Some Projects in Reverse Mathematics

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http://www.nd.edu/~cholak/papers/chicago11.pdf

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Reverse mathematics and infinite traceable graphs

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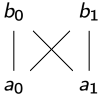
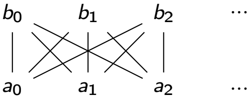
Key Graphs

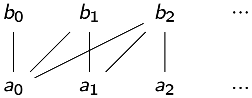
the graph $K_{2,2}$



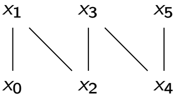
the graph $K_{\omega,\omega}$



the graph A



a finite fence:



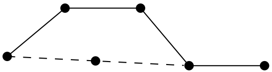
Paths

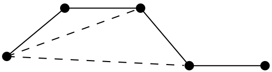
Definition

A *path* through a graph $G = (V, E)$ is a sequence of vertices $\{v_i\}_{i \in I}$ s.t. $E(v_i, v_{i+1})$ for each $i + 1 \in I$.

Definition

A path is *chordless* if $E(v_i, v_j)$ if and only if $i = j \pm 1$.





Two Theorems of Galvin, Rival and Sands

Theorem

Every infinite traceable graph either contains arbitrarily long finite chordless paths or contains a subgraph isomorphic to A.

Theorem

Every infinite traceable graph containing no chordless 4–path contains a subgraph isomorphic to $K_{\omega,\omega}$.

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These are equivalent to ACA_0 over RCA_0.

A graph where subset of V^∞ must compute $\mathbf{0}'$

If $G = (V, E)$ is a graph and $x \in V$, then we say x has *infinite degree* if there are infinitely many y such that $E(x, y)$. Let V^∞ denote the set of vertices with infinite degree in G.

Theorem

There is an infinite computable graph $G = (V, E)$ such that G has a computable tracing function, G has no chordless 4–paths and

$$\forall X \left((\exists e\, (W_e^X \text{ is infinite} \wedge W_e^X \subseteq V^\infty)) \rightarrow 0' \leq_T X \right)$$

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But they indicate an alternate proof of needed corollary is available using the Finite Ramsey Theorem.

A Finite Ramsey Style Result

Theorem (RCA_0)

For all $n \in \mathbb{N}$, there is an m such that for all finite traceable graphs G with $|G| \geq m$, either G contains a copy of $K_{2,2}$ or G contains a chordless n-path.

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There is a constant $c > 0$ such that for all $n \geq 2$ we have $m(n) \leq t_{c\lceil \log n \rceil}(2)$ where the tower function $t_k(x)$ is defined recursively by $t_1(x) = x$ and $t_k(x) = 2^{t_{k-1}(x)}$ for $k > 1$.

Question

Can this be improved?

But the proof uses *WKL*

Theorem (RCA_0)

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As statement in 2nd order arithmetic this theorem is arithmetic.

Theorem

WKL is Π^1_1-conservative over RCA_0.

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Theorem

WKL is Π^1_1-conservative over RCA_0.

Construct a computable tree such that it is computably bounded, infinite, and any path codes an infinite traceable graph contained in the Hasse diagram for the lattice but does not contain a copy of $K_{2,2}$. Since finite chordless paths in this graph are finite fences when viewed in L, L contains arbitrarily long finite fences.

Conclusion

Infinite combinatorics can lead to interesting reverse mathematics.

Computably Enumerable Partial Orderings

Peter Cholak, Damir Dzhafarov, Noah Schweber, Richard Shore

ND, Berkeley, Cornell

Two combinatorial principles

Statement (CAC)

Every infinite partial order has either an infinite chain or an infinite anti-chain.

Statement (ADS)

Every infinite linear order has either an infinite ascending sequence or an infinite descending sequence.

For countable orders,

- Over RCA_0, Hirschfeldt and Shore showed CAC and ADS are weaker than RT^2_2.
- CAC implies ADS over RCA_0, but the converse is open.

Natural generalizations of *CAC* and *ADS*

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In the context of reverse mathematics, a partial or linear ordering of $\mathbb{N}$ may be regarded as being computable. What if we relax this?

Let $A = \langle A_i : i \in \mathbb{N} \rangle$ be a family of sets, possibly with repetition. Let $I \subseteq \mathbb{N}$. We say I is

- a *chain of indices* if $A_i \subseteq A_j$ or $A_j \subseteq A_i$ for all $i, j \in I$;
- an *anti-chain of indices* if $A_i \not\subseteq A_j$ and $A_j \not\subseteq A_i$ for all $i, j \in I$;
- an *ascending sequence of indices* if $A_i \subseteq A_j$ for all $i < j$ in I;
- a *descending sequence of indices* if $A_i \supseteq A_j$ for all $i < j$ in I.

The inclusion ordering on a computable family of sets is Π^0_1 or co-c.e.

Natural generalizations of *CAC* and *ADS*

Statement (*Set-CAC*)

Every family of sets has either an infinite chain of indices or an infinite anti-chain of indices.

Statement (*Set-ADS*)

Every family of sets linearly ordered by inclusion has either an infinite ascending sequence of indices or an infinite descending sequence of indices.

A separation result for set forms

Over RCA_0, *Set-CAC* easily implies *Set-ADS*. Also, each of the two principles implies its original analog.

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Theorem

Over RCA_0, ADS implies Set-ADS.

Theorem

There exists a computable family of sets with no infinite anti-chain of indices and with every chain of indices computing $\emptyset'$.

Corollary

Over RCA_0, Set-CAC implies ACA_0.

