Ramsey Theory and Reverse Mathematics

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The resulting paper

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http://www.nd.edu/~cholak/papers/
http://www.nd.edu/~cholak/papers/italy.pdf

Ramsey's theorem

- $[X]^n = \{Y \subseteq X : |Y| = n\}$.
- A k-coloring C of $[X]^n$ is a function from $[X]^n$ into a set of size k.
- H is *homogeneous* for C if C is constant on $[H]^n$, i.e. all n-element subsets of H are assigned the same color by C.
- RT^n_k is the statement that every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

Reverse mathematics

What are the consequences of Ramsey's theorem (and its natural special cases) as a formal statement in second order arithmetic?

Questions

Question (Relation between 2nd order statements)

Does RT^2_2 imply WKL? Does $RT^2_{<\infty}$ imply WKL?

Question (First order consequences)

Is RT^2_2 Π^1_1-conservative over $RCA_0 + B\Sigma_2$? Is $RT^2_{<\infty}$ Π^1_1-conservative over $RCA_0 + B\Sigma_3$?

Question (Π^0_2 consequences)

Is RT^2_2 Π^0_2-conservative over RCA_0? In particular, does RT^2_2 prove the consistency of $P^- + I\Sigma_1$? Does RT^2_2 prove that Ackerman's function is total?

Computability theory

Study the complexity (in terms of the arithmetical hierarchy or degrees) of infinite homogeneous sets for a coloring C relative to that of C. (For simplicity, assume that C is computable (recursive) and relativize.)

Language

Use the sorted language with the symbols: $=, \in, +, \times, 0, 1, <$;
Number variables: $n, m, x, y, z \ldots$; Set Variables: $X, Y, Z \ldots$.

$$p \text{ is prime.}$$

This just uses bounded quantification.

$$\forall \delta \exists \epsilon [|x - c| < \epsilon \Rightarrow |x^2 - c^2| < \delta]$$

This is an example of a Π^0_2 formula. The negation is Σ^0_2. A formula which is logically equivalent (over our base theory) to both a Π^0_n formula and a Σ^0_n formula is Δ^0_n.

Induction

$I\Sigma_n$ is the following statement: For every $\varphi(x)$, a Σ^0_n formula, if $\varphi(0)$ and $\forall x[\varphi(x) \Rightarrow \varphi(x+1)]$ then $\forall x[\varphi(x)]$.

Over our base theory, $I\Sigma^0_n$ and $I\Pi^0_n$ are equivalent. $I\Sigma_n$ is also equivalent to every Π^0_n-definable set (Σ^0_n set) has a least element.

Comprehension

Δ^0_1 comprehension is the statement: For every $\varphi(x)$, a Δ^0_1 formula, there is an X such that $X = \{x : \varphi(x)\}$.

For example, Δ^0_1 comprehension implies the set of all primes exists.

Arithmetic comprehension is the statement: For every $\varphi(x)$, a Δ^0_n formula, there is an X such that $X = \{x : \varphi(x)\}$.

Bounding

Statement ($B\Sigma_n$)

For every $\varphi(x, y)$, a Σ_n^0 formula, if $\forall x \exists y[\varphi(x, y)]$ then for all a there is a b such that $\forall x \leq a \exists y \leq b[\varphi(x, y)]$.

Every initial segment of a Σ_n^0 function is bounded.

$B\Sigma_{n+1}$ is stronger that $I\Sigma_n$ but not as strong as $I\Sigma_{n+1}$.

2^{nd}-order arithmetic

We work over models of 2^{nd}-order arithmetic.

The intended model: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), +, \times, 0, 1, <)$.

P^- is the theory of finite sets. The base theory, RCA_0, is the logical closure of $P^- + I\Sigma^0_1$ and Δ^0_1 comprehension. PA is P^- plus arithmetic induction.

$$(\mathbb{N}, \{\text{all computable sets}\}, +, \times, 0, 1, <) \models RCA_0.$$

ACA_0 is RCA_0 plus arithmetic comprehension.

$$(\mathbb{N}, \{\text{all arithmetic sets}\}, +, \times, 0, 1, <) \models ACA_0.$$

Statements in 2nd-order arithmetic

Statement (*WKL*)

Every infinite tree of binary strings has an infinite branch.

$\forall T \exists P$[if T is an infinite binary branching tree then P is an infinite path through T].

Statement (RT^n_k)

For every infinite set X and for every k-coloring of $[X]^n$ there is an infinite homogeneous set H.

Statement ($RT^n_{<\infty}$)

For every k, RT^n_k.

Statement (*RT*)

For every n, $RT^n_{<\infty}$.

