

AUTOMORPHISMS OF MANIFOLDS

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0. Introduction

This survey is about homotopy types of spaces of automorphisms of topological and smooth manifolds. Most of the results available are *relative*, that is, they compare different types of automorphisms.

In chapter 1, which motivates the later chapters, we introduce our favorite types of manifold automorphisms and make a comparison by (mostly elementary) geometric methods. Chapters 2,3, and 4 describe algebraic models (using L -theory and/or algebraic K -theory) for certain moduli spaces associated with a manifold M . These algebraic models rely heavily on

- Wall's work in surgery theory, e.g. [Wa1] ,
- Waldhausen's work in h -cobordism theory alias concordance theory, which includes a parametrized version of Wall's theory of the finiteness obstruction, [Wa2] .

The moduli spaces are of interest for the following reason. Suppose that two different notions of automorphism of M are being compared, one more structure-preserving than the other. Let $X_1(M)$ and $X_2(M)$ be the corresponding automorphism spaces, so that $X_1(M) \subset X_2(M)$. As a rule, the space of cosets $X_2(M)/X_1(M)$ is a union of connected components of a suitable moduli space.

Chapter 5 contains the beginnings of a more radical approach in which one tries to calculate the classifying space $BX_1(M)$ in terms of $BX_2(M)$, say, rather than trying to calculate $X_2(M)/X_1(M)$. Chapter 6 contains some examples and calculations.

Not included in this survey is the disjunction theory of automorphism spaces and embedding spaces begun by Morlet, see [BuLaRo], and continued in [Go4], [Go1], [Go2], [We2], [We3], [GoWe], [GoKl]. It calls for a survey of its own.

1. Stabilization and descent

1.1. NOTATION, TERMINOLOGY

1.1.1. Terminology. *Space* with a capital S means *simplicial set*. We will occasionally see simplicial Spaces (=bisimplicial sets). A simplicial Space $k \mapsto Z_k$ deter-

1991 *Mathematics Subject Classification*. 19Jxx, 57R50, 57N37.

Key words and phrases. homeomorphism, diffeomorphism, algebraic K -theory, L -theory, concordance, pseudoisotopy.

Both authors partially supported by NSF grant.

mines a Space $(\coprod_k \Delta^k \times Z_k) / \sim$ where \sim refers to the relations $(f_*x, y) \sim (x, f^*y)$, and Δ^k is the k 0-simplices viewed as a Space (= simplicial set). This is isomorphic to the Space $k \mapsto Z_k(k)$, the *diagonal* of $k \mapsto Z_k$. See [Qui], for example.

A *euclidean k -bundle* is a fiber bundle with fibers homeomorphic to \mathbb{R}^k . Trivial euclidean k -bundles are often denoted ε^k .

The homotopy fiber of a map $B \rightarrow C$ of Spaces, where C is based, will be denoted $\text{hofiber}[B \rightarrow C]$. A *homotopy fiber sequence* is a diagram of spaces $A \rightarrow B \rightarrow C$ where C is based, together with a nullhomotopy of the composition $A \rightarrow C$ which makes the resulting map from A to $\text{hofiber}[B \rightarrow C]$ a (weak) homotopy equivalence.

The term *cartesian square* is synonymous with *homotopy pullback square*. More generally, an *n -cartesian square* ($n \leq \infty$) is a commutative diagram of Spaces and maps

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

such that the resulting map from A to the homotopy pullback of $B \rightarrow D \leftarrow C$ is n -connected.

