

Computability Theory

Domination, Measure, Randomness, and Reverse Mathematics

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The Game Plan

Basic Concepts

Computability

Cantor Space – Category and Measure

Domination

and Measure, again

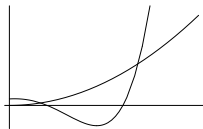
Randomness

Reverse Mathematics

(Given my contribution to this project please consider this as a survey talk not a research talk.)

Domination

Motivation and Definition



Definition

If $f(n) \geq g(n)$ for all but finitely many n , then f *dominates* g , $f \geq^* g$.

Primitive Recursion Functions

Ackermann's Function

Theorem

Ackermann's function dominates all primitive recursive functions.

Ackermann's function is computable. Hence the primitive recursive functions do not capture the informal notion of *computable*.

Computable Functions

Basic Definitions

Definition (Turing Machines)

- $\Phi_{e,s}^X(n) = \Phi_{e,s}(X, n) = \Phi_{e,s}^\emptyset(X, n)$ is the e th Turing machine with oracle X with input n run s stages. Determining if $\Phi_{e,s}^X(n) \downarrow$ is computable in X . The number and type of the inputs can vary.
- $\Phi_e^X(n) \downarrow$ if there is a stage s such that $\Phi_{e,s}^X(n) \downarrow$. Φ_e need not be total. But from now on we will assume we are dealing with total Φ .

Definition (Turing Reducibility)

- $f \leq_T X$ iff $f = \Phi_e^X$, for some e .
- $Y \leq_T X$ iff $\chi_Y \leq_T X$.
- $Y \equiv_T X$ iff $X \leq_T Y$ and $Y \leq_T X$. We write this as the Turing degree, \mathbf{x} .

Cantor Space

Category

We will work in Cantor Space, 2^ω .

Definition

- Let $\sigma \in 2^\omega$. $[\sigma]$ is a basic clopen (open and closed) class.
- A *open* class is a countable union of basic open classes. The complement of a open class is *closed*.
- The countable intersection of (basic) open classes is a G_δ class.
- The countable union of closed classes is a F_σ class.

Cantor Space is compact: if a closed class \mathbb{X} can be covered by open classes, \mathbb{X} can be covered by finitely open classes.

Cantor Space

Computability and Category

Definition

- $\mathbb{X} \subseteq 2^\omega$ is *computable* in Z iff there is a total Φ such that $X \in \mathbb{X}$ iff $\Phi^Z(X) \downarrow = 1$ and $X \notin \mathbb{X}$ iff $\Phi^Z(X) \downarrow = 0$.
- \mathbb{X} is Π_1^Z iff $\mathbb{X} = \{X \mid \forall n (\Phi^Z(X, n) \downarrow = 1)\}$.
- \mathbb{X} is Π_2^Z iff $\mathbb{X} = \{X \mid \forall n \exists m (\Phi^Z(X, n, m) \downarrow = 1)\}$.

Lemma

- \mathbb{X} is clopen iff \mathbb{X} is computable. (Finite Use Principle)
- \mathbb{X} is closed iff \mathbb{X} is Π_1^Z , for some Z . If $Z = \emptyset$ then \mathbb{X} is called a Π_1^0 -class or effectively closed.
- A class \mathbb{X} is G_δ iff \mathbb{X} is Π_2^Z , for some Z . These classes are called boldface Π_2^0 or Π_2^0 .
- Similarly a F_σ class is Σ_2^0 .

Measure Theory on Cantor Space

Definition

- $\mu([\sigma]) = 2^{-|\sigma|}$. This determines the measure of G_δ classes and hence F_σ classes.
- A class \mathbb{X} is measurable iff $\liminf \mu(\mathbb{G})$ exists and is equal to $\mu(\mathbb{X})$, where \mathbb{G} is a G_δ class containing \mathbb{X} .

Definition

A Borel measure $\hat{\mu}$ is *regular* if every measurable (in terms of $\hat{\mu}$) class \mathbb{P} there is a G_δ class $\mathbb{Q} \supseteq \mathbb{P}$ and an F_σ class $\mathbb{S} \subseteq \mathbb{P}$ such that $\hat{\mu}(\mathbb{S}) = \hat{\mu}(\mathbb{P}) = \hat{\mu}(\mathbb{Q})$.

Theorem

μ is regular.

