mathcal{A} \models \gamma_{\vec{a}}(a_1, \ldots, a_k, \vec{c})$. If $\mathcal{A} \models \gamma_{\vec{a}}(b_1, \ldots, b_k, \vec{c})$, then the ordering of $\vec{b}$, $\vec{c}$ is the same as that of $\vec{a}$, $\vec{c}$. Furthermore, because of the way in which we chose parameters, the distribution among the intervals of $\mathcal{A}$ with named parameters must be the same for $\vec{a}$ and $\vec{b}$. Finally for $a_p, \ldots, a_q$ and $b_p, \ldots, b_q$ all falling in the same interval with a named parameter, we can draw the following conclusions:

1. If they all fall in an interval of type ω, ω^*, or $\mathbb{Z}$, then $\gamma_{\vec{a}}$ actually guarantees that $a_p = b_p, \ldots, a_q = b_q$.

2. If they all fall in an interval of type $n \cdot \eta$, then $\gamma_{\vec{a}}$ guarantees that

   (a) for all u, v with $p \leq u, v \leq q$, a_u and a_v fall in the same interval of type n exactly if b_u and b_v do;

(b) the position within the intervals of type n of a_u and b_u are the same;

(c) the ordering of $a_p, \ldots, a_q$ matches that of $b_p, \ldots, b_q$.

Therefore, the ordering properties of η imply that in the second case, there is an automorphism of this entire interval taking $a_p, \ldots, a_q$ onto $b_p, \ldots, b_q$. It follows that $(\mathcal{A}, \vec{a}, \vec{c}) \cong (\mathcal{A}, \vec{b}, \vec{c})$. Hence, it is clear that the set $\{\gamma_{\vec{a}} | \vec{a}$ is a tuple from $\mathcal{A}\}$ is a formally Σ_2^0 Scott family. QED

We now give a proof of the converse of Theorem 3.8.

Theorem 3.9 *Let $\mathcal{A} = (A, <_A)$ be a relatively Δ_2^0-categorical linear ordering. Then $\mathcal{A}$ is a sum of finitely many intervals, each of type n, ω, ω^*, $\mathbb{Z}$, or $n \cdot \eta$, so that each of the $n \cdot \eta$ interval has a supremum and infimum.*

We prove this theorem by dividing it into several lemmas and smaller propositions.

Definition 3.10 *Let $\mathcal{A}$ be a linear ordering. A* maximal interval with property P *is an interval I of $\mathcal{A}$ so that*

1. *I has property P; and*

2. *If a is any other element of $\mathcal{A}$, then $I \cup \{a\}$ is not contained in an interval with property P.*

We will often use expressions such as "maximal ω-interval" to denote a maximal interval of order type ω.

Proposition 3.11 *A relatively Δ_2^0-categorical linear ordering $\mathcal{A}$ must have only finitely many maximal intervals of order type $\omega, \omega^*, \mathbb{Z}$, and cannot have arbitrarily large maximal finite intervals.*

Proof: Assume that $\mathcal{A}$ either has infinitely many maximal intervals of one of the above order types, or has arbitrarily large maximal finite intervals. Let $\vec{c} = c_0, \ldots, c_n$ be the parameters in a putative formally Σ_2 Scott family, and let $\mathcal{A} = I_0 + c_0 + I_1 + \cdots + I_n + c_n + I_{n+1}$. For some $k = 0, \ldots, n+1$, I_k contains infinitely many maximal intervals of one of the above order types, or has arbitrarily large maximal finite intervals. (For simplicity, we write I for this I_k. There are $a_1 < a_2$ in I so that

1. there are infinitely many elements in I to the right of a_1;

2. there are infinitely many elements between a_1 and a_2; and

3. there are infinitely many elements in I to the left of a_2.

We claim that for any Σ_2 formula $\gamma(x_1, x_2, \vec{c})$ satisfied by a_1, a_2, there exist a_1', a_2' so that a_1', a_2' also satisfy γ but have only finitely many elements of $\mathcal{A}$ between them. Since obviously $(\mathcal{A}, a_1, a_2, \vec{c}) \not\cong (\mathcal{A}, a_1', a_2', \vec{c})$, $\mathcal{A}$ cannot have the supposed Scott family.

We can assume, without loss of generality, that the formula γ is of the form $\exists \vec{u} \delta(x_1, x_2, \vec{u}, \vec{c})$, where δ is the conjunction of *all* finitary Π_1 formulas satisfied by a_1, a_2, some $\vec{u}$, and $\vec{c}$. Consequently, we must show that, given a tuple $\vec{u}$, there are $a_1', a_2', \vec{u}\,'$ so that *every* Π_1 formula satisfied by $a_1, a_2, \vec{u}, \vec{c}$ is satisfied by $a_1', a_2', \vec{u}\,', \vec{c}$, or that $(a_1, a_2, \vec{u}, \vec{c}) \leq_1 (a_1', a_2', \vec{u}\,', \vec{c})$. Therefore, by Proposition 3.5, in order to verify our claim, we must demonstrate the following

> for any $\vec{u}$ containing the parameters $\vec{c}$, there exist $a_1', a_2', \vec{u}\,'$ so that $u_i' = u_i$ for $u_i \in \vec{c}$, and the intervals determined by $a_1', a_2', \vec{u}'$ are of size no greater than the corresponding intervals of $a_1, a_2, \vec{u}$.

Assume, without loss of generality, that $\vec{u} = u_1, \ldots, u_n$ is arranged in increasing order, and that the ordering is as follows:

1. $u_1, \ldots, u_{j_1}$ lie to the left of I;

2. $u_{j_1+1}, \dots, u_{j_2}$ lie in the interval I and to the left of a_1, but they are in the same successor chain as an element of $\vec{c}$;

3. $u_{j_2+1}, \dots, u_{j_3}$ lie in the interval I and to the left a_1, and they are not in the same successor chain as an element of $\vec{c}$;

4. $a_1 < u_{j_3+1}, \dots, u_{j_4} < a_2$;

5. $u_{j_4+1}, \dots, u_{j_5}$ lie in the interval I and to the right of a_2, and they are not in the same successor chain as an element of $\vec{c}$;

6. $u_{j_5+1}, \dots, u_{j_6}$ lie in the interval I and to the right of a_2, but they are in the same successor chain as an element of $\vec{c}$;

7. $u_{j_6+1}, \dots, u_n$ lie to the right of the interval I.

Note that the intervals (u_{j_2}, u_{j_2+1}) and (u_{j_5}, u_{j_5+1}) are infinite.

Find a successor chain (not necessarily a maximal successor chain) in I of length $n+2$ so that no element in this chain is in a successor chain with any element of $\vec{c}$. Define $a_1', a_2', \vec{u}\,'$ as follows:

1. for $i \leq j_2$, $u_i' = u_i$;

2. $u_{j_2+1}', \dots, u_{j_3}', a_1', u_{j_3+1}', \dots, u_{j_4}', a_2', u_{j_4+1}', \dots, u_{j_5}'$ is a sequence of successors in the chain of length $n+2$;

3. for $i \geq j_{5+1}$, $u_i' = u_i$.

The desired property of $a_1', a_2'\vec{u}\,'$ follows immediately from the fact we noted above.

QED

Lemma 3.12 *If $\mathcal{B}$ is a countable linear ordering containing no ω, ω^*, or $\mathbb{Z}$ intervals, then $\mathcal{B}$ is of the form $n_1 + \Sigma_{q\in\eta} B_q + n_2$, where n_1 or n_2 could be 0 and each B_q is finite with $|B_q| \geq 1$.*

Proof: If $\mathcal{B}$ has a least element a, then we know that either a doesn't have an immediate successor, or a^+ doesn't, or a^{++} doesn't, etc., because $\mathcal{B}$ doesn't contain an ω interval. Therefore, $\mathcal{B}$ has an initial maximal finite discrete interval of order type n_1. (Of course, $n_1 = 0$ if $\mathcal{B}$ has no least element.) Similarly $\mathcal{B}$ has a terminal maximal finite discrete interval of order type n_2. (Of course, $n_2 = 0$ if $\mathcal{B}$ has no greatest element.) We show that $\mathcal{B}$ has the form $n_1 + \mathcal{B}' + n_2$, where $\mathcal{B}'$ has the form $\Sigma_{q\in\eta} B_q$.

Let $c_1 \in \mathcal{B}'$. By repeatedly applying the successor and predecessor function we can obtain only a finite interval $\{c_{1,0}, \ldots, c_{1,n_1}\}$. Let this finite interval be B_{q_1} for some $q_1 \in \eta$. Note that $c_{1,0}$ must have no immediate predecessor and c_{1,n_1} must have no immediate successor. Furthermore, $c_{1,0}$ is greater than any element of n_1 and c_{1,n_1} is less than any element of n_2.

Let $q_2 \in \eta - \{q_1\}$. If $q_2 < q_1$, then find some c_2 greater than any element of n_1 and less than $c_{1,0}$. Repeatedly apply the successor and predecessor function to obtain a finite interval $\{c_{2,0}, \ldots, c_{2,n_2}\}$. Let this finite interval be B_{q_2}. Note that $c_{2,0}$ must have no immediate predecessor and c_{2,n_2} must have no immediate successor. Furthermore, $c_{2,0}$ is greater than all the elements of n_1 and $c_{2,n_2} < c_{1,0}$. We make the appropriate changes to our construction and our argument if $q_2 > q_1$.

We can continue this back-and-forth argument to construct a $\Sigma_{q\in\eta} B_q \cong \mathcal{B}'$. Consequently, $\mathcal{B}$ is of the form $n_1 + \Sigma_{q\in\eta} B_q + n_2$. QED

Convention Hereafter, when we write $\Sigma_{q\in\eta} B_q$, we assume that each B_q is finite with $|B_q| \geq 1$.

Corollary 3.13 *If $\mathcal{A}$ is a relatively Δ^0_2-categorical linear ordering, then there is an n so that $\mathcal{A}$ is a finite sum of intervals, each of the form $\omega, \omega^*, \mathbb{Z}, m, \Sigma_{q\in\eta}B_q$, where each $|B_q| < n$.*

Proposition 3.14 *If $\mathcal{A}$ is a relatively Δ^0_2-categorical linear ordering (with endpoints), then any maximal interval of the form $\Sigma_{q\in\eta}B_q$ has an infimum and a supremum.*

Proof: We prove the fact for infimums; the argument for supremums is similar. Assume that $\mathcal{A}$ is a relatively Δ^0_2-categorical linear ordering with $\vec{c}$ the parameters in a putative formally Σ_2 Scott family. Let $\mathcal{A}$ have a maximal interval P of the form $\Sigma_{q\in\eta}B_q$. By Corollary 3.13, there must be an interval directly to the left of one of the following forms:

1. finite;

2. ω^*;

3. ω.

In (1) and (2), P has an infimum. We give the argument for (3). Fix $a \in P$ so a has no immediate successor, and there are no members $\vec{c}$ in P to the left of a. We claim that for any Σ_2 formula $\gamma(x, \vec{c})$ satisfied by a, there exists a' in the adjacent ω-interval so that a' also satisfies $\gamma(x, \vec{c})$. Again we must show that

> for any $\vec{u}$ containing $\vec{c}$, there exist $a', \vec{u}\,'$ so that $u'_i = u_i$ for $u_i \in \vec{c}$, and the intervals determined by a', $\vec{u}\,'$ are of size no greater than the corresponding intervals of $a, \vec{u}$.

Assume, without loss of generality, that $\vec{u} = u_1, \ldots, u_n$ is in increasing order, and that the ordering is as follows:

1. $u_1, \ldots, u_{j_1}$ lie to the left of the ω-interval;

2. $u_{j_1+1}, \ldots, u_{j_2}$ lie in the ω-interval;

3. $u_{j_2+1}, \ldots, u_{j_3}$ lie to the left of a in P;

4. $a < u_{j_3+1}, \ldots, u_{j_n}$.

Note that (a, u_{j_3+1}) is infinite. Define $a', \vec{u}\,'$ as follows, where "$x+i$" is the i^{th} successor of x in the ω-interval:

1. for $1 \leq i \leq j_2$, $u'_i = u_i$;

2. for $1 \leq i \leq j_3 - j_2$, $u'_{j_2+i} = u_{j_2} + i$;

3. $a' = u'_{j_3} + 1$;

4. for $j_3 + 1 \leq i \leq n$, $u'_i = u_i$.

The desired property of $a'\vec{u}\,'$ follows immediately from the fact we noted above.

QED

Definition 3.15 *A* finite partition *of η is a finite sequence $q_1 < \ldots < q_j \in \eta$ and open intervals $J_1, \ldots, J_{j+1}$ so that $\eta = J_1 + q_1 + J_2 + \ldots + q_j + J_{j+1}$. A partition P_1 is* finer *than a partition P_2 exactly if every element in the finite sequence for P_2 is in the sequence for P_1.*

Definition 3.16 *Let $\mathcal{B}$ be a linear ordering of the form $\Sigma_{q\in\eta} B_q$.*
*1. $\mathcal{B}$ satisfies **Property 1** iff there is a finite partition of η so that for each open interval J_i in the partition there is n_i so that for all $q \in J_i$, B_q has order type n_i. That is, there are $n_1, n_2, \ldots, n_m \geq 1$ so that $\mathcal{B} = n_1 \cdot \eta + n_2 + n_3 \cdot \eta + \ldots + n_m \cdot \eta$.*
*2. $\mathcal{B}$ satisfies **Property 2** iff there exist $m_1 \neq m_2$ so that for any finite partition of η there is an open interval J in the partition so that for infinitely many $q \in J$, B_q is of order type m_1 and for infinitely many $q' \in J$, $B_{q'}$ is of order type m_2.*

Lemma 3.17 *Fix a linear ordering $\mathcal{B}$ of the form $\Sigma_{q\in\eta}B_q$ with n as in Corollary 3.13. If $\mathcal{B}$ does not satisfy Property 2, then it satisfies Property 1.*

Proof: Let $m_0, \ldots, m_k$ be the complete list of natural numbers so that for each $0 \leq i \leq k$, there is $q \in \eta$ with B_q of order type m_i. For each $m_i \neq m_j$, there is a finite partition of η so that no interval of the partition contains infinitely many copies of m_i and infinitely many copies of m_j. Choose a *finite* partition of η finer than all of these partitions. Now in each open interval of the partition, there exists only one m_i so that B_q is of order type m_i for infinitely many q in that interval. Make the partition finer to obtain finite partition that satisfies the requirements in Property 1.

QED

Proposition 3.18 *If $\mathcal{A}$ is a relatively Δ^0_2-categorical linear order, then no interval of the form $\Sigma_{q\in\eta}B_q$ satisfies Property 2.*

Proof: Assume that $\mathcal{A}$ does have an interval of the form $\Sigma_{q\in\eta}B_q$ with m_1 and m_2 as in Property 2, and let $\vec{c}$ be the parameters in a putative c.e. Scott family of Σ^c_2 formulas. Let m_1 be the largest number for which there is such an m_2. Define a finite partition of η so that for each $m > m_1$ and each open interval J of the partition

1. for all $q \in J$, B_q has type m; or for all $q \in J$, B_q does not have type m; and
2. for all $q \in J$, no parameter appears in B_q.

For one of these open intervals J, there are infinitely many $q \in J$ with B_q of type m_1, and there are infinitely many $q' \in J$ with $B_{q'}$ of type m_2. Consider $q_1 \in J$ and $a \in A$ with $a \in B_{q_1}$ of type m_2. We claim that for any Σ_2 formula $\gamma(x, \vec{c})$ satisfied by a, there exists $q_1' \in J$ and $a' \in A$ so that $a' \in B_{q_1'}$ of type m_1, and a' satisfies $\gamma(x, \vec{c})$. Again, we must show that

for any $\vec{u}$ containing the parameters $\vec{c}$ there exist $a', \vec{u}\,'$ so that $u_i' = u_i$ for

$c_i \in \vec{c}$ and the intervals determined by $a', \vec{u}\,'$ are of size no greater than the corresponding intervals of $a, \vec{u}$.

Assume, without loss of generality, that $\vec{u} = u_1, \ldots, u_n$ is in increasing order, and that the ordering is as follows:

1. for all $q \in J$ and all i with $1 \leq i \leq j_1$ and all $b \in B_q$, $u_i < b$;

2. there exists $r_1 < r_2 < \cdots < r_k < q_1$ in J so that $u_{j_1+1}, \ldots, u_{j_2} \in B_{r_1}$; $u_{j_2+1}, \ldots, u_{j_3} \in B_{r_2}$; $\ldots$; $u_{j_k+1}, \ldots, u_{j_{k+1}} \in B_{r_k}$;

3. $u_{j_{k+1}+1}, \ldots, u_{j_{k+2}}$ lie to the left of a in B_{q_1};

4. $u_{j_{k+2}+1}, \ldots, u_{j_{k+3}}$ lie to the right of a in B_{q_1};

5. there exist $q_1 < s_1 < s_2 < \cdots < s_t$ in J so that $u_{j_{k+3}+1}, \ldots, u_{j_{k+4}} \in B_{s_1}$; $u_{j_{k+4}+1}, \ldots, u_{j_{k+5}} \in B_{s_2}$; $\ldots$; $u_{j_{k+t+2}+1}, \ldots, u_{j_{k+t+3}} \in B_{s_t}$;

6. for all $q \in J$ and all i with $j_{k+t+3} < i < n$ and all $b \in B_q$, $b < u_i$.

Note that there are infinitely many elements between u_{j_1} and u_{j_1+1}; between u_{j_2} and u_{j_2+1}; $\ldots$; between $u_{j_{k+1}}$ and $u_{j_{k+1}+1}$; between $u_{j_{k+3}}$ and $u_{j_{k+3}+1}$; $\ldots$; and between $u_{j_{k+t+3}}$ and $u_{j_{k+t+3}+1}$. Furthermore, note that none of the intervals B_q, B_r, or B_s has more than m_1 elements, because of the construction of our partition.

Now find $r'_1 < r'_2 < \cdots < r'_k < q'_1 < s'_1 < \cdots < s'_t$ in J so that $B_{q'_1}$ has type m_1 and each $B_{r'}, B_{s'}$ has type m_1. Define $a', \vec{u}\,'$ as follows:

1. for $i \leq j_1$, $u'_i = u_i$;

2. for $1 \leq i \leq k$, $u'_{j_i+1}, \ldots, u'_{j_{i+1}}$ are ordered in $B_{r'_i}$ exactly as $u_{j_i+1}, \ldots, u_{j_{i+1}}$ are ordered in B_{r_i};

3. $u'_{j_{k+1}+1}, \ldots, u'_{j_{k+2}}, a', u'_{j_{k+2}+1}, \ldots, u'_{j_{k+3}}$ are ordered in $B_{q'_1}$ exactly as $u_{j_{k+1}+1}, \ldots, u_{j_{k+2}}, a, u_{j_{k+2}+}$ are ordered in B_{q_1};

4. for $1 \leq i \leq t$ $u'_{j_{k+2+i}+1}, \ldots, u'_{j_{k+3+i}}$ are ordered in $B_{s'_i}$ exactly as $u_{j_{k+2+i}+1}, \ldots, u_{j_{k+3+i}}$ are ordered in B_{s_i};

5. for $j_{k+t+3} < i < n$, $u'_i = u_i$.