A *commutative diagram* of Spaces is a functor F from some small category \mathcal{D} to Spaces. We say that \mathcal{D} is the shape of the diagram. When we represent such a diagram graphically, we usually only show the maps $F(g_i)$ for a set $\{g_i\}$ of morphisms generating \mathcal{D} . For example, the commutative square just above is a functor from a category with four objects and five non-identity morphisms to Spaces. The notion of a *homotopy commutative diagram* of shape \mathcal{D} has been made precise by [Vogt]. It is a continuous functor from a certain topological category $\mathcal{W}\mathcal{D}$ (determined by \mathcal{D}) to Spaces. In more detail, $\mathcal{W}\mathcal{D}$ is a small topological category with discrete object set, and comes with a continuous functor $\mathcal{W}\mathcal{D} \rightarrow \mathcal{D}$ which restricts to a homeomorphism (=bijection) of object sets, and to a homotopy equivalence of morphism spaces. Graphically, we represent homotopy commutative diagrams of shape \mathcal{D} like commutative diagrams of shape \mathcal{D} .

1.1.2. Notation. For a topological manifold M , we denote by $\text{TOP}(M)$ the Space of homeomorphisms $f : M \rightarrow M$ which agree with the identity on ∂M . (A k -simplex in $\text{TOP}(M)$ is a homeomorphism $f : M \times \Delta^k \rightarrow M \times \Delta^k$ over Δ^k which agrees with the identity on $\partial M \times \Delta^k$.) References: [BLR], [Bu].

We use the abbreviations $\text{TOP}(n) := \text{TOP}(\mathbb{R}^n)$ and $\text{TOP} = \bigcup_n \text{TOP}(n)$, where we include $\text{TOP}(n)$ in $\text{TOP}(n+1)$ by $f \mapsto f \times \text{id}_{\mathbb{R}}$.

Let $\partial_+ M$ be a codimension zero submanifold of ∂M , closed as a subspace of ∂M . Let $\partial_- M$ be the closure of $\partial M \setminus \partial_+ M$. Let $\text{TOP}(M, \partial_+ M)$ be the Space of homeomorphisms $M \rightarrow M$ which agree with the identity on $\partial_- M$. (This is $\text{TOP}(M)$ if $\partial_+ M = \emptyset$.) Special case: the Space of concordances alias pseudo-isotopies of a compact manifold N , which is $\text{C}(N) := \text{TOP}(N \times I, N \times 1)$ where $I = [0, 1]$. References: [Ce], [HaWa], [Ha], [DIg], [Ig]. Let $\text{G}(M, \partial_+ M)$ be the Space of homotopy equivalences of triads, $(M; \partial_+ M, \partial_- M) \rightarrow (M; \partial_+ M, \partial_- M)$, which are the identity on $\partial_- M$. We abbreviate $\text{G}(M, \emptyset)$ to $\text{G}(M)$. *Warning:* If $\partial M = \emptyset$, then $\text{G}(M)$ is the space of homotopy equivalences $M \rightarrow M$, but in general it is not.

1.1.3. Definitions. An *h-structure* on a closed manifold M^n is a pair (N, f) where N^n is another closed manifold and $f : N \rightarrow M$ is a homotopy equivalence. If f is a *simple* homotopy equivalence, (N, f) is an *s-structure*. An *isomorphism* from an *h-structure* (N_1, f_1) to another *h-structure* (N_2, f_2) on M is a homeomorphism $N_1 \rightarrow N_2$ over M .

We see that the *h-structures* on M form a groupoid. Better, they form a *simplicial groupoid*: Objects in degree k are pairs (N, f) where N^n is another closed manifold and $f : N \times \Delta^k \rightarrow M \times \Delta^k$ is a homotopy equivalence over Δ^k . Morphisms in degree k are homeomorphisms over $M \times \Delta^k$.