These are Π^1_2 sentences: look at the set quantifiers ignore the (inside) number quantifiers.

Conservation

Definition

If T_1 and T_2 are theories and Γ is a set of sentences then T_2 is Γ-conservative over T_1 if $\forall\varphi[(\varphi \in \Gamma \wedge T_2 \vdash \varphi) \Rightarrow T_1 \vdash \varphi]$.

Theorem (H. Friedman)

ACA_0 is arithmeticly conservative over PA.

Computability Theory

$A \leq_T B$ iff there is a computer which using an oracle for B can compute A. For all A, $\emptyset \leq_T A$.

A' (read A-jump) is all those programs e which using A as an oracle halt on input e. $A^{(n)}$ is the n^{th} jump of A. The jump operation is order preserving.

So for all A, $\emptyset^{(n)} \leq_T A^{(n)}$. A is low$_n$ iff $A^{(n)} \leq_T \emptyset^{(n)}$.

Key idea: if A is low$_n$ then sets which are Δ^0_{n+1} in A are Δ^0_{n+1} in $\emptyset$.

WKL

Theorem (Jockusch and Soare)

[The Low Basis Theorem] Every infinite computable tree of binary strings has a low path (working in the standard model).

Theorem (Harrington)

$RCA_0 + WKL$ is Π^1_1-conservative over RCA_0.

Adding a path

Lemma (Harrington)

If $\mathcal{M} = (\mathbb{X}, \mathcal{F}, +, \times, 0, 1, <)$ is a model of RCA$_0$, $T \in \mathcal{F}$ and T codes an infinite tree of binary strings then there is a $G \subset \mathbb{X}$ such that $\mathcal{M}' = (\mathbb{X}, \mathcal{F} \cup G, +, \times, 0, 1, <)$ is a model of $I\Sigma_1$ and P^- and G is an infinite path through T.

Lemma (H. Friedman)

Any model of P^- and $I\Sigma_n$ can be expanded to a model of RCA$_0$ + $I\Sigma_n$ by only adding reals.

Iterating the addition of a path

Corollary (Harrington)

Every countable model $\mathcal{M}$ of RCA_0 is a ω-submodel (the integers do not change) of some countable model $\mathcal{M}'$ of $RCA_0 + WKL$.

By Gödel completeness, this implies Theorem 12. All Σ^1_1 sentences true in $\mathcal{M}$ are true in $\mathcal{M}'$.

Lemma

Every countable model of $RCA_0 + I\Sigma_n$ is a ω-submodel of some countable model of $RCA_0 + I\Sigma_n + WKL$.

Ramsey's Theorem – Known Results

Theorem (Specker)

There is a computable 2-coloring of $[\mathbb{N}]^2$ with no infinite computable homogeneous set.

Corollary (Specker)

RT^2_2 *is* not *provable in* RCA_0.

Results of Jockusch

Theorem (Jockusch)

1. *For any n and k, any computable k-coloring of $[\mathbb{N}]^n$ has an infinite Π^0_n homogeneous set.*
2. *For any $n \geq 2$, there is a computable n-coloring of $[\mathbb{N}]^n$ which has no infinite Σ^0_n homogeneous set.*
3. *For any n and k and any computable k-coloring of $[\mathbb{N}]^n$, there is an infinite homogeneous set A with $A' \leq_T 0^{(n)}$.*
4. *For each $n \geq 2$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that $0^{(n-2)} \leq_T A$ for each infinite homogeneous set A.*

Jockusch's results into Reverse Mathematics

Theorem (Simpson)

1. *For each $n \geq 3$ and $k \geq 2$ (both n and k standard), the statements RT^n_k are equivalent to ACA_0 over RCA_0.*
2. *The statement RT is not provable in ACA_0.*
3. *RT is equivalent to ACA_0 plus for all n, for all X, the n^{th}-jump of X exists.*
4. *RT does not prove ATR.*
5. *ATR proves RT.*

Cone Avoidance

Theorem (Seetapun)

For any computable 2-coloring C of $[\mathbb{N}]^2$ and any noncomputable sets $C_0, C_1, \ldots$, there is an infinite homogeneous set X such that $(\forall i)[C_i \not\leq_T X]$.

Corollary (Seetapun)

RT^2_2 does not imply ACA_0. Hence, over RCA_0, RT^2_2 is strictly weaker than RT^3_2.

First order consequences

Theorem (Hirst)
RT^2_2 proves $B\Sigma_2$.