The desired property of $a', \vec{u}\,'$ follows immediately from the facts we noted above.

QED

Lemma 3.17 immediately implies the following corollary.

Corollary 3.19 *If $\mathcal{A}$ is a relatively Δ^0_2-categorical linear order, then any interval of the form $\Sigma_{q\in\eta} B_q$ satisfies Property 1.*

We now complete the proof of Theorem 3.9. Assume that $\mathcal{A}$ is relatively Δ^0_2-categorical. By Corollary 3.13 and Proposition 3.14, $\mathcal{A}$ can be written as a finite sum of intervals, each of the form n, ω, ω^*, $\mathbb{Z}$, $\Sigma_{q\in\eta} B_q$, so that each interval of the form $\Sigma_{q\in\eta} B_q$ has a supremum and infimum. By Corollary 3.19, for each interval of the form $\Sigma_{q\in\eta} B_q$ there are $n_1, n_2, \ldots, n_m \geq 1$ so that $\Sigma_{q\in\eta} B_q = n_1 \cdot \eta + n_2 + n_3 \cdot \eta + \ldots + n_m \cdot \eta$.

QED

We note that our characterization of Δ^0_2-categorical linear orderings is exactly the same as Michael Moses' characterization of computably categorical 1-decidable linear orderings in [22].

Theorem 3.20 *For a 1-decidable linear ordering $\mathcal{A}$ (with greatest and least element) the following are equivalent:*

1. *Every 1-decidable linear ordering isomorphic to $\mathcal{A}$ is isomorphic by a computable isomorphism;*

2. *$\mathcal{A}$ is of the form given in Theorem 3.9.*

The "reason" for this correlation has not yet been established. Some further observations will appear in Chapter 5.

CHAPTER 4

UNRELATIVIZED Δ^0_2-CATEGORICITY

4.1 Δ^0_2-categoricity in Boolean algebras

In this chapter we offer priority constructions that give, under some extra assumptions, results on Δ^0_2-categoricity (unrelativized) for Boolean algebras and linear orderings. We shall see that with these assumptions, the realtivized and unrelativized notions define the same class of structures. Furthermore, we shall attempt to highlight where the syntactic arguments from the previous chapter provide some insight into the organization, strategies, and hypotheses needed for the priority constructions.

Notation Let $\mathcal{A}$ be a Boolean algebra.
$i)$ the unary predicate $Atom(x)$ denotes that x is an atom of $\mathcal{A}$; i.e.,
$(\mathcal{A}, Atom) \models Atom(x)$ iff $\mathcal{A} \models x \neq 0 \wedge \neg(\exists z \exists y(z \neq x \wedge y \neq x \wedge z \cup y = x))$;
$ii)$ the unary predicate $Atomless(x)$ denotes that x contains no atoms;
i.e., $(\mathcal{A}, Atom, Atomless) \models Atomless(x)$ iff $(\mathcal{A}, Atom) \models \forall z[\exists y(y \cup z = x) \rightarrow \neg Atom(z)]$.

4.1.1 Our first priority construction

Theorem 4.1 *Let $\mathcal{A}$ be a Boolean algebra so that $(\mathcal{A}, Atom, Atomless)$ is a computable structure. If $\mathcal{A}$ is Δ^0_2-categorical, then it is a finite direct sum of atoms, 1-atoms, and an atomless algebra, then $\mathcal{A}$.*

Proof: Assume $\mathcal{A}$ is not of the described form. We attempt to construct a computable $\mathcal{B} \cong \mathcal{A}$ so that there is no Δ^0_2 isomorphism between them. We use $\varphi_e^{\Delta^0_2}$ to denote the e^{th} Δ^0_2 function, and $\varphi_{e,s}^{\Delta^0_2}$ to denote the computable approximation of $\varphi_e^{\Delta^0_2}$ at stage s. (Recall that the limit lemma tells us that any total Δ^0_2 function is the pointwise limit of computable approximation functions.)

Requirements and the true path

We employ a tree construction where each node has two outcomes, I (inactive) and A (active), with $I < A$. Nodes of length e work on requirement

$$R_e : \varphi_e^{\Delta^0_2} \text{ is not an isomorphism from } \mathcal{A} \text{ onto } \mathcal{B}.$$

Moreover, the construction must determine an isomorphism $g : \mathcal{B} \cong \mathcal{A}$.

At stage s we inductively define δ_s, a node of length $s-1$ approximating the **true path** f through the tree. If $\alpha \subset \delta_s$, then α decides its outcome o ($o = I$ or $o = A$) by determining if $\varphi_e^{\Delta^0_2}$ threatens to be an isomorphism. In addition, $\alpha \subset \delta_s$ defines $g_{\alpha,s}$, its contribution to the stage s approximation of the function g, so that $|\alpha|$ is in the domain and range. The construction will ensure that for $\beta \subseteq \alpha \subset \delta_s$, $g_{\beta,s} \subseteq g_{\alpha,s}$; therefore, δ_s defines the stage s approximation $g_s \supseteq \bigcup_{\alpha \subset \delta_s} g_{\alpha,s}$. (Note: at stage s, a node $\alpha \subset \delta_s$ will often first determine $g'_{\alpha,s}$, a *preliminary* version of $g_{\alpha,s}$, then decide its outcome, and finally determine $g_{\alpha,s}$ itself.)

Finally, at the end of stage s, we use g_s to *fix* a Boolean algebra $\mathcal{B}_s$ containing s atoms and containing the elements $0, 1, \ldots 2^s - 1$. Since $\mathcal{B}$ must be computable, all of the relations true among the elements of $\mathcal{B}_s$ must be true in $\mathcal{B}_{s+1}$.

The true path will be defined in the standard way for $0''$ constructions: $\alpha \subset f$ iff α is the left-most node of length $|\alpha|$ so that $\alpha \subset \delta_s$ for infinitely many s. We must show that nodes along the true path succeed in defining an isomorphism $g : \mathcal{B} \cong \mathcal{A}$ and in meeting the requirements R_e.

Challenging pairs and candidate pairs

Consider $\alpha \subset \delta_s$ working on requirement R_e. If $\alpha \supset \lambda$, the empty string, then it receives from its predecessor node β the function $g_{\beta,s}$ that it believes is a correct approximation of g, because all requirements R_i with $i < e$ seem to be met. Let $\vec{d}$ be the domain of $g_{\beta,s}$ with $g_{\beta,s}(\vec{d}) = \vec{c}$. If $\alpha = \lambda$, then there is no β, and $\vec{c} = 0_{\mathcal{A}}, 1_{\mathcal{A}}$.

The node α considers the finite algebra determined by $\vec{c}$ and attempts to determine the first pair $\langle a^1_{\alpha,s}, a^2_{\alpha,s}\rangle$ so that $a^1_{\alpha,s}$ seems infinite, $a^2_{\alpha,s}$ seems to have infinitely atoms, and for some atom a in the finite subalgebra determined by $\vec{c}$, $a^1_{\alpha,s} \cup a^2_{\alpha,s} = a$; α includes $a^1_{\alpha,s}, a^2_{\alpha,s}$ in the range of $g_{\alpha,s}$. Consider a', the atom corresponding to a in the subalgebra determined by the preimage of $\vec{c}$ under $g_{\alpha,s} \circ \varphi^{\Delta^0_2}_{e,s}$. The pair $\langle a^1_{\alpha,s}, a^2_{\alpha,s}\rangle$ challenges this a' to contain elements $a^1_{\alpha,s}{}', a^2_{\alpha,s}{}'$ which have the same properties that $a^1_{\alpha,s}, a^2_{\alpha,s}$ seem to have. Hence we call it a **challenging pair**. A pair $a^1_{\alpha,s}{}', a^2_{\alpha,s}{}'$ which seems to meet the challenge is called a **candidate pair**.

We should note that α may make incorrect guesses about which pairs have the above properties, because it cannot computably determine which elements are infinite or have infinitely many atoms. However, the properties of $\mathcal{A}$ do guarantee that such a pair indeed exists, and eventually a node α along the true path will determine its final challenging pair. Similarly, a pair may appear to be a candidate pair until the enumeration of the diagram of $\langle \mathcal{A}, Atom, Atomless\rangle$ reveals otherwise.

If α is along the true path and its ultimate challenge is never actually met, then the final outcome of α is I. If a true candidate pair $a^1_{\alpha,s}{}', a^2_{\alpha,s}{}'$ is found at stage t, then α attempts to change its approximation so that the image under $g_{\alpha,t} \circ \varphi^{\Delta^0_2}_{e,t}$ of $a^1_{\alpha,s}{}'$ is finite; we thereby meet requirement R_e.

Relationships among nodes

Throughout the construction, a node $\alpha \subset \delta_s$ may **initialize** another node γ; i.e., if $\gamma \subseteq \delta_t$ for some $t < s$, then α cancels the assignment of γ's challenging pair and the approximation g_t. There will be three circumstances when a node will initialize another:

1. if α redefines its challenging pair at stage s, then α initializes each γ with $\alpha \subset \gamma$;
2. if $\alpha \subset \delta_s$ determines its outcome to be I, then α initializes all γ with $\alpha\hat{}\langle I\rangle <_L \gamma$;
3. if α diagonalizes against $\varphi_{e,s}^{\Delta_2^0}$ at stage s, then α initializes all γ with $\alpha\hat{}\langle A\rangle \subseteq \gamma$.

In turn, as α defines $g_{\alpha,s}$, it must respect nodes $\sigma <_L \alpha\hat{}\langle o\rangle$. What exactly does "respect" mean? Consider $\sigma <_L \alpha\hat{}\langle o\rangle$ and some stage $t < s$ so that

1. $\sigma \subseteq \delta_t$;
2. $g_t(\vec{v}) = \vec{u}$; and
3. g_t has not been canceled by stage s.

If at stage s, α defines $g_{\alpha,s}(\vec{v}) = \vec{u}\,'$, then as we determine the Boolean algebra $\mathcal{B}_s$ based on α's approximation of g, we may give an element of the finite algebra determined by $\vec{v}$ a certain number of subsets. However, this definition at stage s must allow for the possibility that at some later stage t' with $\sigma \subset \delta_{t'}$, $g_{t'}(\vec{v}) = \vec{u}$. How can the construction ensure this compatibility? Assume that each atom of the finite algebra determined by $\vec{u}\,'$ contains no more elements than corresponding atom of $\vec{u}$. Then at stage s we do not risk giving some element in the algebra determined by $\vec{v}$ more subsets than its image under g_t actually contains. Consequently, at stage t', $\delta_{t'}$ can redefine $g_{t'}(\vec{v}) = \vec{u}$ and remain in agreement with the algebra $\mathcal{B}_{t'-1}$ we have chosen. In other words, $\vec{u} \leq_1 \vec{u}\,'$. Therefore, the relation crucial to classifying relativized Δ_2^0-categoricity also play a critical role in our examination of the unrelativized notion.

Construction

Stage 1: Fix, for the rest of the construction, $g(0) = 0_{\mathcal{A}}$ and $g(1) = 1_{\mathcal{A}}$. For each σ in the tree, σ is initialized and $\delta_1 = \lambda$. The algebra $\mathcal{B}_1$ is the trivial algebra.

Stage s + 1: Of course, $\lambda \subset \delta_{s+1}$. Let $\alpha \subset \delta_{s+1}$ be a node of length e. Let β be the predecessor node of α, and let $dom(g_{\beta,s}) = \vec{d}$ with $g_{\beta,s+1}(\vec{d}) = \vec{c}$. (If $\alpha = \lambda$, then there is no β, and $\vec{c} = 0_{\mathcal{A}}, 1_{\mathcal{A}}$.) The node α uses the computability of the relations *Atom* and *Atomless* in $\mathcal{A}$ to search for the first pair $\langle a^1_{\alpha,s+1}, a^2_{\alpha,s+1}\rangle$ so that

1. for an atom a of the *finite* algebra determined by $\vec{c}$, $a^1_{\alpha,s+1} \cup a^2_{\alpha,s+1} = a$;

2. $a^1_{\alpha,s+1}$ has $s + 1$ proper subsets;

3. $a^2_{\alpha,s+1}$ contains $s + 1$ atoms of $\mathcal{A}$; and

4. we have not yet discovered either that $a^1_{\alpha,s+1}$ is finite or that $a^2_{\alpha,s+1}$ has only finitely many atoms of $\mathcal{A}$.

This is the challenging pair of α at stage $s + 1$.

Case I: The node α has been initialized since its last active stage or has defined a new challenging pair since its last active stage.

The node α initializes all γ with $\alpha \subset \gamma \subseteq \delta_t$ for some $t < s$, and it chooses outcome I. (Notice, in particular, that any approximation g_t defined in part by α at an earlier stage t is canceled. However, of course, α does not cancel the definition of its challenging pair.) It must define $g_{\alpha,s+1}$ so that the challenging pair is in the range. We may assume by induction on the construction that $\vec{d}$ includes all elements which are in the domain of an uncanceled approximation g_t where $t < s+1$ and $\delta_t <_L \alpha$. Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of $g_{\beta,s+1}$. Again, we may assume by induction on the construction that $g_{\beta,s+1}$ is compatible with the algebra $\mathcal{B}_s$; i.e., $b_{\beta,s+1}$ can be extended to be a relation-preserving map between $\mathcal{B}_s$ and $\mathcal{A}$. Therefore,

α can extend $g_{\beta,s+1}$ so that it includes $\vec{b}$ and e in the domain and the challenging pair and e in the range. Let $b^1_{\alpha,s+1}, b^2_{\alpha,s+1}$ be such that $g_{\alpha,s+1}(b^1_{\alpha,s+1}, b^2_{\alpha,s+1}) = a^1_{\alpha,s+1}, a^2_{\alpha,s+1}$. [[Summary: After redefining its challenging pair, the node α restarts its work on the requirement R_e.]]

Case II: α has not been initialized since its last active stage, and its challenging pair is the same as it was at its last active stage.

There is a stage $t < s+1$ so that

1. $\alpha \subset \delta_t$;

2. either α has been initialized since the previous stage r in which $\alpha \subset \delta_r$, or the definition of the challenging pair changed during stage t;

3. α has not been initialized since stage t; and

4. $\langle a^1_{\alpha,s+1}, a^2_{\alpha,s+1}\rangle = \langle a^1_{\alpha,t}, a^2_{\alpha,t}\rangle$.

(Throughout the rest of this construction, we will refer to this particular t.)
We assume by induction on the construction that since α has not been initialized since stage t, the portion of g_{s+1} defined by nodes above α is the same as the portion of g_t defined by nodes above α. The node α defines $g'_{\alpha,s+1}$, its *tentative* version of $g_{\alpha,s+1}$, to be the same as $g_{\alpha,t}$. We now consider the following subcases.

Case IIa: One of the following is true:

1. An element of $\vec{d}$ or one of the elements $b^1_{\alpha,s+1}, b^2_{\alpha,s+1}$ is not in the range of $\varphi^{\Delta^0_2}_{e,s+1}$. (Otherwise, let $\vec{c}\,'$, $a^1_{\alpha,s+1}{}', a^2_{\alpha,s+1}{}'$ be such that $\varphi^{\Delta^0_2}_{e,s+1}(\vec{c}\,') = \vec{d}$ and $\varphi^{\Delta^0_2}_{e}(a^1_{\alpha,s+1}{}', a^2_{\alpha,s+1}{}') = b^1_{\alpha,s+1}, b^2_{\alpha,s+1}$.)

2. Some relation realized by $\vec{c}, a^1_{\alpha,s+1}, a^2_{\alpha,s+1}$ is not realized by $\vec{c}\,', a^1_{\alpha,s+1}{}', a^2_{\alpha,s+1}{}'$, or vice-versa.

3. If α enumerates the diagram of $\langle \mathcal{A}, Atom, Atomless \rangle$ for $s+1$ steps, then it sees that one of the atoms of the finite algebra determined by $\vec{c}, a^1_{\alpha,s+1}, a^2_{\alpha,s+1}$ has a different size or a different number of atoms (of $\mathcal{A}$) than the corresponding atom in the finite algebra determined by $\vec{c}', {a^1_{\alpha,s+1}}', {a^2_{\alpha,s+1}}'$.

4. If $t' < s+1$ is the greatest stage so that $\alpha \subseteq \delta_{t'}$, then our guess at the preimage of $\vec{d}$ or one of $b^1_{\alpha,s+1}, b^2_{\alpha,s+1}$ under $\varphi_e^{\Delta^0_2}$ has changed since t'. (Otherwise, the pair $\langle {a^1_{\alpha,s+1}}', {a^2_{\alpha,s+1}}' \rangle$ is a candidate pair.)

The outcome of α is I; the node α initializes all γ with $\alpha\hat{\ }\langle I \rangle <_L \gamma$; and $g_{\alpha,s+1} = g^{'}_{\alpha,s+1}$.
[[Summary: If the approximation $g^{'}_{\alpha,s+1}$ defined so far is part of the isomorphism g itself, then $\varphi_e^{\Delta^0_2}$ doesn't appear to be an isomorphism, because it does not seem total, as in (1) and (4), or we can readily see that $g^{'}_{\alpha,s+1} \circ \varphi_{e,s+1}^{\Delta^0_2}$ does not appear to be an automorphism of $\mathcal{A}$, as in (2) and (3).]]

Case IIb: Not Case IIa and all of the following are true:

1. there is a stage r with $t < r < s+1$ so that $\alpha\hat{\ }\langle A \rangle \subset \delta_r$, and $\alpha\hat{\ }\langle A \rangle$ has not been initialized since r;

2. α diagonalized against R_e at stage r;

3. if y is an atom of the finite subalgebra determined by elements in the range of higher priority g_w, and during the diagonalization α guessed that y was infinite, then α finds 2^{s+1} subsets of y and still guesses that y is infinite.

The outcome of α remains A, and $g_{\alpha,s+1} = g_{\alpha,r}$.
[[Summary: The node α seems already to have diagonalized successfully against R_e at an earlier stage, and nothing has injured this work.]]

Case IIc: Neither Case IIa nor Case IIb

First, α initializes each node γ with $\alpha\hat{}\langle A\rangle \subseteq \gamma$. The node α attempts to perform the following diagonalization on the candidate pair $\langle {a^1_{\alpha,s+1}}', {a^2_{\alpha,s+1}}'\rangle$. If it completes the diagonalization, then the outcome of α is A.

Diagonalization: Let w be the *greatest* number so that $t \leq w < s+1$, $\alpha\hat{}\langle I\rangle \subseteq \delta_w$, and g_w is uncanceled. Let $dom(g_w) = \vec{v}$ with $g_w(\vec{v}) = \vec{u}$. By our construction, the approximation g_w must extend $g^{'}_{\alpha,s+1}$.

Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of g_w. By induction on the construction, we can assume that g_w is compatible with this algebra, and α can define an extension of g_w whose domain includes $\vec{b}$. Let the image of $\vec{b}$ be $\vec{a}$. (Note that the atom a of the algebra determined by $\vec{c}$ is *not* necessarily an element of $\vec{a}$.) Throughout the diagonalization, α must leave fixed the mapping $g_{\beta,s+1}(\vec{d}) = \vec{c}$. However, α will attempt to alter the mapping of the rest of $\vec{v}, \vec{b}, b^1_{\alpha,s+1}, b^2_{\alpha,s+1}$ so that it meets requirement R_e while it respects g_w.

This diagonalization follows closely the argument for relativized Δ^0_2-categoricity given in Theorem 3.7: $\vec{c}$, the range of $g_{\beta,s}$, corresponds to the parameters $\vec{c}$ in the potential formally Σ_2 Scott family; $\vec{u}$, the range of g_w, corresponds to the extra tuple $\vec{u}$ in the back-and-forth relation. Here, however, we also need to consider $\vec{a}$, the image of the extra elements of $\mathcal{B}_s$. Nevertheless, note that we are not concerned with the atom size in the finite algebra determined by $\vec{u} \cup \vec{a}$, but only in that determined by $\vec{u}$.

Recall that $g^{'}_{\alpha,s+1}$ is α's tentative contribution to g_{s+1}; $a^1_{\alpha,s+1}, a^2_{\alpha,s+1}$ is a challenging pair with $a^1_{\alpha,s+1} \cup a^2_{\alpha,s+1} = a$, and α has defined $g^{'}_{\alpha,s+1}(b^1_{\alpha,s+1}, b_{\alpha,s+1}) = a^1_{\alpha,s+1}, a^2_{\alpha,s+1}$. Furthermore, $\langle {a^1_{\alpha,s+1}}', {a^2_{\alpha,s+1}}'\rangle$ seemingly has met the challenge, because $\varphi_e^{\Delta^0_2}({a_{\alpha,s+1}}', {a^2_{\alpha,s+1}}') = b^1_{\alpha,s+1} b^2_{\alpha,s+1}$, and ${a^1_{\alpha,s+1}}'$ has 2^{s+1} proper subsets and still appears infinite, and ${a^2_{\alpha,s+1}}'$ has 2^{s+1} atoms of $\mathcal{A}$. The goal is to define $g_{\alpha,s+1}(b^1_{\alpha,s+1}) =$

$a^1_{\alpha,s+1}{}''$ so that $a^1_{\alpha,s+1}{}''$ is a finite Boolean algebra, thus ensuring that $\varphi_e^{\Delta^0_2}$ is not an isomorphism if $\varphi_{e,s+1}^{\Delta^0_2}$ is a correct approximation of $\varphi_e^{\Delta^0_2}$ and $a^1_{\alpha,s+1}{}'$ truly is infinite.

Consider the finite subalgebra determined by $\vec{u}$. Let $k = ((s+1)+|\vec{u}|+|\vec{a}|)$. Let $y_1, y_2, \ldots, y_m$ be atoms of this finite algebra with $y_1 \cup \cdots \cup y_m = a^2_{\alpha,s+1}$. Determine the first among $y_1, \ldots, y_m$ which contains 2^k subsets and still appears to be infinite. (If no such y_j exists, then $a^2_{\alpha,s+1}$is finite, and hence, α must choose a new challenging pair and enter Case I.) Assume, without loss of generality, that it is y_m.

Now consider the finite subalgebra determined by $\vec{u} \cup \vec{a}$. Let $z^0_1, \ldots, z^0_{p_0}, z^1_1, \ldots, z^1_{p_1}, \ldots, z^m_1, \ldots, z^m_{p_m}$ be atoms of this algebra so that $z^0_1 \cup \cdots \cup z^0_{p_0} = a^1_{\alpha,s+1}$ and $z^j_1 \cup \cdots \cup z^j_{p_j} = y_j$ for $j = 1, \ldots, m$. In $a^2_{\alpha,s+1}$ find atoms (of $\mathcal{A}$) $z^0_1{}', \ldots, z^0_{p_0}{}'$ and $z^j_1{}' \ldots z^j_{p_j}{}'$ for $j = 1, \ldots, m-1$. Furthermore, in $a^2_{\alpha,s+1}$ find other atoms (of $\mathcal{A}$) $z^m_1{}', \ldots, z^m_{p_m-1}{}'$, and let $z^m_{p_m}{}' = a - [[\bigcup_{j=0,\ldots,m-1}(\cup_{i=1,\ldots,p_j} z^j_i{}')] \cup (z^m_1{}' \cup \cdots \cup z^m_{p_m-1}{}')]$. Notice that $\bigcup z$-elements $= \bigcup z'$-elements $= a$.

Now we are ready to redefine $g_{\alpha,s+1}$. Let v_1 be an element of $\vec{v} \cup \vec{b}$ currently mapped to u_1. If $u_1 \cap a = 0$, then $g_{\alpha,s+1}(v_1) = u_1$. Otherwise, u_1 is a union of some x disjoint from a and some union of z elements. Let $g_{\alpha,s+1}(v_1) =$ the union of x and the corresponding z' elements. In particular, assume v_1 is an element of $\vec{d}$. If u_1 is disjoint from a, then $g_{\alpha,s+1}(v_1) = u_1$. If u_1 is not disjoint from a, then $u_1 = x \cup a$, since a is an atom of the algebra determined by $\vec{c}$. Consequently, $g_{\beta,s+1}(v_1) = x \cup \bigcup z'$-elements $= u_1$. The node α finishes the definition of $g_{\alpha,s+1}$ by including e in the domain and range. This completes the diagonalization.

After each node $\alpha \subset \delta_{s+1}$ has determined $g_{\alpha,s+1}$, we complete the definition of g_{s+1} and define the finite Boolean algebra $\mathcal{B}_{s+1}$. The algebra $\mathcal{B}_s$ has s atoms, which we designate $b_1, b_2, \ldots, b_s$, and therefore has 2^s distinct elements. The function $\bigcup_{\alpha \subset \delta_{s+1}} g_{\alpha,s+1}$ contains all of these 2^s elements in its domain; we designate the image of each b_i as a_i.

If the function $\bigcup_{\alpha\subset\delta_{s+1}} g_{\alpha,s+1}$ has no more than 2^s elements in its domain, then determine the first a_i so that a_i has a nontrivial subset. Extend $\bigcup_{\alpha\subset\delta_{s+1}} g_{\alpha,s+1}$ to contain this subset and its complement with respect to a_i in the range; this extension is g_{s+1}.

If $\bigcup_{\alpha\subset\delta_{s+1}} g_{\alpha,s+1}$ has more than 2^s elements in its domain, then it must map some b^* to a nontrivial subset of some a_i, which we designate a^*. If necessary, extend $\bigcup_{\alpha\subset\delta_{s+1}} g_{\alpha,s+1}$ so that every element in the finite algebra with atoms $a_1, \ldots, a_{i-1}, a^*, a_i \cap \overline{a^*}, a_{i+1}, \ldots, a_s$ is included in the range; this extension is g_{s+1}. Using g_{s+1}, define $\mathcal{B}_{s+1}$ so that b_i is no longer an atom. This concludes the construction.

4.1.2 Supporting lemmas

Lemma 4.2 *For each s, $\mathcal{B}_s$ and δ_s satisfy the following properties:*

1. *If β is the predecessor node of $\alpha \subset \delta_s$, then $g_{\beta,s} \subseteq g_{\alpha,s}$.*

2. *Let $\alpha\hat{}\langle o\rangle \subset \delta_s$, and let $t < s$ be the greatest stage so that $\delta_t <_L \alpha\hat{}\langle o\rangle$. If $\vec{v}$ is the set of all elements in the domain of g_t, then $\vec{v} \subseteq dom(g_{\alpha,s})$.*

3. *Let o be a fixed outcome, $\alpha\hat{}\langle o\rangle \subset \delta_s$, and $t < s$ be the last stage with $\alpha\hat{}\langle o\rangle \subseteq \delta_t$. If $g_{\alpha,s} \neq g_{\alpha,t}$, then every node $\gamma \supseteq \alpha\hat{}\langle o\rangle$ is initialized at some stage t' with $t < t' \leq s$.*

4. *Let α, $\vec{v}$, and g_t be as in (2). If y is an atom of the finite algebra determined by $g_t(\vec{v})$ and y' is the corresponding atom of the finite subalgebra determined by $g_{\alpha,s}(\vec{v})$, then $|y| \geq min\{|y'|, 2^s\}$.*

5. *$\mathcal{B}_s$ extends $\mathcal{B}_{s-1}$ as a finite Boolean algebra.*

6. *$\mathcal{B}_s$ is compatible with each g_t not canceled by stage s.*

Proof: We use induction on the stage s. For stage 0, (1) - (6) are trivially true. Assume (1) - (6) are true for stage s. We must show them true for stage $s+1$.

We use induction on the length of the node $\alpha \subset \delta_{s+1}$. Assume that (1) - (4) are true for all $\beta \subset \alpha$. We must show them true for α.

1) If the outcome of α is I at stage $s+1$, then α falls either into Case I or Case IIa. If Case I, then (1) is true by construction. If Case IIa, then there is a stage $t < s+1$ so that

a) $\alpha \subset \delta_t$, and α defined a challenging pair at t;

b) α has not been initialized at some t' with $t < t' \leq s+1$; and

c) The challenging pair defined at t is the challenging pair at $s+1$.

At stage t, $g_{\alpha,t} \supseteq g_{\beta,t}$ by construction. If $g_{\beta,t} \neq g_{\beta,s+1}$, then by the induction hypotheses on stages and nodes, α was initialized at some stage t' with $t < t' \leq s+1$, a contradiction. Therefore, $g_{\beta,s+1} = g_{\beta,t} \subseteq g_{\alpha,t} = g_{\alpha,s+1}$.

2) If the outcome of α at stage $s+1$ is I, then (2) is true by the induction hypothesis on nodes and (1). If the outcome is A, then let $r \leq s+1$ be the greatest stage at which α actually performed a diagonalization. If $r = s+1$, then (2) is true by construction. If $r < s+1$, then (2) is true for r by induction hypothesis, and $g_{\alpha,s+1} = g_{\alpha,r}$. If any more elements were added to $\vec{v}$ between r and $s+1$, then $\alpha\hat{\ }\langle A\rangle$ would have been initialized between r and $s+1$, contradicting the definition of r.

3) If $o = I$ and $g_{\alpha,s+1} \neq g_{\alpha,t}$, then α redefines its challenging pair at $s+1$, so (3) is true by construction. If $o = A$ and $g_{\alpha,s} \neq g_{\alpha,t}$, then by construction $\alpha\hat{\ }\langle A\rangle$ was initialized at some stage t' with $t < t' \leq s+1$; therefore, so was any node $\gamma \supseteq \alpha\hat{\ }\langle A\rangle$.

4) If the outcome at stage $s+1$ is I, then (4) is true by induction on nodes and (1) for α. If the outcome of α is A, let $r \leq s+1$ be the last stage at which α actually performed a diagonalization. If $r = s+1$, then the diagonalization guarantees that

only one atom y of the algebra determined by $g_t(\vec{v})$ is possibly made larger when we define $g_{\alpha,s+1}(\vec{v})$; however, such a y must have size at least 2^{s+1}. If $r < s+1$, then at stage $s+1$ we have seen that the y from stage r does indeed have 2^{s+1} proper subsets.

5) Since the algebra $\mathcal{B}_{s+1}$ is determined entirely by g_{s+1}, we need only verify that for each $\alpha \subset \delta_s$, $g_{\alpha,s+1}$ is a relation-preserving map between a subset of $\mathcal{B}_s$ and $\mathcal{A}$. If α is in Case I, then the construction guarantees that $g_{\alpha,s+1}$ respects $\mathcal{B}_s$ as it defines $g_{\alpha,s+1}$. If Case IIa or IIb, then $g_{\alpha,s+1}$ is part of an uncancelled g_t. By induction hypothesis on (6), g_t, and hence $g_{\alpha,s+1}$ is compatible with $\mathcal{B}_s$. Finally, if α is in Case IIc, then the diagonalization ensures that $g_{\alpha,s+1}$ maps atoms of the algebra $\mathcal{B}_s$ to nonzero disjoint elements of $\mathcal{A}$ whose union is 1, and $g_{\alpha,s+1}$ maps combinations of these atoms to the analogous combinations of these disjoint elements.

6) Let $t < s+1$. First, if $\delta_t \subset \delta_{s+1}$, then by (3) either g_t is canceled or $g_{s+1} \supseteq g_t$. If $\delta_{s+1} <_L \delta_t$, then g_t is canceled. If $\delta_t <_L \delta_{s+1}$, then by (2) we know that $dom(g_t) \subseteq dom(g_{s+1})$. Furthermore, g_t completely determines $\mathcal{B}_t$, and g_{s+1} completely determines $\mathcal{B}_{s+1}$, and by (5), $\mathcal{B}_t$ and $\mathcal{B}_{s+1}$ are consistent. Consequently, we know that the finite algebras determined by $g_t(dom(g_t))$and $g_{s+1}(dom(g_t))$ look the same. However, at the end of stage $s+1$, there may be other elements not in $dom(g_t)$ that appear in the algebra $\mathcal{B}_{s+1}$. Can g_t be extended to a relation-preserving map between $\mathcal{B}_{s+1}$ and $\mathcal{A}$? In short, are the atoms of $g_t(dom(g_t))$ big enough to accommodate the images of the new elements of $\mathcal{B}_{s+1}$?

For each atom y determined by $g_t(dom(g_t))$, let y' be the corresponding atom determined by $g_{s+1}(dom(g_t))$. For each such y, the construction guarantees that $|y| \geq min\{|y'|, 2^{s+1}\}$. Since $\mathcal{B}_{s+1}$ is determined entirely by g_{s+1}, and $\mathcal{B}_{s+1}$ contains only $s+1$ atoms, each atom y can be expressed as a finite algebra which contains either more atoms than y' has atoms of $\mathcal{A}$, or more atoms than appear in the entire algebra $\mathcal{B}_{s+1}$. In short, the desired compatibility is guaranteed. QED

Lemma 4.3 *There is a true path f with the following features:*

1. *If $\alpha \subset f$, then α is the left-most node of length $|\alpha|$ so that $\alpha \subset \delta_s$ for infinitely many s.*
2. *If $\alpha \subset f$, then α does not define challenging pairs infinitely often.*
3. *If $\alpha\hat{\ }\langle A\rangle \subset f$, then α does not diagonalize infinitely often.*
4. *If $\alpha\hat{\ }\langle o\rangle \subset f$ and $S = \{s : \alpha\hat{\ }\langle o\rangle \subset \delta_s\}$, then $\lim_{s\in S} g_{\alpha,s} = g_\alpha$; i.e., there are $t \in S$ and a tuple $\vec{d}$ so that for all $s \in S$ with $s \geq t$, $dom(g_{\alpha,s}) = dom(g_{\alpha,t}) = dom(g_\alpha) = \vec{d}$, and $g_{\alpha,s}(\vec{d}) = g_{\alpha,t}(\vec{d}) = g_\alpha(\vec{d})$.*
5. *If $g = \bigcup_{\alpha\subset f} g_\alpha$, then $g : \mathcal{B} \cong \mathcal{A}$.*

Proof: We show (2) - (4) by simultaneous induction on the length of $\alpha \subset f$. Assume (2) - (4) are true for all $\beta \subset \alpha$. Let t be the least stage and $\vec{d}$, $\vec{c}$ be the tuples so that

a) For all stages $s \geq t$, $\alpha \leq_L \delta_s$.

b) For all nodes $\beta \subset \alpha$ and all $s \geq t$, $\langle a^1_{\beta,s}, a^2_{\beta,s}\rangle = \langle a_{\beta,t-1}, a_{\beta,t-1}\rangle$.

c) If $\beta\hat{\ }\langle A\rangle \subset f$, then β does not diagonalize after stage $t-1$.

d) If β is the predecessor node of α, then t and $\vec{d}$ witness that (4) is true for β, and $g_\beta(\vec{d}) = \vec{c}$.

e) Let $\langle a_1, a_2\rangle$ be the least pair so that for some atom a of the finite algebra determined by $\vec{c}$, $a_1 \cup a_2 = a$; a_1 is infinite; and a_2 has infinitely many atoms of $\mathcal{A}$. Then all lesser pairs have been discovered not to have this property by stage t in the enumeration of $\langle \mathcal{A}, Atom, Atomless\rangle$.

First, we claim that α is not initialized at any $s \geq t$. A node α can be initialized at stage s for one of three reasons:

a) $\delta_s <_L \alpha$;

b) a node $\beta \subset \alpha$ redefines its challenging pair;

c) a node β with $\beta\hat{\ }\langle A\rangle \subseteq \alpha$ actively diagonalizes at stage s.

However, none of these can occur at a stage $s \geq t$ by the induction hypothesis. Therefore, by the construction, for all stages $s \geq t$, $\langle a^1_{\alpha,s}, a^2_{\alpha,s}\rangle = \langle a^1_{\alpha,t}, a^2_{\alpha,t}\rangle = \langle a_1, a_2\rangle$. If $\alpha\hat{\ }\langle I\rangle \subset f$, then by the construction $g_\alpha = g_{\alpha,t}$.

If $\alpha\hat{\ }\langle A\rangle \subset f$, let $r \geq t$ be the first stage so that $\alpha\hat{\ }\langle A\rangle \subset \delta_r$, and for all $s \geq r$, $\alpha\hat{\ }\langle I\rangle \not\subseteq \delta_s$. Recall that $w < r$ is the *greatest* number so that $t \leq w < s+1$, $\alpha\hat{\ }\langle I\rangle \subseteq \delta_w$, and g_w is uncanceled by r; $\vec{v} = dom(g_w)$; and $g_w(\vec{v}) = \vec{u}$. Let $r' \geq r$ be the first stage where $\alpha\hat{\ }\langle A\rangle \subset \delta_{r'}$, and y, the atom of the finite subalgebra determined by $\vec{u}$ which we guessed at stage r' to be infinite, is indeed infinite. By the construction $g_\alpha = g_{\alpha,r'}$.

Part (5) now follows almost immediately from Lemma 4.2. First, by part (1) of Lemma 4.2 and (4) of this lemma, $g = \bigcup_{\alpha\subset f} g_\alpha$ is a well-defined function. Since each g_s is 1-1, g is 1-1. Since each g_α contains $|\alpha|$ in its domain and range, g is total and onto. Finally, by parts (5) and (6) of Lemma 4.2, $\mathcal{B}$ is a computable Boolean algebra, and each finite piece g_α (for $\alpha \subset f$) is a relation-preserving function. Consequently, g is an isomorphism. QED

Lemma 4.4 *Each requirement R_e is satisfied.*

Proof: Let $\alpha \subset f$ be of length e . If $e > 0$, then let β be the predecessor node of α, and let $\vec{c}$ consist of the range of g_β and the elements in the challenging pair eventually defined by α.