Let $\mathcal{S}(M)$ be the diagonal nerve (= diagonal of degreewise nerve) of this simplicial groupoid; also let $\mathcal{S}^s(M)$ be the diagonal nerve of the simplicial subgroupoid of *s-structures*. Think of $\mathcal{S}(M)$ and $\mathcal{S}^s(M)$ as the *moduli Spaces* of *h-structures* on M and *s-structures* on M , respectively. (They are actually simplicial *classes*, not simplicial sets, as it stands. The reader can either accept this, or avoid it by working in a Grothendieck “universe”.) The forgetful functor $(N, f) \mapsto N$ induces a map from $\mathcal{S}(M)$ to the diagonal nerve of the simplicial groupoid of all closed n -manifolds and homeomorphisms between such. This map is a Kan fibration. Its fiber over the point corresponding to M is $G(M)$. Hence there is a homotopy fiber sequence

$$\text{TOP}(M) \rightarrow G(M) \rightarrow \mathcal{S}(M).$$

More generally, given compact M and $\partial_+ M \subset \partial M$ as above, there is an *h-structure space* $\mathcal{S}(M, \partial_+ M)$ and an *s-structure space* $\mathcal{S}^s(M, \partial_+ M)$ and a homotopy fiber sequence

$$\text{TOP}(M, \partial_+ M) \rightarrow G(M, \partial_+ M) \rightarrow \mathcal{S}(M, \partial_+ M).$$

We omit the details. Important special cases: the *space of h-cobordisms* and the *space of s-cobordisms* on a compact manifold,

$$\begin{aligned} \mathcal{H}(N) &:= \mathcal{S}(N \times I, N \times 1) \\ \mathcal{H}^s(N) &:= \mathcal{S}^s(N \times I, N \times 1). \end{aligned}$$

Since $G(N \times I, N \times 1) \simeq *$, we have $\Omega\mathcal{H}(N) \simeq C(N)$. There is a stabilization map $\mathcal{H}(N) \rightarrow \mathcal{H}(N \times I)$, *upper stabilization* to be precise [Wah2], [HaWa]. Let

$$\mathcal{H}^\infty(N) := \text{hocolim}_k \mathcal{H}(N \times I^k), \quad C^\infty(N) := \text{hocolim}_k C(N \times I^k).$$

1.1.4. More definitions. The *block automorphism Space* $\widetilde{\text{TOP}}(M)$ has as its k -simplices the homeomorphisms $g : M \times \Delta^k \rightarrow M \times \Delta^k$ which satisfy $g(M \times s) = M \times s$ for each face $s \subset \Delta^k$, and restrict to the identity on $\partial M \times \Delta^k$. References: [ABK], [Bu1] [Br1]. There is also a block *s-structure Space*

$$\widetilde{\mathcal{S}}^s(M),$$

defined as the diagonal nerve of a simplicial groupoid. The objects of the simplicial groupoid in degree k are of the form (N, f) where N^n is closed and f is a *simple*

homotopy equivalence $N \times \Delta^k \rightarrow M \times \Delta^k$ such that $f(N \times t) \subset M \times t$ for each face t of Δ^k , and f restricts to a homeomorphism $\partial N \times \Delta^k \rightarrow \partial M \times \Delta^k$. The morphisms in degree k are homeomorphisms respecting the reference maps to $M \times \Delta^k$. References: [Qun1], [Wa1, §17.A], [Ni], [Rou1]. There is a homotopy fiber sequence

$$\widetilde{\text{TOP}}(M) \rightarrow \widetilde{\text{G}}^s(M) \rightarrow \widetilde{\text{S}}^s(M),$$

where $\widetilde{\text{G}}^s(M)$ is defined like $\widetilde{\text{TOP}}(M)$, but with simple homotopy equivalences instead of homeomorphisms.— These definitions have relative versions (details omitted); for example, there is a homotopy fiber sequence

$$\widetilde{\text{TOP}}(M, \partial_+ M) \rightarrow \widetilde{\text{G}}^s(M, \partial_+ M) \rightarrow \widetilde{\text{S}}^s(M, \partial_+ M).$$

Let $\text{G}^s(M, \partial_+ M) \subset \text{G}(M, \partial_+ M)$ consist of the components containing those f which are simple homotopy automorphisms and induce simple homotopy automorphisms of $\partial_+ M$. The inclusion $\text{G}^s(M, \partial_+ M) \rightarrow \widetilde{\text{G}}^s(M, \partial_+ M)$ is a homotopy equivalence (because it induces an isomorphism on homotopy groups; both Spaces are fibrant).