Corollary (Hirst)

- *RT^2_2 is stronger than RCA_0.*
- *RT^2_2 is not Σ^0_3-conservative over RCA_0.*

$B\Sigma_2$ is strictly between $I\Sigma_1$ and $I\Sigma_2$.

Our Work – Computability Theory

Theorem

For any computable 2–coloring of $[\mathbb{N}]^2$, there is an infinite homogeneous set X which is low$_2$, i.e. $X'' \leq_T 0''$.

Definition

An infinite set X is *r-cohesive* if for each computable set R, $X \subseteq^* R$ or $X \subseteq^* \overline{R}$.

Theorem (Jockusch and Stephan)

There exists a low$_2$ r-cohesive set.

Theorem

For each Δ^0_2 set A there is an infinite low$_2$ set X which is contained in A or $\overline{A}$.

Definition (S)

A k–coloring of $[X]^2$ is called *stable* if for each a, the color assigned to the pair $\{a,b\}$ has a fixed color c_a for all sufficiently large b (i.e., there is a d_a such that for all b greatly than d_a, the color of $\{a,b\}$ is c_a).

Now any computable coloring of pairs (from $\mathbb{N}$) becomes stable when it is restricted to an r-cohesive set.

Lemma

For any computable stable 2-coloring C of $[\mathbb{N}]^2$, there are 2 disjoint Δ_2^0 sets A_i such that $\bigsqcup_{i<2} A_i = \mathbb{N}$ and any infinite subset of some A_i computes an infinite homogeneous set for C.

Our work – Reverse mathematics

Theorem

$RCA_0 + RT^2_2$ is Π^1_1-conservative over $RCA_0 + I\Sigma_2$.

Corollary

RT^2_2 does not imply PA over RCA_0.

This improves Seetapun's result that RT^2_2 does not imply ACA_0 over RCA_0.

$RT_2^2 = COH + SRT_2^2$

Statement (COH)

Let R_i be a sequence of sets. Then there is a set G such that for all i, either $G \subseteq^* R_i$ or $G \subseteq^* \overline{R_i}$. (There "modulo finite" is coded by finite sets in our models of arithmetic.)

Over RCA_0, RT_2^2 follows from $COH + SRT_2^2$.

Theorem (Mileti, Lempp and Jockusch)

Over RCA_0, RT_2^2 and $COH + SRT_2^2$ are equivalent.

COH

Theorem

$RCA_0 + COH$ is Π^1_1-conservative over RCA_0.

Theorem

Every countable model of RCA_0 ($+I\Sigma_2$) is a ω-submodel of some countable model of $RCA_0 + COH$ ($+I\Sigma_2$).

Start with any model of RCA_0. Given a sequence of sets, R_i, add a $\vec{R}$-cohesive set G while preserving $I\Sigma_1$. Close to get a model of RCA_0. Iterate over all such sequences to get a model of RCA_0+COH.

Adding a cohesive set G

Force using the conditions $\langle D, L\rangle$, where D is finite, L is an infinite set in the ground model and $\max D < \min L$. We say $\langle D', L'\rangle$ extends $\langle D, L\rangle$ iff $D \subseteq D' \subset D \cup L$ and $L' \subseteq L$.
If G is any generic set then either $G \subseteq^* R_i$ or $G \subseteq^* \overline{R}_i$.
To preserve $I\Sigma_1$, for all $\psi(x, G)$, a Σ_1^0-formula and all numbers a, we want to ensure either $\forall x \leq a[\psi(x, G)]$ or for some b, $\neg\psi(b, G) \wedge \forall x < b\psi(x, G)]$.

The key to preserving $I\Sigma_1$

Let $\langle D, L\rangle$ be a given condition. By $I\Sigma_1$ in the ground model, there is a least $c \leq a$ (if any) such that $\langle D, L\rangle$ cannot be extended to a condition which forces $\forall y \leq c[\psi(y, G)]$ (whether such an extension of $\langle D, L\rangle$ exists is Σ_1^L). If there is no such c, then $\langle D, L\rangle$ has an extension which forces $\forall x \leq a[\psi(x, G)]$. If there is such a c, then by minimality, there is a condition $\langle D', L'\rangle$ extending $\langle D, L\rangle$ which forces $\forall x < c[\psi(x, G)]$ and $\neg\psi(c, G)$ (since it has no extension which forces $\psi(c, G)$).

COH and *WKL*

Theorem

COH and WKL are independent over RCA_0.

SRT^2_2

Theorem

$RCA_0 + SRT^2_2 + WKL$ is Π^1_1-conservative over $RCA_0 + I\Sigma_2$.