If the final outcome of α is I, then one of the following is true:

1. not every element of $\vec{c}$ is in $ran(g \circ \varphi_e^{\Delta_2^0})$;

2. the algebra determined by $\vec{c}$ does not satisfy the same relations as the algebra determined by its preimage under $g \circ \varphi_e^{\Delta_2^0}$;

3. the size of one of the elements of $\vec{c}$ is different from its preimage under $g \circ \varphi_e^{\Delta_2^0}$.

If the final outcome of α is A, then Lemma 4.2 guarantees that there is some r so that for *all* $s \geq r$ with $\alpha \subset \delta_s$, the outcome of α is A, and α does not actively diagonalize at any stage after r. Our construction then dictates that all approximations of $\varphi_e^{\Delta_2^0}$ including and after r map ${a^1_{\alpha,r}}'$, an infinite element of $\mathcal{A}$, to $b^1_{\alpha,r}$, a finite element of $\mathcal{B}$. Consequently, $\varphi_e^{\Delta_2^0}$ is not an isomorphism. QED

4.2 Δ_2^0-categoricity in linear orderings

Notation Let $\mathcal{A}$ be a linear ordering.

i) The predicate $S(x, y)$ denotes the successor relation; i.e., $(\mathcal{A}, S) \models S(x, y)$ iff $\mathcal{A} \models (x < y \wedge \forall z[(\neg(x < z < y)])$.

ii) The predicate $L^-(x)$ denotes that x has no immediate predecessor; i.e., $(\mathcal{A}, S, L^-) \models L^-(x)$ iff $(\mathcal{A}, S) \models \forall y(\neg S(y, x))$ iff $\mathcal{A} \models \forall y < x \exists z(y < z < x)$.

iii) The predicate $L^+(x)$ denotes that x has no immediate successor.

(Throughout the remainder of this paper, the words "successor" and "predecessor" are intended to mean "immediate successor" and "immediate predecessor," respectively.)

The main theorem of this section is

Theorem 4.5 *Let $\mathcal{A} = (A, <_A)$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is a computable structure. Then $\mathcal{A} = (A, <_A)$ is a sum of finitely many intervals,* *each of type n, ω, ω^*, $\mathbb{Z}$, or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and infimum.*

The proof of this theorem is considerably more complicated than the one for Boolean algebras, just as the arguments concerning relative Δ^0_2-categoricity were.

4.2.1 Some propositions about the maximal discrete pieces

Proposition 4.6 *Let $\mathcal{A}$ be a Δ^0_2-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is a computable structure. Then all of the following conditions hold:*

1. *$\mathcal{A}$ has finitely many maximal ω-intervals and maximal ω^*-intervals.*
2. *$\mathcal{A}$ does not contain arbitrarily large maximal finite discrete intervals.*
3. *There exist intervals $I_0, \dots, I_n$ so that $\mathcal{A} = I_0 + c_1 + I_1 + \cdots + c_n + I_n$ and NO interval I_k is of the form $J_1 + \mathbb{Z} + J_2 + \mathbb{Z} + J_3$, where each J_i contains an element with either no successor or no predecessor.*

Proof: Assume $\mathcal{A}$ does not satisfy one of the three conditions. We attempt to construct a computable $\mathcal{B} \cong \mathcal{A}$ so that there is no Δ^0_2 isomorphism between them.

We employ a tree construction with the same requirements and outcomes as those in Theorem 4.1: R_e says that $\varphi_e^{\Delta^0_2}$ is not an isomorphism between $\mathcal{A}$ and $\mathcal{B}$; and the outcomes are I (inactive) and A (active). At stage s, we define the approximations δ_s and g_s, and we *fix* an ordering $\mathcal{B}_s$, which orders the elements $0, 1, \dots, s-1$ in $\mathcal{B}$.

Challenging pairs and candidate pairs

Consider $\alpha \subset \delta_s$ working on requirement R_e. If $\alpha \supset \lambda$, the empty string, then it receives from its predecessor node β the function $g_{\beta,s}$ that it believes to be a correct approximation of g, because all requirements R_i with $i < e$ seem to be satisfied. Let

$\vec{d}$ be the domain of $g_{\beta,s}$ with $g_{\beta,s}(\vec{d}) = \vec{c} = c_1 < \ldots < c_j$. For $\alpha \supseteq \lambda$, let c_0 be the least element of $\mathcal{A}$ and c_{j+1} be the greatest element of $\mathcal{A}$.

In each interval (c_i, c_{i+1}), α searches for **challenging pairs** $\langle a_i^1, a_i^2 \rangle$ so that $(c_i, a_i^1), (a_i^1, a_i^2)$, and (a_i^2, c_{i+1}) each contains an element with either no successor or no predecessor. (The assumption that $\mathcal{A}$ does not satisfy one of the conditions in the statement of the theorem guarantees that α will always be able to find an interval with such a pair.) The node α next includes a_i^1, a_i^2 in the range of $g_{\alpha,s}$. The pair $\langle a_i^1, a_i^2 \rangle$ challenges the preimage under $g_{\alpha,s} \circ \varphi_{e,s}^{\Delta_2^0}$ of each of the three intervals $(c_i, a_i^1), (a_i^1, a_i^2), (a_i^2, c_{i+1})$ to contain an element with either no successor or no predecessor. If all challenges seem to be met at some later stage t by **candidate pairs**, then α attempts to change its approximation of g so that the image under $g_{\alpha,t} \circ \varphi_{e,t}^{\Delta_2^0}$ of one of the candidate pairs is a pair of elements in the same successor chain; we thereby meet R_e.

The challenging pairs in this construction differ from those in Theorem 4.1 in two ways. First, a node α may have multiple challenging pairs associated with it *at the same time*. This feature is necessary because we cannot computably determine which intervals of $\mathcal{A}$ have elements with no successor or no predecessor. The set of challenging pairs associated with a node α (whether or not $\alpha \subset \delta_s$) at stage s is designated by $ch_{\alpha,s}$. There is a bit of ambiguity in this notation, as the challenging pairs associated with a node α may change during the stage s. However, an easy convention should dispel any confusion: if we are currently working in stage s of the construction, then $ch_{\alpha,s}$ denotes the set associated with α at the actual moment; if we are working in a later stage t, or if we are describing some feature of the construction as a whole, then $ch_{\alpha,s}$ represents the pairs associated with α at the *end* of s.

Second, a node α never makes an incorrect guess about what is a challenging pair, because we assume that the relations S, L^+, L^- are all computable. Therefore, only the initialization of a node α will remove a pair's status as a challenging pair.

Relationships among nodes

Throughout the construction, a node $\alpha \subset \delta_s$ may **initialize** another node γ; i.e., if $\gamma \subseteq \delta_t$ for some $t < s$, then α defines $ch_{\gamma,s} = \emptyset$, and it cancels the approximation g_t. There will be two circumstances when a node will initialize another:

1. if α defines new challenging pairs at stage s, then α initializes each γ with $\alpha \subset \gamma$;

2. if $\alpha \subset \delta_s$ determines its outcome to be I, then α initializes all γ with $\alpha\hat{\ }\langle I \rangle <_L \gamma$.

(We will see that here, unlike in the proof of Theorem 4.1, a node will make no incorrect decisions about how to diagonalize after it ceases to be initialized and has defined all of its possible challenging pairs. Therefore, there is no need for the third occasion of initialization given in Theorem 4.1.)

In turn, as α defines $g_{\alpha,s}$, it must respect nodes $\sigma <_L \alpha\hat{\ }\langle o \rangle$. What exactly does "respect" mean? Consider $\sigma <_L \alpha\hat{\ }\langle o \rangle$ and some stage $t < s$ so that

1. $\sigma \subseteq \delta_t$;

2. $g_t(v_1, v_2) = u_1, u_2$; and

3. g_t has not been canceled by stage s.

If at stage s, α defines $g_{\alpha,s}(v_1, v_2) = u_1', u_2'$, then as we decide the ordering $\mathcal{B}_s$ based on α's approximation of g, we may order a certain number of elements between v_1 and v_2. However, this ordering must allow for the possibility that at some later stage t' with $\sigma \subset \delta_{t'}$, $g_{t'}(v_1, v_2) = u_1, u_2$. How can the construction ensure this compatibility?

Assume that the interval (u_1', u_2') contains no more elements than the interval (u_1, u_2). Then at stage s we do not risk placing more elements between v_1 and v_2 than actually exist between u_1 and u_2. Consequently, at stage t', $\delta_{t'}$ can redefine $g_{t'}(v_1, v_2) = u_1, u_2$ and remain in agreement with the ordering $\mathcal{B}_{t'-1}$ we have chosen. Again, we note the significance of the $\leq_1$ relation in our priority construction.

Construction and supporting lemmas

Stage 0: For each σ in the tree, $ch_{\sigma,0} = \emptyset$. $g_0 = \emptyset$. $\delta_0 = \lambda$.

Stage $s+1$: For each σ in the tree, we define $ch_{\sigma,s+1} = ch_{\sigma,s}$.

Of course, $\lambda \subset \delta_{s+1}$. Let $\alpha \subset \delta_{s+1}$ be a node of length e. Let β be the predecessor node of α, and let $dom(g_{\beta,s+1}) = \vec{d}$ with $g_{\beta,s+1}(\vec{d}) = \vec{c} = c_1 < \ldots < c_j$. (If $\alpha = \lambda$, then there is no β and $g_{\beta,s+1} = \emptyset$.) For $\alpha \supseteq \lambda$, let c_0 be the least element of $\mathcal{A}$ and c_{j+1} be the greatest element of $\mathcal{A}$. If α has no associated challenging pairs, then it searches until it finds a pair. Furthermore, if any interval (c_i, c_{i+1}) is without a challenging pair associated with α, then α searches $s+1$ steps to see if the interval has a pair $\langle a_i^1, a_i^2 \rangle$ to add to $ch_{\alpha,s+1}$.

Case I: α finds a pair to add to $ch_{\alpha,s+1}$.

The node α initializes all γ with $\alpha \subset \gamma \subseteq \delta_t$ for some $t < s$, and it chooses outcome I. (Notice, in particular, that any approximation g_t defined in part by α at an earlier stage t is canceled. However, of course, α does not define $ch_{\alpha,s+1}$ to be $\emptyset$.) It must define $g_{\alpha,s+1}$ so that these challenging pairs are in the range. We may assume by induction on the construction that $\vec{d}$ includes all elements which are in the domain of an uncanceled approximation g_t where $t < s+1$ and $\delta_t <_L \alpha$. Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of $g_{\beta,s+1}$. Again, we may assume by induction on the construction that $g_{\beta,s+1}$ is compatible with the ordering $\mathcal{B}_s$; therefore, α can extend $g_{\beta,s+1}$ so that it includes $\vec{b}$ and e in the domain and all challenging pairs and e in the range. Let b_i^1, b_i^2 be such that $g_{\alpha,s+1}(b_i^1, b_i^2) = a_i^1, a_i^2$.

[[Summary: After redefining $ch_{\alpha,s+1}$, the node α restarts its work on the requirement R_e.]]

Case II: α finds no such challenging pair to add to $ch_{\alpha,s+1}$.

There is a stage $t < s+1$ so that

1. $\alpha \subset \delta_t$;
2. α defined a challenging pair at stage t;
3. α has not been initialized since stage t; and
4. $ch_{\alpha,s+1} = ch_{\alpha,t}$.

(Throughout the rest of this construction, we will refer to this particular t.)

We assume by induction on the construction that since α has not been initialized since t, the portion of g_{s+1} defined by nodes above α is the same as the portion of g_t defined by nodes above α. The node α defines $g'_{\alpha,s+1}$, its *tentative* version of $g_{\alpha,s+1}$, to be the same as $g_{\alpha,t}$. We now consider the following subcases.

Case IIa: One of the following is true

1. An element of $\vec{d}$ is not in the range of $\varphi^{\Delta^0_2}_{e,s+1}$, or for some i, b_i^1, b_i^2 is not in the range of $\varphi^{\Delta^0_2}_{e,s+1}$. (Otherwise, let $\vec{c}'$, $a_i^1{}', a_i^2{}'$ be such that $\varphi^{\Delta^0_2}_{e,s+1}(\vec{c}') = \vec{d}$ and $\varphi^{\Delta^0_2}_{e,s+1}(a_i^1{}', a_i^2{}') = b_i^1, b_i^2$.)
2. The ordering of $\vec{c}$ and the challenging pairs does not match the ordering of the inverse images under $g'_{\alpha,s+1} \circ \varphi^{\Delta^0_2}_{e,s+1}$.

3. The size of some interval determined by $\vec{c}$ and the challenging pairs does not match the sizes of corresponding interval determined by their inverse images under $g'_{\alpha,s+1} \circ \varphi^{\Delta^0_2}_{e,s+1}$.

4. If $t' < s+1$ is the greatest stage so that $\alpha \subseteq \delta_{t'}$, then our guess at the preimage of $\vec{d}$ or b^1_i, b^2_i for some i under $\varphi^{\Delta^0_2}_e$ has changed since t'.

5. For some i where (c_i, c_{i+1}) has a challenging pair, after enumerating the open diagram of $(\mathcal{A}, S, L^-, L^+)$ for $s+1$ steps, one of the intervals $(c'_i, a^1_i{}')$, $(a^1_i{}', a^2_i{}')$, $(a^2_i{}', c'_{i+1})$ appears to have only elements with successors and predecessors. (Otherwise, the pairs $\langle a^1_i{}', a^2_i{}' \rangle$ are candidate pairs.)

The outcome of α is I, and $g_{\alpha,s+1} = g'_{\alpha,s+1}$.
[[Summary: If the approximation $g'_{\alpha,s+1}$ defined so far is part of the isomorphism g itself, then $\varphi^{\Delta^0_2}_e$ doesn't appear to be an isomorphism, because it does not seem total, as in (1) and (4), or we can readily see that $g'_{\alpha,s+1} \circ \varphi^{\Delta^0_2}_{e,s+1}$ does not appear to be an automorphism of $\mathcal{A}$, as in (2), (3), and (5).]]

Case IIb: Not Case IIa and both of the following are true:

1. there is a stage r with $t < r < s+1$ so that $\alpha\hat{\ }\langle A \rangle \subset \delta_r$, and $\alpha\hat{\ }\langle A \rangle$ has not been initialized since r;

2. at stage r, α diagonalized against $\varphi^{\Delta^0_2}_e$ by defining $g_{\alpha,r}$ so that $g_{\alpha,r} \circ \varphi^{\Delta^0_2}_{e,r}(a^1_i{}', a^2_i{}')$ is a pair in the same successor chain.

The outcome of α remains A, and $g_{\alpha,s+1} = g_{\alpha,r}$.
[[Summary: The node α has already successfully diagonalized against $\varphi^{\Delta^0_2}_e$ at an earlier stage, and nothing has injured this work.]]

Case IIc: Neither Case IIa nor Case IIb

The node α attempts to perform the following diagonalization for each interval (c_i, c_{i+1}) containing a challenging pair. If there are intervals containing no challenging pairs, then α simultaneously searches for challenging pairs in them. If it finds a challenging pair in an interval without one before completing a diagonalization, then α adds it to $ch_{\alpha,s+1}$ and enters Case I. If it completes a single one of the diagonalizations, then the outcome of α is A.

Diagonalization: Let w be the *greatest* number so that $t \leq w < s+1$, $\alpha\hat{}\langle I\rangle \subseteq \delta_w$, and g_w is uncanceled. Let $dom(g_w) = \vec{v}$ with $g_w(\vec{v}) = \vec{u}$. By our construction, the approximation g_w must extend $g'_{\alpha,s+1}$.

Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of g_w. By induction on the construction, we can assume that g_w is compatible with this ordering, and α can define an extension of g_w whose domain includes $\vec{b}$. Let the image of $\vec{b}$ be $\vec{a}$. Throughout the diagonalization, α must leave fixed the mapping $g_{\beta,s+1}(\vec{d}) = \vec{c}$. However, α will attempt to alter the mapping of the rest of $b_i^1, b_i^2, \vec{v}, \vec{b}$ so that it meets requirement R_e while it respects g_w.

Note that, as in Theorem 4.1, the diagonalization follows closely the argument for relativized Δ_2^0-categoricity given in Proposition 3.11.

Recall that $g'_{\alpha,s+1}$ is α's tentative contribution to g_{s+1}, and for each challenging pair $\langle a_i^1, a_i^2\rangle$ α has defined $g'_{\alpha,s+1}(b_i^1, b_i^2) = a_i^1, a_i^2$. Furthermore, $\langle a_i^1{}', a_i^2{}'\rangle$ has met the challenge, because $\varphi_{e,s+1}^{\Delta_2^0}(a_i^1{}', a_i^2{}') = b_i^1, b_i^2$, and each of the intervals $(c_i', a_i^1{}')$, $(a_i^1{}', a_i^2{}')$, $(a_i^2{}', c_{i+1}')$ has an element with either no successor or no predecessor. The goal is to define $g_{\alpha,s+1}(b_i^1, b_i^2) = a_i^1{}'', a_i^2{}''$ so that $a_i^1{}'', a_i^2{}''$ are in the same successor chain, thus ensuring that $\varphi_e^{\Delta_2^0}$ is not an isomorphism if $\varphi_{e,s+1}^{\Delta_2^0}$ is a correct approximation of $\varphi_e^{\Delta_2^0}$.

Let $v_1, \ldots, v_k$ be the elements of $\vec{v} \cup \vec{b}$ currently mapped to elements $u_1, \ldots, u_k$ in (c_i, c_{i+1}). Assume, without loss of generality, that the ordering of these elements in $\mathcal{A}$ is of the form

$$c_i < u_1 < \cdots < u_l < a_i^1 < u_{l+1} < \cdots < u_m < a_i^2 < u_{m+1} < \cdots < u_k < c_{i+1}.$$

By the very definition of challenging pairs, there is an element x that has no successor or no predecessor and thereby guarantees one of the following intervals is infinite:

1. (c_i, u_1);
2. (u_j, u_{j+1}) for some $j \in \{1, l-1\}$;
3. (u_l, a_i^1).

Similarly, there is an element y that has no successor or no predecessor and thereby guarantees that one of the following intervals is infinite:

1. (a_i^2, u_{m+1});
2. (u_j, u_{j+1}), for some $j \in \{m+1, \ldots, k-1\}$;
3. (u_k, c_{i+1}).

Find in (c_i, c_{i+1}) a successor chain of $k+2$ elements so that to the left of this chain there is an element x' in (c_i, c_{i+1}) with no successor or no predecessor, and to the right of this chain there is an element y' in (c_i, c_{i+1}) with no successor or no predecessor. ("To the right" and "to the left" are not intended to mean "directly to the right" or "directly to the left.")