(Uniformly) Almost Everywhere Dominating

Definition (Dobrinen and Simpson)

- A Turing degree \mathbf{a} is *almost everywhere (a.e.) dominating* if for almost all Z for all $g \leq_T Z$ there is function f of degree \mathbf{a} which dominates g .
- A Turing degree \mathbf{a} is *uniformly almost everywhere (u.a.e.) dominating* if there is function f of degree \mathbf{a} such that

$$\mu \left(\{Z \in 2^\omega : (\forall g)[g \leq_T Z \Rightarrow g \leq^* f]\} \right) = 1.$$

We also call such a function f *uniformly a.e. dominating*.

Lemma

U.a.e. dominating implies a.e. dominating.

Existence

Theorem (Martin)

An uniformly a.e. dominating function f dominates all computable functions and hence must be high. I.e. $f' \equiv_T \mathbf{0}''$, where $f' = \{e \mid \Phi_e^f(e) \downarrow\}$.

Theorem (Kurtz)

There is a uniformly a.e. dominating function of degree $\mathbf{0}'$.

Theorem (Cholak, Greenberg, Miller)

There is an incomplete (c.e.) uniformly a.e. dominating degree.

Positive Measure Dominating

Definition (Kjos-Hanssen)

- $\text{Tot}(\Phi) = \{X \mid \forall n \exists s \Phi_s(X, n) \downarrow \text{ is total}\}$
- $\Phi < \mathbf{a}$ iff either $\mu(\text{Tot}(\Phi)) = 0$ or there is an $f \leq_T \mathbf{a}$ such that

$$\mu(\{X \in \text{Tot}(\Phi) \mid f \geq \Phi(X)\}) > 0.$$

- If, for all Φ , $\Phi < \mathbf{a}$ then \mathbf{a} is *positive measure (p.m.) dominating*.

Lemma

U.a.e. dominating implies a.e dominating implies p.m. dominating.

Theorem (Binns, Kjos-Hanssen, Miller, Soloman)

Converse holds. P.m. dominating implies a.e dominating which implies u.a.e. dominating.

U.a.e. Domination and Measure

Theorem (Dobrinen and Simpson)

A Turing degree \mathbf{a} is u.a.e. dominating iff for every Π_2^0 class $\mathbb{Q} \subseteq 2^\omega$ there is a $\Sigma_2^{\mathbf{a}}$ class $\mathbb{S} \subseteq \mathbb{Q}$ such that $\mu(\mathbb{S}) = \mu(\mathbb{Q})$.

$$(\Rightarrow) \quad \mathbb{Q} = \{X \mid \forall n (\Phi_e(X, n) \downarrow)\},$$

for some e . Let Ψ be such that $\Psi^X(n)$ is the least s where $\Phi_{e,s}(X, n) \downarrow$. f dominates Ψ^X for almost all $X \in \mathbb{Q}$.

$$\mathbb{S} = \{X : \exists k \forall n (\Phi_{e, f(n)+k}(X, n) \downarrow)\}.$$

P.m. Domination and Measure

Theorem (Kjos-Hanssen after Dobrinen and Simpson)

A Turing degree \mathbf{a} is p.m. dominating iff $\text{Tot}(\Phi)$ has a $\Pi_1^{\mathbf{a}}$ subclass, \mathbb{S} , of positive measure.

(\Leftarrow) By compactness, $\{\Phi(X, n) \mid X \in \mathbb{S}\}$ is finite for all n . Therefore $\{\langle n, m \rangle : \forall X (X \in \mathbb{S} \rightarrow \Phi(X, n) < m)\}$ is $\Sigma_1^{\mathbf{a}}$. Hence by $\Sigma_1^{\mathbf{a}}$ uniformization there is a function $f \leq \mathbf{a}$ such that $\forall n \forall X (X \in \mathbb{S} \rightarrow \Phi(X, n) < f(n))$.

Goal Check I: We have related domination and measure. Now lets add randomness to this mixture.

1-Random Reals

Want to miss all “effectively null classes”.

Definition (Martin-Löf)

- A *Martin-Löf test (relative to X)* is a computable (in X) collection of Σ_1^0 open classes $\{\mathbb{U}_e\}$ with $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- R misses a test, $\{\mathbb{U}_e\}$, iff $R \notin \bigcap_e \mathbb{U}_e$.
- R is *1-random (relative to X)* iff R misses all Martin-Löf tests (relative to X).

Theorem (Martin-Löf, Solovay, Levin, Chaitin, Kolmogorov)

The definition of 1-randomness is very robust.