1.2. OPEN STABILIZATION VERSUS CLOSED STABILIZATION

Let M^n be compact, $M_0 = M \setminus \partial M$. *Open stabilization* refers to the map

$$\text{TOP}(M, \partial M) \rightarrow \bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k)$$

given by $f \mapsto f|_{M_0}$. We include $\text{TOP}(M_0 \times \mathbb{R}^k)$ in $\text{TOP}(M_0 \times \mathbb{R}^{k+1})$ by $g \mapsto g \times \text{id}_{\mathbb{R}}$. *Closed stabilization* refers to the inclusion

$$\text{TOP}(M, \partial M) \rightarrow \bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k)).$$

Open stabilization factors through closed stabilization, by means of the restriction maps $\text{TOP}(M \times I^k, \partial(M \times I^k)) \rightarrow \text{TOP}(M_0 \times I_0^k)$ and an identification $I_0 \cong \mathbb{R}$. Here in §1.2 we describe the homotopy type of $\bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k)$, and descend from there to $\bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k))$. For a more algebraic version of this, see §5.2.

Let $\hat{\tau} : M_0 \rightarrow \text{BTOP}(n)$ classify the tangent bundle [Mi1], [Kis], [Maz2], [KiSi, IV.1]. We map $\text{G}(M_0)$ to the mapping Space $\text{map}(M_0, \text{BTOP})$ by $f \mapsto \hat{\tau}f$ and we map $\bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k)$ to $\text{G}(M_0)$ by $f \mapsto pfi$, where $p : M_0 \times \mathbb{R}^k \rightarrow M_0$ and $i : M_0 \cong M_0 \times \mathbf{0} \rightarrow M_0 \times \mathbb{R}^k$ are projection and inclusion, respectively.

1.2.1. Theorem [CaGo]. *The resulting diagram*

$$\bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k) \longrightarrow \text{G}(M_0) \longrightarrow \text{map}(M_0, \text{BTOP})$$

is a homotopy fiber sequence.

The proof uses immersion theory [Gau], general position, and the half-open s -cobordism theorem [Sta]. See also [Maz1].

Choose a collar for M , that is, an embedding $c : \partial M \times I \rightarrow M$ extending the map $(x, 0) \rightarrow x$ on $\partial M \times 0$. Reference: [Brn], [KiSi, I App. A]. Any homeomorphism $f : M_0 \rightarrow M_0$ determines an h -cobordism W_f on ∂M : the region of M enclosed by ∂M and $fc(\partial M \times 1)$. The bundle on the geometric realization of $\text{TOP}(M_0)$ with fiber W_f over the vertex f is a bundle of h -cobordisms, classified by a map v from $\text{TOP}(M_0)$ to $\mathcal{H}(\partial M)$.

If $f : M_0 \rightarrow M_0$ is the restriction of some homeomorphism $g : M \rightarrow M$, then $W_f \cong gc(\partial M \times I)$ is trivialized. Conversely, a trivialization of W_f can be used to construct a homeomorphism $g : M \rightarrow M$ with an isotopy from $g|_{M_0}$ to f . Therefore: the diagram

$$(1.2.2) \quad \text{TOP}(M, \partial M) \xrightarrow{\text{res}} \text{TOP}(M_0) \xrightarrow{v} \mathcal{H}(\partial M)$$

is a homotopy fiber sequence. See [Cd] for details. The special case where $M = \mathbb{D}^n$ is due to [KuLa].