Let $x'' = \min\{x, x'\}$ and $y'' = \max\{y, y'\}$. Either c_i or some u_j is the left endpoint of an interval guaranteed to be infinite by x''. For each u_n up to and including this endpoint, leave the mapping of v_n as it is. Similarly, either c_{i+1} or some u_j is the right endpoint of an interval guaranteed to be infinite by y''. For each u_n including and after this endpoint, leave the mapping of v_n as it is. Thus we are left with a sequence of $r \leq k+2$ elements in the universe of $\mathcal{A}$. The node α defines $g_{\alpha,s+1}$ so as to remap the current preimages of these elements to appropriate elements in the successor chain and finishes the definition of $g_{\alpha,s+1}$ by including e in the domain and range. This concludes the diagonalization.

After each $\alpha \subset \delta_{s+1}$ has determined $g_{\alpha,s+1}$, we define $g_{s+1} = \cup_{\alpha \subset \delta_{s+1}} g_{\alpha,s+1}$ and use it to define $\mathcal{B}_{s+1}$. This concludes the construction.

Lemma 4.7 *For each s, δ_s and $\mathcal{B}_s$ satisfy the following properties:*

1. *If β is the predecessor node of some $\alpha \subset \delta_s$, then $g_{\beta,s} \subseteq g_{\alpha,s}$.*
2. *Let $\alpha\hat{\ }\langle o\rangle \subset \delta_s$, and let $t < s$ be the greatest stage so that $\delta_t <_L \alpha\hat{\ }\langle o\rangle$.*

   If $\vec{v}$ is the set of all elements in the domain of g_t, then $\vec{v} \subseteq dom(g_{\alpha,s})$.
3. *Let o be a fixed outcome, $\alpha\hat{\ }\langle o\rangle \subset \delta_s$, and $t < s$ be the last stage with $\alpha\hat{\ }\langle o\rangle \subseteq \delta_t$. If $g_{\alpha,s} \neq g_{\alpha,t}$, then every node $\gamma \supseteq \alpha\hat{\ }\langle o\rangle$ is initialized at some stage t' with $t < t' \leq s$.*
4. *Let α, $\vec{v}$, and g_t be as in (2). Then the intervals of the partition of $\mathcal{A}$ determined by $g_t(\vec{v})$ are at least as large as those of the partition determined by $g_{\alpha,s}(\vec{v})$.*
5. *$\mathcal{B}_s$ extends $\mathcal{B}_{s-1}$ as a finite ordering.*
6. *$\mathcal{B}_s$ is compatible with each g_t not canceled by stage s.*

Proof: We use induction on the stage s. For stage 0, (1) - (6) are trivially true. Assume that (1) - (6) are true for stage s. We must show them true for stage $s+1$.

We use induction on the length of the node $\alpha \subset \delta_{s+1}$. The arguments for (1) - (4) are essentially the same as those in Lemma 4.2.

5) Since the algebra $\mathcal{B}_{s+1}$ is determined entirely by g_{s+1}, we need only verify that for each $\alpha \subset \delta_s$, $g_{\alpha,s+1}$ is an order-preserving map between a subset of $\mathcal{B}_s$ and $\mathcal{A}$. If α is in Case I, then the construction guarantees that $g_{\alpha,s+1}$ respects $\mathcal{B}_s$ as it defines $g_{\alpha,s+1}$. If Case IIa or IIb, then $g_{\alpha,s+1}$ is part of an uncancelled g_t. By induction hypothesis on (6), g_t, and hence $g_{\alpha,s+1}$ is compatible with $\mathcal{B}_s$. Finally, if α is in Case IIc, then the diagonalization ensures that $g_{\alpha,s+1}$ preserves the order of the elements of $\mathcal{B}_s$.

6) Let $t < s+1$. If $\delta_t \subset \delta_{s+1}$, then by (3) either g_t is canceled or $g_{s+1} \supseteq g_t$. If $\delta_{s+1} <_L \delta_t$, then g_t is canceled. If $\delta_t <_L \delta_{s+1}$, then by (2) and (4) we know that $dom(g_t) \subseteq dom(g_{s+1})$ and the intervals determined by $g_{s+1}(dom(g_t))$ are no larger than those determined by $g_t(dom(g_t))$. Since the ordering of $\mathcal{B}_{s+1}$ is determined entirely by g_{s+1}, the desired compatibility is guaranteed. QED

Lemma 4.8 *There is a true path f with the following features*

1. *If $\alpha \subset f$, then α is the left-most node of length $|\alpha|$ so that $\alpha \subset \delta_s$ for infinitely many s.*

2. *If $\alpha \subset f$, then α does not define challenging pairs infinitely often.*

3. *If $\alpha\hat{\ }\langle o\rangle \subset f$ and $S = \{s : \alpha\hat{\ }\langle o\rangle \subset \delta_s\}$, then $\lim_{s\in S} g_{\alpha,s} = g_\alpha$; i.e., there are $t \in S$ and a tuple $\vec{d}$ so that for all $s \in S$ with $s \geq t$, $dom(g_{\alpha,s}) = dom(g_{\alpha,t}) = dom(g_\alpha) = \vec{d}$, and $g_{\alpha,s}(\vec{b}) = g_{\alpha,t}(\vec{d}) = g_\alpha(\vec{d})$.*

4. *If $g = \bigcup_{\alpha\subset f} g_\alpha$, then $g : \mathcal{B} \cong \mathcal{A}$.*

Proof: We show (2) and (3) by simultaneous induction on the length of $\alpha \subset f$. Assume (2) and (3) are true for all $\beta \subset \alpha$. Let t be the least stage and $\vec{b}$ be the tuple so that

a) for all stages $s \geq t$, $\alpha \leq_L \delta_s$;

b) for all nodes $\beta \subset \alpha$ and all $s \geq t$, $ch_{\beta,s} = ch_{\beta,t-1}$;

c) if β is the predecessor node of α, then t and $\vec{d}$ witness that (3) is true for β; and

d) for each interval determined by $g_\beta(\vec{d})$ containing a challenging pair, α has found such a pair by stage t.

First, we claim that α is not initialized at any stage $s \geq t$. A node can be initialized at stage s for one of two reasons:

a) $\delta_s <_L \alpha$;

b) a node $\beta \subset \alpha$ defines a challenging pair at s.

However, neither of these can occur at a stage $s \geq t$, by the induction hypothesis. Therefore, by the construction, for all $s \geq t$, $ch_{\alpha,s} = ch_{\alpha,t}$. If $\alpha\hat{\ }\langle I \rangle \subset f$, then by the construction $g_\alpha = g_{\alpha,t}$. If $\alpha\hat{\ }\langle A \rangle \subset f$, then let $r \geq t$ be the first stage so that

a) $\alpha\hat{\ }\langle A \rangle \subset \delta_r$, and $\alpha\hat{\ }\langle A \rangle$ is not initialized after r;

b) α performed a diagonalization on a candidate pair $\langle a_i^1{}', a_i^2{}' \rangle$ at stage r.

By the construction, $g_\alpha = g_{\alpha,r}$.

The argument for (4) is the same as that given for (5) of Lemma 4.3. QED

Lemma 4.9 *Each requirement R_e is satisfied.*

Proof: Let $\alpha \subset f$ be of length e . If $e > 0$, then let β be the predecessor node of α, and let $\vec{c}$ consist of the range of g_β and all challenging pairs eventually defined by α.

If the final outcome of α is I, then one of the following is true:

1. not every element of $\vec{c}$ is in $ran(g \circ \varphi_e^{\Delta_2^0})$;

2. the ordering of $\vec{c}$ is not consistent with the ordering of its preimage under $g \circ \varphi_e^{\Delta_2^0}$;

3. the size of some interval determined by $\vec{c}$ is different from its preimage under $g \circ \varphi_e^{\Delta_2^0}$;

4. $g \circ \varphi_e^{\Delta_2^0}$ maps an interval without candidate pairs to one with challenging pairs.

If the final outcome of α is A, then there is some r so that for *all* $s \geq r$ with $\alpha \subset \delta_s$, the outcome of α is A. Our construction then dictates that all approximations of $\varphi_e^{\Delta_2^0}$ including and after r map a candidate pair to elements in the same successor chain in $\mathcal{B}$. Consequently, $\varphi_e^{\Delta_2^0}$ is not an isomorphism. QED

Definition 4.10 *A* maximal $\mathbb{Z}$-cluster *is a maximal interval I of the form $\mathbb{Z} \cdot \mathcal{C}$ for some linear ordering $\mathcal{C}$.*

Corollary 4.11 *Let $\mathcal{A}$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Then $\mathcal{A}$ has finitely many maximal $\mathbb{Z}$-clusters.*

Proof: Let $c_1, \ldots, c_n$ partition $\mathcal{A}$ so that condition (iii) of Proposition 4.6 holds. Each of the open intervals determined by the must contain at most three maximal $\mathbb{Z}$-clusters. Therefore, $\mathcal{A}$ contains less than $4n + 4$ maximal $\mathbb{Z}$-clusters. QED

Corollary 4.12 *Let $\mathcal{A}$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Then there is an n so that $\mathcal{A}$ can be written as a finite sum of intervals of the following forms:*

1. *m for some $m < n$;*

2. *maximal ω-intervals, maximal ω^*-intervals;*

3. *maximal $\mathbb{Z}$-clusters;*

4. *maximal intervals of the form $\Sigma_{q\in\eta}B_q$, where each $B_q < n$.*

Proof: Order the finitely many maximal ω-intervals, maximal ω^*-intervals, and maximal $\mathbb{Z}$-clusters in $\mathcal{A}$. There remain finitely many intervals, each of which contains no ω-intervals, no ω^*-intervals, and no $\mathbb{Z}$ intervals. Furthermore, there is n so that all maximal finite discrete intervals have order type $< n$. By Lemma 3.12, each of these intervals is either finite or of the form $m_1 + \Sigma_{q\in\eta}B_q + m_2$, where $0 \leq m_1, m_2 < n$, and each $B_q < n$. QED

4.2.2 Linear orderings with intervals of the form $\Sigma_{q\in\eta}B_q$

The propositions in this subsection are analogous to Proposition 3.14 and Proposition 3.18. The diagonalizations employed are relatively straightforward implementations of the strategies described in these two results, so only sketches are provided.

Proposition 4.13 *Let $\mathcal{A}$ be a linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Let n be such that one of the following is true:*

1. *$\mathcal{A}$ has the form $\omega + \Sigma_{q\in\eta}B_q + 1$, and for all $q \in \eta$, $|B_q| < n$;*

2. *$\mathcal{A}$ has the form $\omega + \mathbb{Z}$-cluster $+\Sigma_{q\in\eta}B_q + 1$, and for all $q \in \eta$, $B_q < n$.*

Then $\mathcal{A}$ is not Δ^0_2-categorical.

Sketch: We will discuss the proof for (2), as it is the more difficult case. The argument will involve a priority construction very much like the one in Proposition 4.6. Differences will appear only in the details of candidate assignment and diagonalization.

Consider $\alpha \subset \delta_s$ working on requirement R_e. If $\alpha \supset \lambda$, the empty string, then it receives from its predecessor node β the function $g_{\beta,s}$ that it believes is a correct approximation of g. Let $\vec{d}$ be the domain of $g_{\beta,s}$ with $g_{\beta,s}(\vec{d}) = \vec{c} = c_1, \ldots, c_j$. For $\alpha \supseteq \lambda$, let c_0 = the "0" of the ω summand and c_{j+1} = the "0" to the right of the $\Sigma_{q\in\eta} B_q$ summand.

The node α first determines which interval (c_i, c_{i+1}) contains both elements from the $\mathbb{Z}$-cluster and from the $\Sigma_{q\in\eta} B_q$ summand. This can be done computably, since, given an element a, α can see if there is a successor chain of at least $n+1$ elements directly to the left of or to the right of a. The node α then determines if the preimage of this interval under $g_{\beta,s} \circ \varphi_{e,s}^{\Delta_2^0}$ has the same property. If so, then α includes in the range of $g_{\alpha,s}$ a **challenging element** a_i which is in the $\Sigma_{q\in\eta} B_q$ summand but not in the same successor chain as c_{i+1}. The node α determines if a_i', the preimage of a_i under the function $g_{\alpha,s} \circ \varphi_{e,s}^{\Delta_2^0}$, is a true **candidate element**; i.e., if it has properties analogous to those of a_i. If so, then α changes its approximation of g so that the image under $g_{\alpha,s} \circ \varphi_{e,s}^{\Delta_2^0}$ of a_i' is in ω or the $\mathbb{Z}$-cluster. We thereby meet requirement R_e. QED

Corollary 4.14 *Let $\mathcal{A}$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Then every maximal $\Sigma_{q\in\eta} B_q$ interval has a supremum and an infimum.*

Recall Definition 3.16 and Lemma 3.17, which are about linear orderings with intervals of the form $\Sigma_{q\in\eta} B_q$.

Proposition 4.15 *Let $\mathcal{A}$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Any maximal $\Sigma_{q\in\eta} B_q$ interval satisfies Property 1.*

Sketch: We must show that an ordering satisfying Property 2 cannot be Δ_0^2-categorical. First, as in Proposition 3.18, let m_1 be the largest number for which there is an m_2 as in Property 2. Define a partition of η so that for each $m > m_1$ and each open interval J of η defined by the partition, either $B_q = m$ for all $q \in J$, or $B_q \neq m$ for all $q \in J$.

There is an open interval $J = (q_1, q_2)$ (q_1 might be $-\infty$, and q_2 might be ∞) of η so that for all $m > m_1$ and all $q \in J$, $B_q \neq m$, and $\Sigma_{q \in J} B_q$ satisfies Property 2 with m_1 and some other m_2. We will build a computable copy $\mathcal{B}$ and $g : \mathcal{B} \cong \mathcal{A}$ so that

1. for all $a \notin \Sigma_{q \in J} B_q$, $g(a) = a$; and

2. there is no Δ^2_0 isomorphism between $\mathcal{B}$ and $\mathcal{A}$.

Again, the argument will involve a priority construction very much like the one in Proposition 4.6, and differences will appear mainly in the details of candidate assignment and diagonalization.

Consider $\alpha \subset \delta_s$ working on requirement R_e. If $\alpha \supset \lambda$, the empty string, then it receives from its predecessor node β the function $g_{\beta,s}$ that it believes is a correct approximation of g. Let $\vec{d}$ be the domain of $g_{\beta,s}$ with $g_{\beta,s}(\vec{d}) = \vec{c} = c_1, \ldots, c_j$. For $\alpha \supseteq \lambda$, define c_0 as follows:

1. $c_0 =$ the greatest element of B_{q_1} if $q_1 \neq -\infty$;

2. $c_0 =$ the greatest bound of $\Sigma_{q \in \eta} B_q$ in $\mathcal{A}$ if $q_1 = -\infty$.

We define the element c_{j+1} similarly.

In each interval (c_i, c_{i+1}), α searches for **challenging pairs** $\langle a_i^1, a_i^2 \rangle$ where a_i^1 is in a maximal successor chain of length m_1 but not in the same successor chain as c_i or c_{i+1}; and a_i^2 is in a maximal successor chain of length m_2 but not in the same successor chain as c_i or c_{i+1}. (The hypotheses on $\Sigma_{q \in \eta} B_q$ guarantee that α will always be able to find an interval with such a pair.) The node α then includes a_i^1, a_i^2 in the range of $g_{\alpha,s}$. The node α then determines if the pair $a_i^{1\prime}, a_i^{2\prime}$, the preimage of the pair a_i^1, a_i^2 under $g_{\alpha,s} \circ \varphi_{e,s}^{\Delta^0_2}$, is a true **candidate pair**; i.e., if it has analogous properties. If so, then α attempts to change its approximation of g so that the image under $g_{\alpha,t} \circ \varphi_{e,t}^{\Delta^0_2}$ of $a_i^{2\prime}$ is an element in a maximal successor chain of length m_1. We thereby meet R_e.

Corollary 4.16 *Let $\mathcal{A}$ be a Δ^0_2-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Then there is $n \in \omega$ so that $\mathcal{A}$ can be written as a finite sum of maximal intervals of the following forms:*

1. *m for some $m < n$;*
2. *$m \cdot \eta$ for some $m < n$;*
3. *ω, ω^*;*
4. *$\mathbb{Z}$-clusters,*

where each maximal $m \cdot \eta$ interval has an infimum and a supremum.

4.2.3 $\mathbb{Z}$-clusters

When we examined relativized Δ^0_2-categoricity, we showed, in a single proposition, that a relatively Δ^0_2-categorical linear ordering could have only finitely many maximal $\omega, \omega^*, \mathbb{Z}$ intervals. There we were not concerned with computability issues in our arguments; we merely needed to demonstrate the existence of certain tuples without describing how to find them. Throughout our study of Δ^0_2-categoricity, however, such computability concerns are central, because the copy $\mathcal{B}$ we produce via the priority construction must be computable.

Each maximal ω and ω^* interval has an element with either no successor or no predecessor, so under the extra decidability assumptions on $\mathcal{A}$, we can easily identify certain infinite intervals. This property played a key role in the proof of Proposition 4.6. A $\mathbb{Z}$-cluster, of course, has no such element. Consequently, it may be impossible to identify the infinite intervals computably, and thus it is harder to formulate a diagonalization to prove that $\mathcal{A}$ has only finitely many $\mathbb{Z}$-intervals.

Proposition 4.17 *Let $\mathcal{A}$ be a Δ_2^0-categorical linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Then any $\mathbb{Z}$-cluster contained in $\mathcal{A}$ actually contains only finitely many $\mathbb{Z}$-intervals.*

The argument must be broken down into a few subpropositions, one of which will require a slightly more complicated priority construction in its proof.

Proposition 4.18 *Let $\mathcal{A}$ be a linear ordering so that $(\mathcal{A}, S, L^-, L^+)$ is computable. Furthermore, let $\mathcal{A}$ contain a maximal $\mathbb{Z}$-cluster I of the form $\mathbb{Z}\cdot\mathcal{C}$, where $\mathcal{C}$ is infinite and discrete. Then $\mathcal{A}$ is not Δ_2^0-categorical.*

Proof: Since $\mathcal{C}$ is infinite and discrete, I must be a sum of the form $c_1(\mathbb{Z}\cdot\omega) + c_2[(\mathbb{Z}\cdot\mathbb{Z})\cdot\mathcal{D}] + c_3(\mathbb{Z}\cdot\omega^*)$, where $\mathcal{D}$ is some linear ordering and each c is either 0 or 1. Furthermore, if there is anything in $\mathcal{A}$ to the left of I, then by Corollary 4.16, either I has an infimum, or I is directly to the right of a maximal ω-interval. Similarly, if there is anything in $\mathcal{A}$ to the right of I, then either I has a supremum, or I is directly to the left of a maximal ω^*-interval.

Case I: $c_2 = 0$. Then $\mathcal{A}$ contains an interval K with endpoints of one of the following forms:

a) $1 + c_1(\mathbb{Z}\cdot\omega) + c_3(\mathbb{Z}\cdot\omega^*) + 1$;

b) $1 + c_1(\mathbb{Z}\cdot\omega) + c_3(\mathbb{Z}\cdot\omega^*) + \omega^*$;

c) $\omega + c_1(\mathbb{Z}\cdot\omega) + c_3(\mathbb{Z}\cdot\omega^*) + 1$;

d) $\omega + c_1(\mathbb{Z}\cdot\omega) + c_3(\mathbb{Z}\cdot\omega^*) + \omega^*$.