Low for 1-Random

Definition

- A is *low for 1-random* iff the class of 1-randoms is the class of 1-randoms relative to A .
- A is *low for 1-random over Z* iff the class of 1-randoms relative to Z is the class of 1-randoms relative to A and Z (or equivalently $A \oplus Z$) iff $A \leq_{LR} Z$.

Theorem (Downey, Hirschfeldt, Nies, Solovay, Stephan, Terwijn)

The class of A such that A is low for 1-random is a nontrivial robust class. Furthermore for all such A , $A' \leq \emptyset'$.

Lowness and Domination

Theorem (Binns, Kjos-Hanssen, Lerman, Solomon)

If B is a.e. dominating then $\mathbf{0}' \leq_{LR} B$.

Theorem (Kjos-Hanssen)

$A \leq_{LR} \mathbf{0}'$ iff every Π_2^0 class of positive measure has a Π_1^A subclass of positive measure iff A is positive measure dominating.

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

If $A \leq_T B'$ and $A \leq_{LR} B$ then every Σ_2^A class has a Σ_2^B subclass of the same measure.

Question (Aside)

What is needed for this theorem in reverse math?

Lowness and Domination, II

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

B is a.e. dominating iff $\mathbf{0}' \leq_{LR} B$.

Proof.

(\Leftarrow) Let P be any Π_2^0 class. P is a Σ_3^0 class, so it has a $\Sigma_2^{0'}$ subclass Q of the same measure (Kurtz, 1981). But, by the above theorem, Q has a Σ_2^B class of the same measure. \square

Corollary

U.a.e. dominating iff a.e. dominating iff p.m. dominating.

Weakly 2-random

- A *Martin-Löf test (relative to X)* is a computable (in X) collection of Σ_1^0 open classes $\{\mathbb{U}_e\}$ with $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- We will remove the restriction that $\mu(\mathbb{U}_e) \leq 2^{-e}$.
- Consider a computable (in X) collection of Σ_1^0 open classes $\{\mathbb{U}_e\}$ with $\lim_e \mu(\mathbb{U}_e) = 0$.
- $\bigcap_e \mathbb{U}_e$ is a measure zero Π_2^0 class (G_δ).
- R is *weakly 2-random (relative to X)* iff R misses all measure zero Π_2^0 (Π_2^X) classes.

Theorem (Kurtz)

Weakly 2 random implies 1-random but converse does not hold.

Low for weakly 2-random

- A is *low for weakly 2-random* if every weak 2-random is weak 2-random over A .
- A is *low for weak 2-tests* iff every Π_2^A nullclass is covered by a Π_2^0 nullclass.

Lemma

Low for weak 2-tests implies low for weak 2-random. (Shortly we will see the converse also holds)

Theorem (Downey, Nies, Weber, Yu)

- *If A is low for weakly 2-random then A is low for random.*
- *There is a noncomputable c.e. A that is low for weak 2-tests.*

Low for random and weakly 2-random

Theorem (Binns, Kjos-Hanssen, Miller, Solomon)

If A is low for random then A is low for weak 2-tests.

Corollary

A is low for random iff A is low for weak 2-tests iff A is low for weakly 2-randoms.

The corollary has also shown independently by Nies and Miller.

Reverse Mathematics

Statement (G_δ -REG)

For every G_δ class $\mathbb{Q} \subseteq 2^\omega$ there is a F_σ class $\mathbb{S} \subseteq \mathbb{Q}$ such that $\mu(\mathbb{S}) = \mu(\mathbb{Q})$.

Theorem (Dobrinen and Simpson)

ACA_0 *implies* G_δ -REG.

Theorem (Kjos-Hanssen)

$RCA_0 + G_\delta$ -REG *does not imply* ACA_0 .

G_δ -REG and traditional systems

G_δ -REG seems to be “orthogonal” to the traditional systems.

Theorem (Referee of Dobrinen and Simpson)

WKL_0 *does not imply* G_δ -REG.

Theorem (Cholak, Greenberg, Miller)

$RCA_0 + G_\delta$ -REG *does not imply* DNR_0 .

Theorem (Cholak, Greenberg, Miller)

$WKL_0 + G_\delta$ -REG *does not imply* ACA_0 ; $WWKL_0 + G_\delta$ -REG *does not imply* WKL_0 .

Question

Does $DNR_0 + G_\delta$ -REG *imply* $WWKL_0$?