This observation can be stabilized. Let $u : \mathcal{H}(\partial M) \rightarrow \mathcal{H}(\partial(M \times I))$ be the composition of stabilization $\mathcal{H}(\partial M) \rightarrow \mathcal{H}(\partial M \times I)$ with the map induced by the inclusion of $\partial M \times I$ in $\partial(M \times I)$. Then

$$\begin{array}{ccc} \text{TOP}(M_0) & \xrightarrow{v} & \mathcal{H}(\partial M) \\ \downarrow & & \downarrow u \\ \text{TOP}(M_0 \times I_0) & \longrightarrow & \mathcal{H}(\partial(M \times I)) \end{array}$$

is homotopy commutative. The homotopy colimit of the $\mathcal{H}(\partial(M \times I^k))$ under the u -maps becomes $\simeq \mathcal{H}^\infty(M)$. Therefore:

1.2.3. Theorem. *There exists a homotopy fiber sequence*

$$\bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k)) \xrightarrow{\text{res}} \bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k) \longrightarrow \mathcal{H}^\infty(M).$$

Let $Q = I^\infty$ be the Hilbert cube. The product $M \times Q$ is a *Hilbert cube manifold* [Cha] without boundary. Let $\text{TOP}(M \times Q)$ be the Space of homeomorphisms $M \times Q \rightarrow M \times Q$ and let $G(M \times Q)$ be the Space of homotopy equivalences $M \times Q \rightarrow M \times Q$. Chapman and Ferry have shown [Bu2] that an evident map from $\bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k))$ to the homotopy fiber of the composition

$$\text{TOP}(M \times Q) \xrightarrow{\hookrightarrow} G(M \times Q) \simeq G(M_0) \rightarrow \text{map}(M_0, B\text{TOP})$$

(last arrow as in 1.2.1) is a homotopy equivalence. Therefore

$$\bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k)) \longrightarrow \text{TOP}(M \times Q) \longrightarrow \text{map}(M_0, B\text{TOP})$$

is a homotopy fiber sequence. Comparison with 1.2.1 gives the next result.

1.2.4. Theorem. *The following homotopy commutative diagram is cartesian:*

$$\begin{array}{ccc} \bigcup_k \text{TOP}((M \times I^k, \partial(M \times I^k))) & \xrightarrow{\text{res}} & \bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k) \\ \downarrow & & \downarrow \\ \text{TOP}(M \times Q) & \longrightarrow & \text{G}(M_0). \end{array}$$

This suggests that the map $\bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k) \rightarrow \mathcal{H}^\infty(M)$ in 1.2.3 factors through $\text{G}(M_0)$. We will obtain such a factorization in 1.5.3.

Remark. Looking at horizontal homotopy fibers in 1.2.4, and using 1.2.3, and the homotopy equivalence $\text{G}(M_0) \simeq \text{G}(M \times Q)$, one finds that the homotopy fiber of the inclusion $\text{TOP}(M \times Q) \rightarrow \text{G}(M \times Q)$ is $\text{C}^\infty(M)$. This can also be deduced from [Cha2], [Cha3].

1.3. BOUNDED STABILIZATION VERSUS NO STABILIZATION

Let M^n be compact. A homeomorphism $f : M \times \mathbb{R}^k \rightarrow M \times \mathbb{R}^k$ is *bounded* if $\{p_2 f(z) - p_2(z) \mid z \in M \times \mathbb{R}^k\}$ is a bounded subset of \mathbb{R}^k , where $p_2 : M \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is the projection. Let $\text{TOP}^b(M \times \mathbb{R}^k)$ be the Space of bounded homeomorphisms $M \times \mathbb{R}^k \rightarrow M \times \mathbb{R}^k$ which agree with the identity on $\partial M \times \mathbb{R}^k$. Note $\text{TOP}(M) = \text{TOP}^b(M \times \mathbb{R}^0)$. *Bounded stabilization* refers to the inclusion

$$\text{TOP}(M) \rightarrow \bigcup_k \text{TOP}^b(M \times \mathbb{R}^k).$$

Surgery theory gives a description of the homotopy type of $\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)$, modulo the mysteries of $\text{G}(M)$. See §2.4; here in §1.3 we analyze the difference between $\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)$ and $\text{TOP}(M)$.