In any case, we can construct $\mathcal{A}'$, a computable copy of $\mathcal{A}$, where K', the image of this interval K, is such that

1. K' contains infinitely many $\mathbb{Z}$-intervals; and

2. for $a, b \in K'$ we can computably decide if the interval (a, b) is infinite or finite.

We can show that $\mathcal{A}'$, and hence $\mathcal{A}$, is not Δ^0_2-categorical. The argument is almost exactly like that given in Proposition 4.6, except that we replace searching for elements with no successor or predecessor with searching for intervals which are infinite.

Case II: $c_2 = 1$. If the linear ordering $\mathcal{D} = 1$ or η, then we can argue that there is $\mathcal{A}'$ and K' with the properties $i)$ and $ii)$ listed above. However, if the linear ordering $\mathcal{D}$ has a successor pair, then $\mathcal{A}$ has an interval, with endpoints, of the type $\omega + (\mathbb{Z} \cdot \omega) + (\mathbb{Z} \cdot \omega^*) + \omega^*$. Hence, we can again argue for the existence of $\mathcal{A}'$ and K'. Therefore, $\mathcal{A}$ is not Δ^0_2-categorical.

Proposition 4.19 *Let $\mathcal{A}$ be a Δ^0_2-categorical linear ordering so that $(\mathcal{A}, S)$ is computable, and let $I = \mathbb{Z} \cdot \mathcal{C}$ be a maximal $\mathbb{Z}$-cluster, where $\mathcal{C}$ is not discrete. Then $\mathcal{A}$ is not Δ^0_2-categorical.*

Proof: This argument requires a bit more than the minor adjustments to Proposition 4.6 needed by Proposition 4.13 and 4.15. Therefore, we provide a more detailed construction and a subsequent list of lemmas.

Of course, we attempt to construct a computable $\mathcal{B} \cong \mathcal{A}$ so that there is no Δ^0_2 isomorphism between them. We employ a tree construction where each node has two outcomes, I (inactive) and A (active), with $I < A$. Nodes of length e work on requirement R_e. Finally, the construction must determine an isomorphism $g : \mathcal{B} \cong \mathcal{A}$. At stage s we will define the approximations δ_s and g_s, and the ordering $\mathcal{B}_s$.

First, without loss of generality, let J be a $\mathbb{Z}$-interval in I so that there is a $\mathbb{Z}$-interval in I to the right of J, but there is no next $\mathbb{Z}$-interval to the right of it. We fix $c_{-\infty}$, some element in J, and c_{∞}, some element in I to the right of J. For all

$a \in A - (c_{-\infty}, c_\infty)$, we define $g(a) = a$. Throughout the construction, we tacitly assume that any tuple $\vec{a}$ from A or $\vec{b}$ from B has elements only from $(c_{-\infty}, c_\infty)$.

Challenging elements and initialization

Consider $\alpha \subset \delta_s$ working on requirement R_e. If $\alpha \supset \lambda$, the empty string, then it receives from its predecessor node β the function $g_{\beta,s}$ that it believes is a correct approximation of g. Let $\vec{d}$ be the domain of $g_{\beta,s}$ with $g_{\beta,s}(\vec{d}) = \vec{c} = c_1 < \ldots < c_j$. For $\alpha \supseteq \lambda$, let $c_0 = c_{-\infty}$ and $c_{j+1} = c_\infty$.

The node α guesses at which is the first interval (c_i, c_{i+1}) to be infinite. (Thus, $c_0, c_1, \ldots, c_i$ are all in the same $\mathbb{Z}$-interval.) In this interval α locates a **challenging element** $a_{\alpha,s}$ so that $(c_i, a_{\alpha,s})$ and $(a_{\alpha,s}, c_{i+1})$ both seem infinite. The node α includes it in the range of $g_{\alpha,s}$. The element $a_{\alpha,s}$ challenges (c'_i, c'_{i+1}), the preimage under $g \circ \varphi_e^{\Delta_2^0}$ of (c_i, c_{i+1}), to contain an element $a'_{\alpha,s}$ with the analogous property. If the challenge seems to be met at stage t, then α changes its approximation so that the image under $g_{\alpha,t} \circ \varphi_{e,t}^{\Delta_2^0}$ of $a'_{\alpha,s}$ is in the same $\mathbb{Z}$-interval as c_0; we thereby meet requirement R_e.

In the definition of challenging elements, this construction resembles Theorem 4.1 more than Proposition 4.6. First, at stage s, α will define at most a single challenging element in *one* interval, rather than challenging pairs in multiple intervals. Second, α will make incorrect guesses as to which element has the above properties. In Proposition 4.6, α could determine computably that an element had no successor or no predecessor, and so it never mistakenly assigned pairs; any cancelling of a challenging pair was done by a node of higher priority. Here, however, α may incorrectly guess the first element of $\vec{c}$ to be in a $\mathbb{Z}$-interval different from c_0. Furthermore, even when α eventually guesses the correct interval (c_i, c_{i+1}), it may incorrectly guess that an element is a challenging element when it actually is in the same $\mathbb{Z}$-interval as either c_i or c_{i+1}. Nevertheless, as in Theorem 4.1 both of these incorrect guesses should introduce only finitely much injury, and a $0''$ construction still suffices.

A node $\alpha \subset \delta_s$ may **initialize** another node γ: if $\gamma \subseteq \delta_t$ for some $t < s$, then α cancels the assignment of $a_{\gamma,t}$ and the approximation g_t. There will be three circumstances when a node will initialize another:

1. if α redefines its challenging element at stage s, then α initializes each γ with $\alpha \subset \gamma$;
2. if α determines its outcome to be I, then α initializes all γ with $\alpha\hat{}\langle I\rangle <_L \gamma$;
3. if α diagonalizes against $\varphi_e^{\Delta_2^0}$ at stage s, then it initializes all γ with $\alpha\hat{}\langle A\rangle \subseteq \gamma$.

Construction and supporting lemmas

Stage 0: $g_0 = \emptyset$. $\delta_0 = \lambda$. Assign no challenging elements.

Stage $s+1$: Of course, $\lambda \subset \delta_{s+1}$. Let $\alpha \subset \delta_{s+1}$ be a node of length e. Let β be the predecessor node, and let $dom(g_{\beta,s}) = \vec{b}$, with $g_{\beta,s}(\vec{d}) = \vec{c} = c_1 < \ldots < c_j$. (If $\alpha = \lambda$, then there is no β and $g_{\beta,s+1} = \emptyset$.) For $\alpha \supseteq \lambda$, let $c_0 = c_{-\infty}$ and $c_{j+1} = c_\infty$ as elements of $\mathcal{A}$. As elements of $\mathcal{B}$ let $d_0 = c_{-\infty}$ and $d_{j+1} = c_\infty$. The node α searches for the first interval (c_i, c_{i+1}), and the first element $a_{\alpha,s+1}$ so that

1. after $s+1$ steps of enumerating the diagram of $(\mathcal{A}, S)$, c_{i+1} appears to be in a different $\mathbb{Z}$-interval from c_0;
2. $a_{\alpha,s+1} \in (c_i, c_{i+1})$ appears to be in a $\mathbb{Z}$-interval different from both c_i and c_{i+1};
3. in the course of this searching for $a_{\alpha,s+1}$, α does not discover that c_{i+1} is in the same interval as c_0.

This element $a_{\alpha,s+1}$ is the challenging element for α at stage $s+1$.

Case I: The node α has been initialized since its last active stage or has defined a new challenging element since its last active stage.

The node α initializes all γ with $\alpha \subset \gamma$ and chooses outcome I. It must include $a_{\alpha,s+1}$ in the range of $g_{\alpha,s+1}$. We may assume by induction on the construction that $\vec{d}$ includes all elements which are in the domain of an uncanceled approximation g_t where $t < s+1$ and $\delta_t <_L \alpha$. Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of $g_{\beta,s+1}$. Again, we may assume by induction on the construction that $g_{\beta,s+1}$ is compatible with the ordering $\mathcal{B}_s$; therefore, α can extend $g_{\beta,s+1}$ so that it includes $\vec{b}$ and e in the domain, and $a_{\alpha,s+1}$ and e in the range. Let $b_{\alpha,s+1}$ be such that $g_{\alpha,s+1}(b_{\alpha,s+1}) = a_{\alpha,s+1}$.
[[Summary: After redefining $a_{\alpha,s+1}$, the node α restarts its work on the requirement R_e.]]

Case II: α does not need to redefine its challenging pair.

There is a stage $t < s+1$ so that

1. $\alpha \subset \delta_t$;
2. either α has been initialized since the previous stage r in which $\alpha \subset \delta_r$, or the definition of $a_{\alpha,t}$ changed during stage t;
3. α has not been initialized since stage t; and
4. $a_{\alpha,s+1} = a_{\alpha,t}$;

(Throughout the rest of this construction, we will refer to this particular t.)
We assume by induction on the construction that since α has not been initialized since t, the portion of g_{s+1} defined by nodes above α is the same as the portion of g_t defined by nodes above α. The node α defines $g'_{\alpha,s+1}$, its *tentative* version of $g_{\alpha,s+1}$, to be the same as $g_{\alpha,t}$. We now consider the following subcases.

Case IIa: One of the following is true:

1. An element of $\vec{d}$ or one of $d_0, d_{j+1}, b_{\alpha,s+1}$ is not in the range of $\varphi_{e,s+1}^{\Delta_2^0}$. (Otherwise, let $\vec{c}\,'$, $c_0', c_{j+1}', a_{\alpha,s+1}{}'$ be such that $\varphi_{e,s+1}^{\Delta_2^0}(\vec{c}\,') = \vec{d}$, $\varphi_e^{\Delta_2^0}(a_{\alpha,s+1}{}') = b_{\alpha,s+1}$, and $\varphi_e^{\Delta_2^0}(c_0', c_{j+1}') = d_0, d_{j+1}$.)

2. The ordering of $\vec{c}$, c_0, c_{j+1}, $a_{\alpha,s+1}$ does not match that of $\vec{c}\,'$, $a_{\alpha,s+1}{}'$, c_0', c_{j+1}'.

3. The size of some interval determined by $\vec{c}$, c_0, c_{j+1}, and $a_{\alpha,s+1}$ does not match the size of the corresponding interval determined by $\vec{c}\,'$, c_0', c_{j+1}', $a_{\alpha,s+1}{}'$.

4. If $t' < s+1$ is the greatest stage so that $\alpha \subseteq \delta_{t'}$, then our guess at the preimage of $\vec{d}$ or one of $b_{\alpha,s+1}, d_0, d_{j+1}$ under $\varphi_e^{\Delta_2^0}$ has changed since t'.

   (Otherwise, the element $a_{\alpha,s+1}{}'$ is a candidate element.)

The outcome of α is I, and $g_{\alpha,s+1} = g_{\alpha,s+1}^{'}$.
[[Summary: If the approximation $g_{\alpha,s+1}^{'}$ defined so far is part of the isomorphism g itself, then $\varphi_e^{\Delta_2^0}$ doesn't appear to be an isomorphism, because it does not seem total, as in (1) and (4), or $g_{\alpha,s+1}^{'} \circ \varphi_{e,s+1}^{\Delta_2^0}$ does not appear to be an automorphism of $\mathcal{A}$, as in (2) and (3).]]

Case IIb: Not Case IIa and all of the following are true:

1. there is a stage r with $t < r < s+1$ so that $\alpha\hat{\ }\langle A\rangle \subset \delta_r$, and $\alpha\hat{\ }\langle A\rangle$ has not been initialized since r;

2. α performed a diagonalization on a candidate element $(a_{\alpha,s+1})'$ at stage r;

3. if K is an interval determined by $a_{\alpha,s+1}$ and elements in the range of higher priority g_w, and during the diagonalization α guessed that K was infinite, then α can find $s+1$ elements in K and still guesses that K is infinite.

The outcome of α remains A, and $g_{\alpha,s+1} = g_{\alpha,r}$.

[[Summary: The node α seems already to have diagonalized successfully against R_e at an earlier stage, and nothing has injured this work.]]

Case IIc: Neither Case IIa nor Case IIb

First, α initializes each node γ with $\alpha\hat{}\langle A\rangle \subseteq \gamma$. The node α attempts to perform the following diagonalization on the candidate element $a_{\alpha,s+1}{}'$. If it completes the diagonalization, then the outcome of α is A.

Diagonalization: Let w be the *greatest* number so that $t \leq w < s+1$, $\alpha\hat{}\langle I\rangle \subseteq \delta_w$, and g_w is uncanceled. Let $dom(g_w = \vec{v}$ with $g_w(\vec{v}) = \vec{u}$. By our construction, the approximation g_w must extend $g^{'}_{\alpha,s+1}$.

Let $\vec{b}$ be the set of elements in $\mathcal{B}_s$ but not in the domain of g_w. By induction on the construction, we can assume that g_w is compatible with this ordering, and α can define an extension of g_w whose domain includes $\vec{b}$. Let the image of $\vec{b}$ be $\vec{a}$. Throughout the diagonalization, α must leave fixed the mapping $g_{\beta,s+1}(\vec{d}) = \vec{c}$. However, α will attempt to alter the mapping of the rest of $\vec{v}, \vec{b}, b_{\alpha,s+1}$ so that it meets requirement R_e while it respects g_w.

Recall that $g^{'}_{\alpha,s+1}$ is α's tentative contribution to g_{s+1}, and for the challenging element $a_{\alpha,s+1}$, α has defined $g^{'}_{\alpha,s+1}(b_{\alpha,s+1}) = a_{\alpha,s+1}$. Furthermore, $a_{\alpha,s+1}{}'$ seemingly has met the challenge, because $\varphi_e^{\Delta_2^0}(a_{\alpha,s+1}{}') = b_{\alpha,s+1}$, and each of the intervals $(c_i', a_{\alpha,s+1}{}')$, $(a_{\alpha,s+1}{}', c_{i+1}')$ appears to be infinite. The goal is to define $g_{\alpha,s+1}(b_{\alpha,s+1}) = a_{\alpha,s+1}{}''$ so that $c_i, a_{\alpha,s+1}{}''$ are in the same $\mathbb{Z}$-interval, thus ensuring that $\varphi_e^{\Delta_2^0}$ is not an isomorphism if $\varphi_{e,s+1}^{\Delta_2^0}$ is a correct approximation of $\varphi_e^{\Delta_2^0}$.

Let $v_1, \ldots, v_k$ be the elements of $\vec{v}$ currently mapped to elements $u_1, \ldots, u_k$ in (c_i, c_{i+1}). Assume, without loss of generality, that the ordering of these elements in $\mathcal{A}$ is of the form

$$c_i < u_1 < \ldots < u_l < a_{\alpha,s+1} < u_{l+1} < \ldots < u_k < c_{i+1}.$$

In each of the following intervals, the node α attempts to find $s+1$ elements or to determine that the interval is finite:

1. $(a_{\alpha,s+1}, u_{l+1})$;
2. (u_j, u_{j+1}) for some $j \in \{l+1, \ldots, k-1\}$;
3. (u_k, c_{i+1}).

Let u_j^* be the right endpoint of the leftmost interval in which α can locate $s+1$ elements without discovering that it is finite. (There must be such an interval. Otherwise α discovers that $(a_{\alpha,s+1}, c_{i+1})$ is finite, redefines its challenging element and enters Case I.) Let $\vec{u}\,''$ be the list of elements of $\vec{u} \cup \vec{a}$ which are in (c_i, u_j^*), and $\vec{u}\,'''$ the list of elements of $\vec{u} \cup \vec{a}$ which are in $[u_j^*, c_{i+1})$.

For each element of $\vec{u}\,'''$, leave the mapping of the corresponding element of $\vec{v} \cup \vec{b}$ as it is. The node α then defines $g_{\alpha,s+1}$ so that for every element of $\vec{u}\,''$, the corresponding element of $\vec{v} \cup \vec{b}$ is now mapped to an element in the same $\mathbb{Z}$-interval as c_i (and hence, as c_0). Finally, α finishes the definition of $g_{\alpha,s+1}$ by including e in the domain and range. This concludes the diagonalization.

After each $\alpha \subset \delta_{s+1}$ has determined $g_{\alpha,s+1}$, we define $g_{s+1} = \cup_{\alpha \subset \delta_{s+1}} g_{\alpha,s+1}$ and use it to define $\mathcal{B}_{s+1}$. This concludes the construction.

Lemma 4.20 *For each s, δ_s and $\mathcal{B}_s$ satisfy the following properties:*

1. *If β is the predecessor node of $\alpha \subset \delta_s$, then $g_{\beta,s} \subseteq g_{\alpha,s}$.*
2. *Let $\alpha\hat{\ }\langle o \rangle \subset \delta_s$, and let $t < s$ be the greatest stage so that $\delta_t <_L \alpha\hat{\ }\langle o \rangle$. If $\vec{v}$ is the set of all elements in the domain of g_t, then $\vec{v} \subseteq dom(g_{\alpha,s})$.*
3. *Let o be a fixed outcome, $\alpha\hat{\ }\langle o \rangle \subset \delta_s$, and $t < s$ be the last stage with $\alpha\hat{\ }\langle o \rangle \subseteq \delta_t$. If $g_{\alpha,s} \neq g_{\alpha,t}$, then every node $\gamma \supseteq \alpha\hat{\ }\langle o \rangle$ is initialized at some stage t' with $t < t' \leq s$.*

4. *Let α, $\vec{v}$, and g_t be as in (2). If K is an interval determined by $g_t(\vec{d})$ and K' is the corresponding interval determined by $g_{\alpha,s}(\vec{d})$, then $|K| \geq min\{|K'|, s\}$.*

5. *$\mathcal{B}_s$ extends $\mathcal{B}_{s-1}$.*

6. *$\mathcal{B}_s$ is compatible with each g_t not canceled by stage s.*

Proof: We again use induction on the stage s. For stage 0, (1) - (6) are trivially true. Assume (1) - (6) are true for stage s. We must show them true for stage $s+1$.

We use induction on the length of the node $\alpha \subset \delta_{s+1}$. Assume that (1) - (4) are true for all $\beta \subset \alpha$. We must show them true for α.

The arguments for (1) - (4) are essentially the same as those in Lemma 4.2.

The argument for (5) is the same as in Lemma 4.7.

6) Let $t < s+1$. First, if $\delta_t \subset \delta_{s+1}$, then by (3) either g_t is canceled or $g_{s+1} \supseteq g_t$. If $\delta_{s+1} <_L \delta_t$, then g_t is canceled. If $\delta_t <_L \delta_{s+1}$, then by (2) we know that $dom(g_t) \subseteq dom(g_{s+1})$. Furthermore, g_t completely determines $\mathcal{B}_t$, and g_{s+1} completely determines $\mathcal{B}_{s+1}$, and by (5), $\mathcal{B}_t$ and $\mathcal{B}_{s+1}$ are consistent. Consequently, we know that the ordering of $g_t(dom(g_t))$and $g_{s+1}(dom(g_t))$ are the same. However, at the end of stage $s+1$, there may be other elements not in $dom(g_t)$ that are ordered in $\mathcal{B}_{s+1}$. Can g_t be extended to an order-preserving map between $\mathcal{B}_{s+1}$ and $\mathcal{A}$? In short, are the intervals determined by $g_t(dom(g_t))$ big enough to accommodate the images of the new elements of $\mathcal{B}_{s+1}$?