Connections to c.e. and co-c.e. partial orderings

Theorem

Every co-c.e. partial ordering of ω is isomorphic to an inclusion ordering on a computable family of sets, and conversely.

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Theorem

There is a co-c.e. partial ordering of ω with no computable copy.

Define the *degree spectrum* of a partial ordering $\leq_P$ of ω to be the set of all degrees containing a copy of $(\omega, \leq_P)$.

The previous theorem says that there is a co-c.e. partial order whose degree spectrum does not contain $\mathbf{0}$.

What other degree spectra are possible?

Theorem

There exists a c.e. partial ordering of ω not isomorphic to any co-c.e. such partial ordering, and conversely. But the classes of degree spectra of c.e. and co-c.e. partial orderings of ω coincide.

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Theorem

There exists a c.e. partial ordering of ω not isomorphic to any co-c.e. such partial ordering, and conversely. But the classes of degree spectra of c.e. and co-c.e. partial orderings of ω coincide.

Every degree spectrum contains the upper cone above $\mathbf{0}'$.

Theorem

There exists a c.e. partial ordering of ω whose degree spectrum consists just of the degrees $\mathbf{d} \geq \mathbf{0}'$.

A new structure satisfying Slaman-Wehner

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A new structure satisfying Slaman-Wehner

Theorem

There exists a c.e. partial ordering of ω whose degree spectrum consists just of the degrees $\mathbf{d} > \mathbf{0}$.

We use a result of Hirschfeldt that if $T \subseteq 2^{<\omega}$ is a computable tree with no dead ends and all paths computable then every non-zero degree knows a listing of the isolated paths of T.

We code the leftmost path ℓ_σ above every $\sigma \in T$ into $\leq_P$, ensuring that each ℓ_σ has infinitely many copies in the coding. We reserve a special point, **not-isolated**, that we use as follows.

- We put every other copy of ℓ_σ above **not-isolated**.
- If and when we see σ split on T, we also put every remaining copy of ℓ_σ above **not-isolated**.

A Ramsey-type König's Lemma

Stephen Flood

Notre Dame

http://nd.edu/~sflood/RKL.pdf

The Pigeon Hole Principle

Statement (RT_2^1)

For every 2-coloring on $\mathbb{N}$ there is a homogeneous (always infinite) set H.

Given an infinite binary tree T we can think of each infinite path as giving 2-coloring of $\mathbb{N}$. Want the existence of a homogeneous set for some such coloring without the existence of the path.

A Ramsey-type König's Lemma

Definition

Given σ a binary branching finite string, H is *homogeneous for* σ *with color* c iff $\forall x \in H$ if $x < |\sigma|$ then $\sigma(x) = c$.

Definition

H is *homogeneous for a path through* T iff there is a color c, such that, for arbitrarily long $\sigma \in T$, H is homogeneous for σ with color c.

Statement (*RKL*)

For every infinite binary tree T*, there is an infinite homogeneous set* H *for a path through* T*.*

The Strength of *RKL*

Theorem (RCA_0)

1. *WKL implies RKL.*
2. *SRT^2_2 implies RKL.*
3. *RKL implies DNR. (Construct a tree where if $|W_e| \geq e + 3$ then W_e cannot be homogeneous.)*

Corollary (RCA_0)

- *RKL does not imply WKL (Liu),*
- *RKL does not imply RT^2_2 (Cholak, Jockusch, Slaman).*

SRT^2_2 implies RKL

Fix T infinite binary tree.

- For each y, let σ_y be the $<_{lex}$-least string in T of length y.
- Given $x < y$, define $f(x, y) = \sigma_y(x)$.
- Note that f is stable.
- Take H homogeneous, infinite for f.
- For each $y \in H$, H is homogeneous for σ_y.

A Strengthening

Definition

Given a set of strings $\Sigma \subseteq 2^{<\mathbb{N}}$, define $T_\Sigma = \{\tau : (\exists \sigma \in \Sigma)[\tau \preceq \sigma]\}$.

Lemma

A binary tree T is Δ^0_2 iff there is a computable set of strings Σ s.t. $[T] = [T_\Sigma]$.

Statement ($RKL^{(1)}$)

For every infinite set of strings Σ, there exists an infinite homogeneous set for a path through T_Σ.

The Strength of $RKL^{(1)}$

Theorem (RCA)

1. $RT^2_2 \implies RKL^{(1)}$, *and*
2. $RKL^{(1)} + B\Sigma_2 \implies SRT^2_2$.

By results of Liu and Cholak, Jockusch, and Slaman,

Corollary

$RKL^{(1)}$ is incomparable with WKL over RCA.

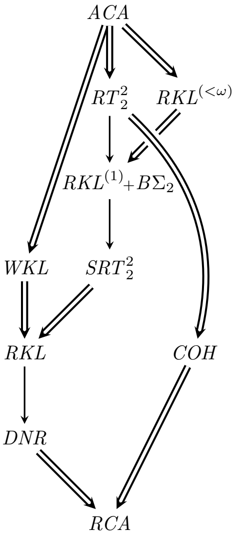
Arithmetic Sets of Strings

Statement ($RKL^{(<\omega)}$)

For every arithmetic tree T with unbounded height, there exists an infinite homogeneous set for a path through T.

Theorem

- $RKL^{(<\omega)}$ *implies* D^2_2 *and hence* $B\Sigma_2$.
- $RKL^{(<\omega)}$ *does not imply WKL. (Following Liu.)*
- RT^2_2 *does not imply* $RKL^{(<\omega)}$. *(Compare* ω*-models.)*


ACA
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COH
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