The Space $\text{TOP}^b(M \times \mathbb{R}^{k+1})/\text{TOP}^b(M \times \mathbb{R}^k)$ for $k \geq 0$ and fixed M is the k -th Space in a spectrum $\mathbf{H}(M)$, by analogy with the sphere spectrum, which is made out of the spaces $\text{O}(\mathbb{R}^{k+1})/\text{O}(\mathbb{R}^k)$. Compare [BuLa1]. Anderson and Hsiang, who introduced bounded homeomorphisms in [AH1], [AH2] showed that $\Omega^{\infty+1}(\mathbf{H}(M)) \simeq \text{C}^\infty(M)$. In more detail: they introduced *bounded concordance spaces*

$$\text{C}^b(M \times \mathbb{R}^k) = \text{TOP}^b(M \times I \times \mathbb{R}^k, M \times 1 \times \mathbb{R}^k)$$

and proved the following. See also [WW1, §1+App.5], [Ha, App.II].

1.3.1. Theorem [AH1], [AH2]. *Assume $n > 4$. Then*

- i) $\Omega(\text{TOP}^b(M \times \mathbb{R}^{k+1})/\text{TOP}^b(M \times \mathbb{R}^k)) \simeq \text{C}^b(M \times \mathbb{R}^k)$;
- ii) $\Omega \text{C}^b(M \times \mathbb{R}^k) \simeq \text{C}^b(M \times I \times \mathbb{R}^{k-1})$.

Part ii) of 1.3.1 shows that the spaces $C^b(M \times \mathbb{R}^k)$ for $k \geq 0$ form a spectrum, with structure maps

$$C^b(M \times \mathbb{R}^{k-1}) \xrightarrow{\text{stab.}} C^b(M \times I \times \mathbb{R}^{k-1}) \simeq \Omega C^b(M \times \mathbb{R}^k);$$

then part i) of 1.3.1 with some extra work [WW1,§1] identifies the new spectrum with $\Omega \mathbf{H}(M)$. It is also shown in [AH1] that the homotopy groups $\pi_j \mathbf{H}(M)$ for $j \leq 0$ are lower K -groups [Ba]:

1.3.2. Theorem. *Let $j \leq 0$ be an integer, then*

$$\pi_j C^b(M \times \mathbb{R}^k) = \begin{cases} K_{j-k+2}(\mathbb{Z}\pi_1(M)) & (j < k-2) \\ \tilde{K}_0(\mathbb{Z}\pi_1(M)) & (j = k-2) \\ \text{Whitehead gp. of } \pi_1(M) & (j = k-1). \end{cases}$$

Remark. Madsen and Rothenberg [MaRo1], [MaRo2] have proved equivariant analogs of 1.3.1 and 1.3.2, and Chapman [Cha4], Hughes [Hu1] have proved a Hilbert cube analog. Carter [Ca1], [Ca2], [Ca3] has shown that $K_r(\mathbb{Z}\pi)$ vanishes if π is finite and $r < -1$.

Remark. It is shown in [WW1,§5] that $\Omega^\infty \mathbf{H}(M) \simeq \mathcal{H}^\infty(M)$; this improves slightly on $\Omega^{\infty+1} \mathbf{H}(M) \simeq C^\infty(M)$.