For each interval K determined by $g_t(dom(g_t))$, let K' be the corresponding interval determined by $g_{s+1}(dom(g_t))$. For each such $|K|$, the construction guarantees that $|K| \geq min\{|K'|, s+1\}$. Since $\mathcal{B}_{s+1}$ is determined entirely by g_{s+1}, and only $0, \ldots, s+1$ are ordered in $\mathcal{B}_{s+1}$, each interval K is big enough to accommodate the images of all elements of $\mathcal{B}_{s+1}$. In short, the desired compatibility is guaranteed.

QED

Lemma 4.21 *There is a true path f with the following features:*

1. *If $\alpha \subset f$, then α is the left-most node of length $|\alpha|$ so that $\alpha \subset \delta_s$ for infinitely many s.*

2. *If $\alpha \subset f$, then α does not define challenging pairs infinitely often.*

3. *If $\alpha\hat{}\langle A\rangle \subset f$, then α does not actively diagonalize infinitely often.*

4. *If $\alpha\hat{}\langle o\rangle \subset f$ and $S = \{s : \alpha\hat{}\langle o\rangle \subset \delta_s\}$, then $\lim_{s\in S} g_{\alpha,s} = g_\alpha$; i.e., there are $t \in S$ and a tuple $\vec{d}$ so that for all $s \in S$ with $s \geq t$, $dom(g_{\alpha,s}) = dom(g_{\alpha,t}) = dom(g_\alpha) = \vec{d}$, and $g_{\alpha,s}(\vec{d}) = g_{\alpha,t}(\vec{d}) = g_\alpha(\vec{d})$.*

5. *If $g = \bigcup_{\alpha\subset f} g_\alpha$, then $g : \mathcal{B} \cong \mathcal{A}$.*

Proof: We show (2) - (4) by simultaneous induction on the length of $\alpha \subset f$. Assume (2) - (4) are true for all $\beta \subset \alpha$. Let t be the least stage and $\vec{d}$, $\vec{c}$ be the tuples so that

a) for all stages $s \geq t$, $\alpha \leq_L \delta_s$;

b) for all nodes $\beta \subset \alpha$ and all $s \geq t$, $a_{\beta,s} = a_{\beta,t-1}$;

c) if $\beta\hat{}\langle A\rangle \subset f$, then β does not diagonalize after stage $t-1$;

d) if β is the predecessor node of α, then t and $\vec{d}$ witness that (3) is true for β, and $g_\beta(\vec{d}) = \vec{c}$; and

e) $a_{\alpha,t+1}$ is in (c_i, c_{i+1}), c_{i+1} is the least element of $\vec{c}$ not in the same $\mathbb{Z}$-interval as c_0, and $a_{\alpha,t+1}$ is not in the same $\mathbb{Z}$-interval as either c_i nor c_{i+1}.

First, we claim that α is not initialized at any stage $s \geq t$. A node can be initialized at stage s for one of three reasons:

a) $\alpha <_L \delta_s$;

b) a node $\beta \subseteq \alpha$ redefines $a_{\beta,s}$;

c) a node $\beta\hat{}\langle A\rangle \subset \alpha$ actively diagonalizes at stage s.

However, none of these can occur at $s \geq t$, by the induction hypothesis. Therefore, by the construction, for all $s \geq t$, $a_{\alpha,s} = a_{\alpha,t}$. If $\alpha\hat{}\langle I\rangle \subset f$, then by the construction $g_\alpha = g_{\alpha,t}$.

If $\alpha\hat{}\langle A\rangle \subset f$, let $r \geq t$ be the first stage so that $\alpha\hat{}\langle A\rangle \subset \delta_r$, and for all $s \geq r$, $\alpha\hat{}\langle I\rangle \not\subseteq \delta_s$. Recall that $w < r$ is the *greatest* number so that $t \leq w < s+1$, $\alpha\hat{}\langle I\rangle \subseteq \delta_w$, and g_w is uncanceled by r; $\vec{v}$ is the domain of g_w; and $g_w(\vec{v}) = \vec{u}$. Let $r' \geq r$ be the first stage where $\alpha\hat{}\langle A\rangle \subset \delta_{r'}$, and u_j^* chosen by α is the right endpoint of the leftmost infinite interval determined by $a_{\alpha,r}$ and the elements of $\vec{u}$ between $a_{\alpha,r'}$ and c_{i+1}. By the construction, $g_\alpha = g_{\alpha,r'}$.

The argument for (5) is the same as that given in Lemma 4.3. QED

Lemma 4.22 *Each requirement R_e is satisfied.*

Proof: Let $\alpha \subset f$ be of length e . If $e > 0$, then let β be the predecessor node of α, and let $\vec{c}$ consist of the range of g_β and all challenging elements eventually defined by α.

If the final outcome of α is I, then one of the following is true:

1. not every element of $\vec{c}$ is in $ran(g \circ \varphi_e^{\Delta_2^0})$;

2. the ordering of $\vec{c}$ is not consistent with the ordering of its preimage under $g \circ \varphi_e^{\Delta_2^0}$;

3. the size of one of the intervals determined by $\vec{c}$ is different from its preimage under $g \circ \varphi_e^{\Delta_2^0}$.

If the final outcome of α is A, then there is some r so that for *all* $s \geq r$ with $\alpha \subset \delta_s$, the outcome of α is A, and α does not actively diagonalize at any stage after r. Our construction then dictates that all approximations of $\varphi_e^{\Delta_2^0}$ including and after r map $c'_{-\infty}$ and $a'_{\alpha,r}$, two elements not in the same $\mathbb{Z}$-interval in $\mathcal{A}$, to d_∞ and $b_{\alpha,r}$, two elements in the same $\mathbb{Z}$-interval in $\mathcal{B}$. Consequently, $\varphi_e^{\Delta_2^0}$ is not an isomorphism.

QED

4.3 Some open questions about Δ_2^0-categoricity

We pose a few questions about Δ_2^0-categoricity, suggested by our results and the works of others cited here:

1. Can the hypotheses about the effectiveness of one copy of our structure be weakened? Can we produce examples to show what the weakest hypotheses might be?

2. The extra hypotheses we formulated are implied if we have 2-decidability. Goncharov showed that the existence of a single 2-decidable copy of a structure is enough to establish the equivalence of relativized and unrelativized computable categoricity. What is true for the Δ_2^0 level?

3. What exactly is the relationship between 1-decidable computable categoricity and Δ_2^0-categoricity? (See Theorem 3.20.) Is the correspondence here an instance of a more general result?

4. Is there a general notion for Δ_2^0-categoricity similar to Goncharov and Dzgoev's branching?

CHAPTER 5

PARTIAL RESULTS Δ^0_3-CATEGORICITY

We discuss results which suggest that Δ^0_3-categoricity may be very difficult to characterize for linear orderings and possible to characterize for Boolean algebras. In fact, we characterize relativized Δ^0_3-categoricity for Boolean algebras.

5.1 Linear orderings

In every definition of categoricity and dimension we have given thus far, we have assumed that our original structure $\mathcal{A}$ is computable. The definition of relative computably categoricity and the syntactic characterization given in [4] actually apply to structures that are not computable, and even to those with no computable copy.

Definition 5.1 *A structure* $\mathcal{A}$ *is* relatively Δ^0_α-categorical *if for every* $\mathcal{B} \cong \mathcal{A}$*, there is a* $\Delta^0_\alpha(\mathcal{D}(\mathcal{B}), \mathcal{D}(\mathcal{A}))$ *isomorphism* $\varphi_e^{\Delta^0_\alpha(\mathcal{D}(\mathcal{B}) \oplus \mathcal{D}(\mathcal{A}))} : \mathcal{B} \cong \mathcal{A}$.

Definition 5.2 *A* relatively *computable* Σ_α *formula for* $\mathcal{A}$*, or* $\Sigma^c_\alpha(\mathcal{D}(\mathcal{A}))$ *formula, is a* Σ_α *formula involving disjunctions and conjunctions over sets computably enumerable relative to* $\mathcal{D}(\mathcal{A})$.

Definition 5.3 *A relatively formally* Σ^0_α *Scott family for* $\mathcal{A}$ *is a c.e. family of* $\Sigma^c_\alpha(\mathcal{D}(\mathcal{A}))$ *formulas with the same two properties as in Definition 1.9.*

Theorem 5.4 *A structure* $\mathcal{A}$ *is relatively* Δ^0_α*-categorical iff it has a relatively formally* Σ^0_α *Scott family.*

When we proved that a linear ordering or Boolean algebra $\mathcal{A}$ with certain properties was not Δ_2^0-categorical, we used neither the assumption that $\mathcal{A}$ was computable, nor that the supposed Scott family had computable Σ_2 formulas. Rather, what we really proved was that any $\mathcal{A}$ with these properties had no Scott family consisting only of Σ_2 formulas. Similarly, when we proved that a linear ordering or boolean algebra $\mathcal{A}$ of a certain isomorphism type had a formally Σ_2^0 Scott family, we never used the assumption that it was a computable copy of this structure. Consequently, we completely characterized the relatively Δ_2^0-categorical linear orderings and Boolean algebras under this new definition as well. It is simply the case that each of the orderings in our class has a computable copy.

Using this more general definition of relativized computable categoricity, Knight established the following result.

Theorem 5.5 *1) For each subset S of ω, there exists a relatively Δ_3^0-categorical linear ordering $\mathcal{A}_S$ so that $\mathcal{A}_S \not\cong \mathcal{A}_T$ if $S \neq T$. Therefore, there are $2^{\aleph_0}$ different relatively Δ_3^0-categorical linear orderings.*
2) For each properly Σ_3^0 subset S of ω, there exists a computable linear ordering $\mathcal{A}_S$ so that

- *a) $\mathcal{A}_S$ is a shuffle sum of finite discrete linear orderings of size $2n+1$ for each $n \in \omega$, and $2n+2$ for each $n \in S$;*
- *b) $\mathcal{A}_S$ is relatively Δ_3^0-categorical; and*
- *c) $\mathcal{A}_S$ has no 2-decidable copy.*

Proof: Part 1): Let S be an arbitrary subset of ω. Construct for this set S the shuffle sum $\mathcal{A}_S$ as described in Part 2) for a Σ_3^0 set. First, if $S \neq T$, then $\mathcal{A}_S \not\cong \mathcal{A}_T$, because if $n \in S - T$, then $\mathcal{A}_S$ has a maximal successor chain of length $2n+2$, and $\mathcal{A}_T$ does not. Furthermore, $\mathcal{A}_S$ has a formally Σ_3^0 Scott family, because we can say all of the following with finitary Σ_3 formulas:

1. two elements x_1 and x_2 have an element between them with no successor;

2. the elements $x_1, \ldots, x_k$ are the $m_1^{th}, \ldots, m_k^{th}$ elements in a maximal successor chain of length exactly n.

Part 2): We construct the linear ordering stage-wise: at the end of each stage t, we define the finite linear ordering $\mathcal{A}_t$, and $\mathcal{A} = \bigcup_{t \in \omega} \mathcal{A}_t$.

Let ρ be a Δ_3^0 enumeration of S. By the Limit Lemma there are there are a Δ_2^0 function $f(k,s)$ so that $\lim_{s \to \infty} f(k,s) = \rho(k)$, and a Δ_1^0 function $g(k,s,t)$ so that $\lim_{t \to \infty} g(k,s,t) = f(k,s)$.

Each element that we include in the ordering at stage s will be part of a string of elements designated as a **chain**. The idea is that we guess that elements in the same chain at stage s actually comprise a maximal successor chain in the structure $\mathcal{A}$ we build.

Construction

Stage t: Let the chains at the end of stage $t-1$ be listed $c_1, \ldots, c_l$. (The chains are considered to be listed by the order in which they are designated, not by their order in $\mathcal{A}$. Therefore, the chain c_i at this stage is also the i^{th} chain at any subsequent stage; however, as we shall see, the exact elements *in* the chain c_i may change in later stages.)

For each $k, s \leq t$, compute $g(k,s,t)$. For each $k \leq t$, find the least number $s \leq t$ so that

1. $g(k,s,t) = n$; and

2. the set $S_{s,n,k,t} = \{r : s \leq r \leq t \text{ and for all } q \text{ with } s \leq q \leq r,\ g(k,q,t) = n\}$ is larger than $S_{s,n,k,u}$ for all $u < t$.

Denote this s by $s_{k,t}$. (Note that $s_{k,t}$ always exists, because if $n = g(k,t,t)$, then $S_{t,n,k,t} = \{t\}$, and $S_{t,n,k,u} = \emptyset$ for all $u < t$ by definition.)

Case I: There is already a chain $c_i, i \leq l$, marked with $\langle k, s_{k,t}, n \rangle$. For each such chain, do the following:

1. if the chain is of length $2n+2$, then leave it alone;

2. if the chain is of length $2n+3$, then split the last element from the chain, to make c_i a chain of length $2n+2$ and to make a *new* chain c_j of length 1.

Case II: There is not a chain marked with $\langle k, s_{k,t}, n \rangle$. Add a *new* chain c_j of length $2n+2$ to the ordering and mark it with $\langle k, s_{k,t}, n \rangle$.

For a chain $c_i, i \leq l$, with marking $\langle k, r, m \rangle$ do the following:

1. if $r > s_{k,t}$, then make c_i a chain of length $2m+3$ (if it is not already so) by adding a *new* element to the end of c_i, and remove any markings;

2. if $r = s_{k,t}$ and $m \neq n$, then make c_i a chain of length $2m+3$ (if it is not already so) by adding a *new* element to the end of c_i, and remove any markings;

3. if $r < s_{k,t}$, then make c_i a chain of length $2m+3$ (if it is not already so) by adding a *new* element to the end of c_i, but do NOT remove any markings.

This concludes the work we do for each $k \leq t$.

For each $m \leq t$, let a new chain of length of $2m+1$ be added to the ordering. "Shuffle" the chains currently in the ordering; if a chain is marked with $\langle k, s_{k,t}, n \rangle$, then so is every copy of it added to the ordering. This concludes the construction of $\mathcal{A}_t$.

We note the following obvious but important facts about our construction:

1. If an element a is in chain c_i at some stage t and in $c_j, j \neq i$, at another stage $u > t$, then:

   (a) at stage t, the element a is the last element in c_i, which has odd length at this stage;

   (b) and at some stage $v \leq u$ the element a is removed from the marked chain c_i to form the unmarked chain c_j of length 1.

2. Chains never change or regain markings; i.e., if a chain c_i is marked with $\langle k, s, n \rangle$ at stage t, and this marking is removed at a stage $u \geq t$, then it is *unmarked* at all stages $v \geq u$.

3. If a chain c_i is unmarked at any stage u, then at the end of stage u it is a chain of odd length $2m + 1$; it remains unmarked forever; and, in $\mathcal{A}$, it forms a maximal discrete chain of length $2m + 1$.

4. If an element a is the last element in a chain c_i of length $2m + 1$ at some stage u, then either a is the last element of a maximal discrete chain of length $2m+1$ in $\mathcal{A}$, or a has neither predecessor nor successor in $\mathcal{A}$.

Lemma 5.6 *Fix k. By the limit lemma, there is a least number s so that $f(k, r) = \rho(k)$ for all $r \geq s$. This number s is also the least number so that $s_{k,t} = s$ for infinitely many t.*

Proof: Let s be as stated, and let $u \geq s, k$ be such that for all $t \geq u$ and all $q \leq s$, $g(k, q, t) = f(k, q)$.

Let $r < s$. Therefore, $r \leq s - 1 < s$ and $f(k, s - 1) \neq f(k, s)$. For all $t \geq u$, $g(k, r, t) = f(k, r)$, and $S_{r,f(k,r),k,t} \subseteq \{r, \ldots, s-1\}$. Therefore, there must be a $t' \geq u$ so that $|S_{r,f(k,r),k,t'}| \geq |S_{r,f(k,r),k,t}|$ for all $t \geq t'$. For all $t > t'$, $s_{k,t} \neq r$.

Assume that there is some $t' \geq u$ so that for all $t \geq t'$, $|S_{s,f(k,s),k,t'}| \geq |S_{s,f(k,s),k,t}|$. But $|S_{s,f(k,s),k,t'}| \leq t'+1$ by definition. Let $u' \geq u, (s+t'+1)$ be such that for all $t \geq u'$ and all q with $s \leq q \leq (s+t'+1)$, $g(k,q,t) = f(k,q) = f(k,s)$. Then $g(k,s,t) = f(k,s)$ and $S_{s,f(k,s),k,u'} \supseteq \{s, \ldots, s+t'+1\}$, so $|S_{s,f(k,s),k,u'}| \geq (t'+2)$, a contradiction. Therefore, there are infinitely many stages t at which $g(k,s,t) = f(k,s)$ and $S_{s,f(k,s),k,t}$ is bigger than $S_{s,f(k,s),k,v}$ for all $v \leq t$. And so, by the definition of $s_{k,t}$ and the above paragraph, s is the least number for which there are infinitely many stages t with $s_{k,t} = s$. QED

Lemma 5.7 *An element $a \in \mathcal{A}$ is in a maximal discrete chain of length $2n+2$ iff there are k, s, n, u, t and a chain c_i so that*

1. *s is the least number for which $f(k,r) = \rho(k) = n$ for all $r \geq s$;*
2. *t is a stage for which*
   (a) *$s_{k,t} = s$;*
   (b) *for all $r < s$ and $v > t$, $s_{k,v} \neq r$;*
   (c) *for all $v \geq t, g(k,s,v) = f(k,s)$; and*
3. *at the end of some stage $u \geq t$, the chain c_i has length at least $2n+2$; the element a is one of the first $2n+2$ elements in c_i; and c_i is marked with $\langle k,s,n \rangle$.*

Proof: ($\Rightarrow$) Assume that a is in a maximal chain of length $2n+2$. The facts at the end of the construction guarantee that there exist a chain c_i, a stage u, and a mark $\langle k,s,n \rangle$ so that the element a is one of the first $2n+2$ elements in c_i, and c_i is marked with $\langle k,s,n \rangle$ at all stages $v \geq u$.

If there is $r < s$ so that at some stage $v > u$, $r = s_{k,v}$, then all marks $\langle k,s,n \rangle$ are removed at stage v. Thus, c_i becomes a chain of length $2n+3$ and its mark is removed at stage v; therefore, in $\mathcal{A}$, a is in a maximal discrete chain of length $2n+3$, a contradiction.