Theorem

Every countable model of $RCA_0 + I\Sigma_2$ is a ω-submodel of some countable model of $RCA_0 + I\Sigma_2 + WKL + SRT^2_2$.

Theorem

SRT^2_2 proves $B\Sigma_2$. Hence SRT^2_2 is not Σ^0_3-conservative over RCA_0.

Why $I\Sigma_2$ and *WKL*?

- Which color? Red or Blue?
- Now whether we can extend a condition to another which forces a Σ^0_1 (Π^0_1) statement is no longer Σ^0_1 (Π^0_1).
- But using *WKL*, whether we can extend a condition to another which forces a Σ^0_2 (Π^0_2) statement is Σ^0_2 (Π^0_2).

RT^2_2

Theorem

$RCA_0 + RT^2_2 + WKL$ is Π^1_1-conservative over $RCA_0 + I\Sigma_2$.

Theorem

Every countable model of $RCA_0 + I\Sigma_2$ is a ω-submodel of some countable model of $RCA_0 + I\Sigma_2 + WKL + SRT^2_2 + COH$.

Finitely many colors

Theorem

$RCA_0 + RT^2_{<\infty}$ is Π^1_1-conservative over $RCA_0 + I\Sigma_3$.

Theorem

Every countable model of $RCA_0 + I\Sigma_3$ with a real of greatest Turing degree is a ω-submodel of some countable model of $RCA_0 + I\Sigma_3 + WKL + SRT^2_{<\infty} + COH$.

Theorem

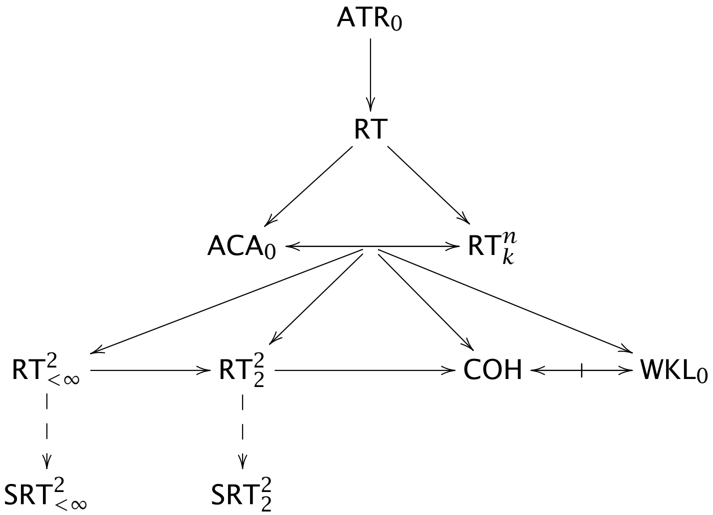
$RCA_0 + SRT^2_{<\infty} \vdash B\Sigma_3$.

Since $B\Sigma_3$ is stronger that $I\Sigma_2$, we have that RT^2_2 does not imply $RT^2_{<\infty}$.

Why $I\Sigma_3$ and a real of greatest Turing degree?

- To determine which color we need $I\Sigma_3$ to hold in the ground model.
- Again using *WKL*, whether we can extend a condition to another which forces a Σ^0_2 (Π^0_2) statement is Σ^0_2 (Π^0_2).
- But we need to preserve $I\Sigma_3$. This requires us to quantify over the conditions.

Some second order consequences



Some first order consequences

Theorem

Let $(\varphi)^1$ *be the first order consequences of* $\varphi + RCA_0$.

1. $(RCA_0)^1 = (WKL_0)^1 = (COH)^1$.
2. $(RCA_0)^1 \subsetneq (B\Sigma_2)^1 \subseteq (SRT^2_2)^1 \subseteq (RT^2_2)^1 \subseteq (I\Sigma_2)^1$.
3. $(I\Sigma_2)^1 \subsetneq (B\Sigma_3)^1 \subseteq (SRT^2_{<\infty})^1 \subseteq (RT^2_{<\infty})^1 \subseteq (I\Sigma_3)^1$.
4. $(I\Sigma_3)^1 \subsetneq PA = (RT^3_2)^1 = (RT^k_n)^1$ *(for any standard* $k \geq 3$ *and* $n \geq 2$*).*

Questions

Question (Relation between 2nd order statements)
Does RT^2_2 imply WKL? Does $RT^2_{<\infty}$ imply WKL?

Question (First order consequences)
Is RT^2_2 Π^1_1-conservative over $RCA_0 + B\Sigma_2$? Is $RT^2_{<\infty}$ Π^1_1-conservative over $RCA_0 + B\Sigma_3$?

Question (Π^0_2 consequences)
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