Theorems 1.3.1–2 say a lot about descent from $\text{TOP}^b(M \times \mathbb{R}^{k+1})$ to $\text{TOP}^b(M \times \mathbb{R}^k)$. For instant descent from $\text{TOP}^b(M \times \mathbb{R}^{k+1})$ to $\text{TOP}(M)$, there is the *hyperplane test* [WW1,§3], [We4]. Think of $\mathbb{R}P^k$ as the Grassmannian of codimension one linear subspaces $W \subset \mathbb{R}^{k+1}$. Let Γ_k be the Space of sections of the bundle $E(k) \rightarrow \mathbb{R}P^k$ with fibers

$$E(k)_W := \text{TOP}^b(M \times \mathbb{R}^{k+1}) / \text{TOP}^b(M \times W)$$

(see the remark just below). Note that $E(k) \rightarrow \mathbb{R}P^k$ has a trivial section picking the coset [id] in each fiber; so Γ_k is a based Space.

Remark. The “bundle” $E(k) \rightarrow \mathbb{R}P^{k+1}$ is really a *twisted cartesian product* [Cu] with base Space equal to the singular simplicial set of $\mathbb{R}P^k$, and with fibers $E(k)_W$ over a vertex W as stated.

We define a map $\Phi_k : \text{TOP}^b(M \times \mathbb{R}^{k+1}) / \text{TOP}(M) \rightarrow \Gamma_k$ by taking the coset $f \cdot \text{TOP}(M)$ to the section $W \mapsto f \cdot \text{TOP}^b(M \times W)$. For $k > 0$, it is easy to produce an embedding v_k making the square

$$\begin{array}{ccc} \text{TOP}^b(M \times \mathbb{R}^k) / \text{TOP}(M) & \xrightarrow{\Phi_{k-1}} & \Gamma_{k-1} \\ \cap \downarrow & & v_k \downarrow \\ \text{TOP}^b(M \times \mathbb{R}^{k+1}) / \text{TOP}(M) & \xrightarrow{\Phi_k} & \Gamma_k \end{array}$$

commutative. Let $\Phi : \bigcup_k \text{TOP}^b(M \times \mathbb{R}^k) / \text{TOP}(M) \rightarrow \bigcup_k \Gamma_k$ be the union of the Φ_{k-1} for $k \geq 0$. It turns out that Φ is highly connected (1.3.5 below), under mild conditions on M .

1.3.3. Definition. An integer j is in the *topological, resp. smooth, concordance stable range* for M if the upper stabilization maps $C(M \times I^r) \rightarrow C(M \times I^{r+1})$, resp. the smooth versions, are j -connected, for all $r \geq 0$.

1.3.4. Theorem [Ig]. *If M is smooth and $n \geq \max\{2j + 7, 3j + 4\}$, then j is in the smooth and in the topological concordance stable range for M . (The equality of smooth and concordance stable range is due to Burghilea–Lashof and Goodwillie. See [Ig, Intro.])*

1.3.5. Proposition [WW1]. *If j is in the topological concordance stable range for M , and $n > 4$, then $\Phi : \bigcup_k \text{TOP}^b(M \times \mathbb{R}^k) / \text{TOP}(M) \rightarrow \bigcup_k \Gamma_{k-1}$ is $(j + 1)$ -connected.*

Outline of proof. For $-1 \leq \ell \leq k$ let $\Gamma_{k,\ell} \subset \Gamma_k$ consist of the sections s for which $s(W) = *$ whenever W contains the standard copy of $\mathbb{R}^{\ell+1}$ in \mathbb{R}^{k+1} . Let $\Phi_{k,\ell}$ be the restriction of Φ_k to $\text{TOP}^b(M \times \mathbb{R}^{\ell+1}) / \text{TOP}(M)$, viewed as a map with codomain $\Gamma_{k,\ell}$. One shows by induction on ℓ that the $\Phi_{k,\ell}$ for fixed ℓ define a highly connected map

$$\text{TOP}^b(M \times \mathbb{R}^{\ell+1}) / \text{TOP}(M) \longrightarrow \bigcup_{k \geq \ell} \Gamma_{k,\ell}. \quad \square$$

1.3.6 Theorem [WW1]. *There exists a homotopy equivalence*

$$\bigcup_k \Gamma_k \simeq \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2})$$

for some involution on $\mathbf{H}(M)$. Here $\mathbf{H}(M)_{h\mathbb{Z}/2} := (E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} \mathbf{H}(M)$ is the homotopy orbit spectrum.