If there is a stage $u' \geq u$ so that $s_{k,v} \neq s$ for all $v \geq u'$, then c_i is chain of length $2n+3$ at stage u', and the last element is never "removed" from the chain. Consequently, in $\mathcal{A}$, a is in a maximal discrete chain of length $2n+3$, a contradiction. Thus, s is the least number so that $s_{k,v} = s$ for infinitely many v. By the previous lemma, s is the least number so that $f(k,r) = \rho(k)$ for all $r \geq s$. Therefore, there is a stage t which satisfies condition 2, and we can, if necessary, redefine u to be greater than or equal to t.

Finally, if $n \neq \rho(k)$, then let $v \geq u$ so that $s_{k,v} = s$, and $g(k,s,v) = f(k,s) = \rho(k) \neq n$. Then c_i, which is marked with $\langle k,s,n\rangle$, becomes a chain of length $2n+3$ (if it is not so already), and its mark is removed. Consequently, in $\mathcal{A}$, a is in a maximal discrete chain of length $2n+3$, a contradiction. Therefore, $n = \rho(k)$.

($\Leftarrow$) Assume there are k, s, n, u, t, and c_i satisfying the above conditions for $a \in \mathcal{A}$. By the construction, c_i cannot lose its marking after stage u. By the previous lemma, there is $t' \geq u$ with $s_{k,t'} = s$; at this stage, the chain $c_i = a_1 < \cdots < a_{2n+1}$, and one of these elements is a. At later stages, we possibly add elements only to the end of this chain, not to the beginning or between elements, and we "shuffle" chains at the end of every stage. Therefore, a_1 definitely has no immediate predecessor, and a_{i+1} is the direct successor of a_i for $i = 1, \ldots, 2n$. And so, we need only verify that a_{2n+2} has no immediate successor. If $b > a_{2n+2}$ is in another chain some stage, then certainly it is not an immediate successor, because we never combine chains in our construction, and we shuffle chains at the end of every stage. If $b > a_{2n+2}$ is in the same chain at some stage $u' \geq t'$, then at the first stage $t'' \geq u'$ in which $s_{k,t''} = s$, this element b is removed from the chain with a_{2n+1} and placed in a chain of length 1. QED

We are now ready to finish our proof. Assume that $n \in S$. Then there are k, s, n, t so that

1. s is the least number for which $f(k, r) = \rho(k) = n$ for all $r \geq s$; and

2. t is a stage for which

   (a) $s_{k,t} = s$;

   (b) for all $r < s$ and $v > t$, $s_{k,v} \neq r$;

   (c) for all $v \geq t, g(k, s, v) = f(k, s)$.

At the end of this stage t, we define a chain c_i of exactly $2n + 2$ and mark it with $\langle k, s, n \rangle$. By the previous lemma and its proof, we know that this chain is, in $\mathcal{A}$, a maximal discrete chain of length $2n + 2$.

By condition 1 of the previous lemma and the properties of the functions $f(k, r)$ and $\rho(k)$, if there is a maximal discrete chain of length $2n + 2$ in $\mathcal{A}$, then $n \in S$.

The actions performed at the end of each stage guarantee that there are maximal discrete chains of every odd length and that $\mathcal{A}$ is indeed a shuffle sum of the desired discrete chains.

Finally, we note that $\mathcal{A}$ has no 2-decidable copy. If it did, then we could computably list the elements of S, since we could determine the exact length of any maximal finite chain in $\mathcal{A}$. Of course, this is a contradiction if S is a properly Σ_3^0 set.

QED

The first part of this theorem shows that the relativized and unrelativized Δ_3^0-categorical linear orderings comprise two very different classes. The second part suggests that perhaps the task of characterizing the Δ_3^0-categorical linear orderings will prove extremely difficult, perhaps impossible.

In addition, the second part reveals that Δ^0_3-categoricity and computable categoricity of 2-decidable linear orderings do not define the same class of orderings. Therefore, there is not, in general, a correspondence between the Δ^0_{n+1}-categorical linear orderings and the n-decidable computably categorical linear orderings, as Theorem 3.20 and Theorem 4.5 might have suggested.

5.2 Boolean algebras

Recall the definition of $\mathcal{A}/\sim$ given in Definition 3.6.

Definition 5.8 *A Boolean algebra $\mathcal{A}$ is* rank 1 *iff $\mathcal{A}/\sim$ is a nontrivial atomless Boolean algebra.*

Definition 5.9 *A Boolean algebra $\mathcal{A}$ is atomic iff for each $a \in \mathcal{A}$, there is an atom $b \leq a$.*

Proposition 5.10 *Any countable rank 1 atomic Boolean algebra is isomorphic to $I(2 \cdot \eta)$.*

Proof: Let $\mathcal{A}$ be a rank 1 atomic Boolean algebra, and let $\mathcal{B} = I(2 \cdot \eta)$. Let $p_0(0_{\mathcal{A}}) = 0_{\mathcal{B}}$, and $p_0(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Assume p_n is given and isomorphically maps a finite subalgebra $\mathcal{A}_n$ onto a finite subalgebra $\mathcal{B}_n$ so that p_n preserves the size of elements. Let one of the atoms a of $\mathcal{A}_n$ be split to obtain a new finite subalgebra whose atoms include a_1, a_2, where $a_1 \cup a_2 = a$. Because a and $p_n(a)$ have the same size, and any infinite element of $\mathcal{A}$ or $\mathcal{B}$ has infinitely many atoms and can be split into two infinite elements, we can easily find b_1, b_2 so that $b_1 \cup b_2 = p_n(a)$, b_1 has the same size as a_1, and b_2 has the same size as a_2. The "back" argument works in precisely the same way. QED

Proposition 5.11 *Let $\mathcal{A}$ be a countable Boolean algebra so that*

1. *$\mathcal{A}$ has infinitely many atoms, but cannot be split into 2 algebras each having infinitely many atoms; and*

2. *any element* $a \in \mathcal{A}$ *bounding infinitely many atoms must bound an atomless element, as well.*

Then $\mathcal{A}$ *is isomorphic to* $I(\omega + \eta)$.

Proof: Let $\mathcal{A}$ have the above properties, and let $\mathcal{B} = I(\omega + \eta)$. Let $p_0(0_\mathcal{A}) = 0_\mathcal{B}$, and $p_0(1_\mathcal{A}) = 1_\mathcal{B}$. Assume p_n is given and isomorphically maps a finite subalgebra $\mathcal{A}_n$ onto a finite subalgebra $\mathcal{B}_n$ so that p_n preserves the size of elements and the number of atoms bound by elements. Let one of the atoms a of $\mathcal{A}_n$ be split to obtain a new finite subalgebra whose atoms include a_1, a_2, where $a_1 \cup a_2 = a$. If a is finite or contains only finitely many atoms, then we can easily find images b_1, b_2 so that $b_1 \cup b_2 = p_n(a)$, and each b_i matches a_i both in size and number of atoms. If a contains infinitely many atoms, then exactly one of a_1, a_2 does; assume that it is a_1. If a_2 is finite or infinite, we can find images b_1, b_2 so that so that $b_1 \cup b_2 = p_n(a)$ and each b_i matches a_i both in size and number of atoms. The "back" argument works the same way.

QED

Theorem 5.12 *Let* $\mathcal{A}$ *be a Boolean algebra which can be expressed as a direct sum of finitely many algebras of the following form: 1) an atom; 2) an atomless element; 3) a 1-atom; 4) a rank 1 atomic element; 5) an element isomorphic to the interval algebra* $I(\omega+\eta)$. *Then* $\mathcal{A}$ *has a formally* Σ^0_3 *Scott family, and hence is* Δ^0_3*-categorical.*

Proof: If the summands are all either atomless or atoms, then it is Δ^0_1-categorical by Goncharov. If $\mathcal{A}$ has summands of type 3), 4), or 5), then absorb all of the atoms into one of these, so we write $\mathcal{A} = c_1 \vee \cdots \vee c_n$, where each c_i is of type 2) - 5). Let $\vec{c}$ consist of the summands of the type listed above. Let $\vec{a} = a_1, \ldots, a_j \in \mathcal{A}$, and let $b_1, \ldots, b_{2^j}$ the atoms in the *formal* finite subalgebra generated by $\vec{a}$. (Some of the b_i's might equal 0.) For each b_k and each c_i, we construct the following formulas:

1) If c_i is atomless or a 1-atom, then $\theta_i^{b_k}(y_k, c_i)$ is constructed as in Chapter 3.

2) If c_i is a rank 1 atomic element, then $\theta_i^{b_k}(y_k, c_i)$ is constructed to say that $y_k \cap c_i$ is finite of size n, or cofinite in c_i with complement of size n, or infinite and coinfinite in c_i, depending on what is true of b_k. As we have seen, there is a finitary Σ_2 formula $\theta_n(x)$ saying that x is of size n. Therefore, the formula $\bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}} \neg\theta_n(x)$ is a Π_2^c formula which expresses that x is infinite.

3) If c_i is an element isomorphic to the interval algebra $I(\omega + \eta)$, then $\theta_i^{b_k}(y_k, c_i)$ is constructed to say that $y_k \cap c_i$ is finite of size n; contains all but n atoms of c_i, but is not cofinite in c_i; is sum of n atoms and an atomless element; or is cofinite in c_i, depending on what is true of b_k. First, we note that the formula which says "z is an atom" is finitary Π_1. Therefore, the formula $\gamma_n(x) = \exists z_1 \cdots \exists z_n(\bigwedge\!\!\!\bigwedge_{m=1\ldots n}$ "z_m is an atom" $\wedge\, z_m < c_i\ \wedge z_m \not< x)$ is a finitary Σ_2 formula expressing that there are n atoms of c_i not contained in x. Consequently, $\gamma_n(x) \wedge \neg\gamma_{n+1}(x)$ is equivalent to a finitary Σ_3 formula which expresses that x contains all but n of the atoms of c_i. By our argument in 2) above, there is a Π_2^c which expresses that x is coinfinite in c_i. Finally, the formula $\rho(z) = \forall w(w \not< z \vee \exists v(v < w))$ is a finitary Π_2 formula expressing that z is atomless. Consequently, $\exists z_1 \cdots \exists z_{n+1}(\bigwedge\!\!\!\bigwedge_{m\neq n+1}$ "z_m is an atom" $\wedge$ "z_{n+1} is atomless" $\wedge z_1 \cup \ldots \cup z_{n+1} = x$ is a finitary Σ_3 formula which expresses that x is a join of n atoms and an atomless piece.

For a tuple of variables $\vec{x} = x_1, \ldots, x_j$, let $\vec{y} = y_1, \ldots, y_{2^j}$ be the terms in the formal finite subalgebra determined by $\vec{x}$. Let $\psi_{\vec{a}}(\vec{x}, \vec{c}) = \bigwedge\!\!\!\bigwedge_{i\in\{1,\ldots,2^j\}} \theta_1^{b_i} \wedge \cdots \wedge \theta_n^{b_i}$. Of course, $\mathcal{A} \models \psi_{\vec{a}}(\vec{a}, \vec{c})$. Furthermore, if for some $\vec{a}\,'$, $\mathcal{A} \models \psi_{\vec{a}}(\vec{a}\,', \vec{c})$, then we immediately have $(\mathcal{A}, \vec{a}, \vec{c}) \cong (\mathcal{A}, \vec{a}\,', \vec{c})$. Consequently, $\{\psi_{\vec{a}} | \vec{a} \in \mathcal{A}\}$ is a c.e. Scott family of Σ_3^c formulas. QED

Lemma 5.13 *Let $\mathcal{A}$ be a Boolean algebra. Suppose that for each $\vec{c}$, we can find an element a, bound by of one of the atoms c of the subalgebra determined by $\vec{c}$, so that the following is true:*

Let $\vec{u}$ be given. Consider the subalgebra determined by $a, \vec{u}, \vec{c}$; let $\alpha_1, \ldots, \alpha_n$ be the atoms whose union is a, and let $\beta_1, \ldots, \beta_m$ be the atoms whose union is $c - a$. Then we can find $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_m$ so that

1. *if α_i bounds finitely many atoms, then α'_i bounds the same number of atoms; same for β_j;*

2. *α_i is finite iff α'_i is;*

3. *$\alpha'_1 \cup \cdots \cup \alpha'_n \cup \beta'_1 \cup \beta'_m = c$; and*

4. *if $a' = \alpha'_1 \cup \ldots \cup \alpha'_n$, then $(\mathcal{A}, a, \vec{c}) \not\cong (\mathcal{A}, a', \vec{c})$.*

Then $\mathcal{A}$ has no formally Σ^0_3 Scott family.

Proof: In order to show that $\mathcal{A}$ has no formally Σ^0_3 Scott family, we fix $\vec{c}$. Then we must show that there is an $\vec{a}$ so that

$\forall \vec{u}\, \exists \vec{a}\,'\, \vec{u}\,' \forall \vec{v}\,' \exists \vec{v}$(each atom of the finite subalgebra determined by $\vec{a}\,', \vec{u}\,', \vec{v}\,', \vec{c}$ is at least as large as the corresponding atom of $\vec{a}, \vec{u}, \vec{v}, \vec{c}$, but $(\mathcal{A}, a, \vec{c}) \not\cong (\mathcal{A}, a', \vec{c})$).

Fix $\vec{c}$, and let a be as in the hypothesis of the lemma. Then if $\vec{u}$ is given, let $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_m$ be as in the hypothesis, and define $a', \vec{u}\,'$ accordingly. Let $\vec{v}\,'$ be given, and consider the finite subalgebra determined by $\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_m, \vec{v}\,'$. Assume $\alpha'_i = \delta'_{i1} \cup \cdots \cup \delta'_{in_i}$, where the δ' elements are atoms of the subalgebra. If we consider only the δ'_{ik} elements which are finite, then we conclude that α_i must bound at least as many atoms of $\mathcal{A}$ as compose the δ_{ik} elements, and we define $\delta_{i1}, \ldots, \delta_{in_i}$ accordingly. We make the same argument and construction for any β'_j. We can then define $\vec{v}$ based on the designation of these atoms of the finite subalgebra. Consequently, each atom of the subalgebra determined by $a', \vec{u}', \vec{v}', \vec{c}$ is at least as large as the corresponding atom in the subalgebra determined by $a, \vec{u}, \vec{v}, \vec{c}$. However, the hypothesis guarantees that $(\mathcal{A}, a, \vec{c}) \ncong (\mathcal{A}, a', \vec{c})$. So $\mathcal{A}$ has no formally Σ^3_0 Scott family. QED

Proposition 5.14 *Let $\mathcal{A}$ be a Boolean algebra. If $\mathcal{A}$ has infinitely many disjoint 1-atoms, then it has no formally Σ^0_3 Scott family.*

Proof: Let $\vec{c}$ be given. One of the atoms c of the subalgebra determined by $\vec{c}$ must bound infinitely many disjoint 1-atoms. Let a be one of these 1-atoms. For any $\vec{u}$ as in the statement of Lemma 5.13, exactly one of the α_i bounds infinitely many atoms. Let α'_i be a union of two 1-atoms contained in c. We can define α' and β' elements to satisfy the hypotheses of Lemma 5.13. QED

Proposition 5.15 *Let $\mathcal{A}$ be a Boolean algebra. Suppose that for any $\vec{c}$, there is an atom c of the finite subalgebra determined by $\vec{c}$ so that c bounds an element a where*

1. *a bounds infinitely many atoms and an atomless element; and*
2. *$c - a$ bounds infinitely many atoms and an atomless element.*

Then $\mathcal{A}$ has no formally Σ^0_3 Scott family.

Proof: Fix $\vec{c}$, and consider a as the hypothesis. Let $\vec{u}$ be given, and consider $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_m$ as in Lemma 5.13. Define each α'_i to bound finitely many atoms, and an atomless element if α_i is infinite. Find β_j which bounds infinitely many atoms. For $k \neq j$, define β_k to bound finitely many atoms, and an atomless element if β_k is infinite. Let $\beta'_j = c - (\bigcup_{1 \leq i \leq n} \alpha'_i \cup \bigcup_{j \neq k} \beta'_k)$. So for each $\vec{c}$, there is an a as in Lemma 5.13. QED

Proposition 5.16 *Let $\mathcal{A}$ be a countable Boolean algebra. If $\mathcal{A}$ has a rank 1 atomic element but no maximal rank 1 atomic element, then $\mathcal{A}$ has no formally Σ^0_3 Scott family.*

Proof: Let $\vec{c}$ be the tuple of the Scott family. Assume, without loss of generality, that $\vec{c}$ includes all of the disjoint 1-atoms of $\mathcal{A}$ in the following sense: if d is a 1-atom of $\mathcal{A}$, then it is equal to one of the elements of $\vec{c}$ up to a finite difference. (By Proposition 5.14 there can be only finitely many disjoint 1-atoms in $\mathcal{A}$.) Thus, every atom of the finite subalgebra determined by $\vec{c}$ must satisfy one of the following conditions:

1. it is bound by one of the 1-atoms of $\vec{c}$;

2. it is finite;

3. it is rank 1.

Otherwise, an infinite atom c, disjoint from of all of the 1-atoms of $\mathcal{A}$, bounds a b so that $[b]$ is an atom of $\mathcal{A}/\sim$. By definition, b is a 1-atom, a contradiction.

If one of the atoms bounds a rank 1 atomic element and an atomless element, then by Proposition 5.15, $\vec{c}$ is not the tuple in a formally Σ_3^0 Scott family. Consequently, any atom of the subalgebra bounding a rank 1 atomic element is itself rank 1 atomic. The union of all such rank 1 atomic elements of $\mathcal{A}$ is a maximal rank 1 atomic element.

QED

Theorem 5.17 *Let $\mathcal{A}$ be a countable Boolean algebra with a formally Σ_3^0 Scott family. Then $\mathcal{A}$ can be expressed as a direct sum of finitely many algebras of the following form: 1)finite; 2) atomless; 3) 1-atoms; 4) rank 1 atomic; 5) $I(\omega+\eta)$.*

Proof: Let $\vec{c}$ be the tuple of the Scott family. Assume, without loss of generality, that $\vec{c}$ includes all of the 1-atoms of $\mathcal{A}$ and the maximal rank 1 atomic element, if it exists. Every atom of the finite subalgebra determined by $\vec{c}$ must satisfy one of the following conditions:

1. it is bound by one of the 1-atoms or the maximal rank 1 atomic element;

2. it is finite;

3. it is rank 1 but bounds no rank 1 atomic element.

Any atom c satisfying 3) must bound either finitely many atoms or infinitely many. If finitely many, then we are done. If c bounds infinitely many, then any $d \leq c$ which bounds infinitely many atoms must also bound an atomless element; otherwise, d is rank 1 atomic, a contradiction. If c can be split into two disjoint elements each bounding infinitely many atoms, then by the previous statement, each one of these elements bounds infinitely many atoms and an atomless element. By Proposition 5.15, $\vec{c}$ cannot be the tuple in a formally Σ_3^0 Scott family. Consequently, c cannot be split into two such elements, each having infinitely many atoms. By Proposition 5.11, each such c is an algebra isomorphic to $I(\omega + \eta)$. QED

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