Sketch proof. Note $\bigcup_k \Gamma_k = \bigcup_{\ell > 0} \bigcup_{k > \ell} \Gamma_{k,\ell}$ and $\bigcup_{k > \ell} \Gamma_{k,\ell} \simeq \Omega^\infty(\mathbb{S}_+^\ell \wedge_{\mathbb{Z}/2} \mathbf{H}(M))$ by Poincaré duality, using 1.3.1 and 1.3.2. \square

1.3.7. Summary. There exist a spectrum $\mathbf{H}(M)$ with involution and a $(j + 1)$ -connected map

$$\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k) / \text{TOP}(M) \longrightarrow \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2})$$

where j is the largest integer in the topological concordance stable range for M . Further, $\Omega^\infty \mathbf{H}(M) \simeq \mathcal{H}^\infty(M)$ and

$$\pi_r \mathbf{H}(M) := \begin{cases} \text{Wh}_1(\pi_1(M)) & r = 0 \\ \tilde{K}_0(\mathbb{Z}\pi_1(M)) & r = -1 \\ K_{r+1}(\mathbb{Z}\pi_1(M)) & r < -1. \end{cases}$$

(See §3 for information about $\pi_r \mathbf{H}(M)$ when $r > 0$.)

1.4. BLOCK AUTOMORPHISM SPACES

The property $\pi_k \widetilde{\text{TOP}}(M) \cong \pi_0 \widetilde{\text{TOP}}(M \times \Delta^k)$ is a consequence of the definitions and has made the block automorphism Spaces popular. See also §2. In homotopy theory terms, the block automorphism Space of M is more closely related to $\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)$ than to $\text{TOP}(M)$. To explain this we need the bounded block automorphism Spaces

$$\widetilde{\text{TOP}}^b(M \times \mathbb{R}^k)$$

(definition left to the reader). The following *Rothenberg* type sequence is obtained by inspection, using 1.3.2. For notation, see 1.3.7. Compare [Sha], [Ra1, §1.10].

1.4.2. Proposition [AnPe], [WW1]. *For $k \geq 0$ there exists a long exact sequence*

$$\begin{aligned} \cdots \rightarrow \pi_r \widetilde{\text{TOP}}^b(M \times \mathbb{R}^k) \rightarrow \pi_r \widetilde{\text{TOP}}^b(M \times \mathbb{R}^{k+1}) \rightarrow H_{r+k}(\mathbb{Z}/2; \pi_{-k} \mathbf{H}(M)) \cdots \\ \rightarrow \pi_{r-1} \widetilde{\text{TOP}}^b(M \times \mathbb{R}^k) \rightarrow \pi_{r-1} \widetilde{\text{TOP}}^b(M \times \mathbb{R}^{k+1}) \cdots \end{aligned}$$

The inclusion $\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k) \rightarrow \bigcup_k \widetilde{\text{TOP}}^b(M \times \mathbb{R}^k)$ is a homotopy equivalence [WW1, 1.14]. Together with 1.4.2, this shows for example that

$$\widetilde{\text{TOP}}(M) \simeq \bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)$$

if M is simply connected, because then $\pi_k \mathbf{H}(M) = 0$ for $k \leq 0$.

Another way to describe the relationship between $\widetilde{\text{TOP}}(M)$ and $\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)$ is to use a filtered version of Postnikov's method for making highly connected covers. Let X be a fibrant Space with a filtration by fibrant subSpaces

$$X(0) \subset X(1) \subset X(2) \subset \dots$$

so that X is the union of the $X(k)$. Call an i -simplex in X *positive* if its characteristic map $\Delta^i \rightarrow X$ is filtration-preserving, i.e., takes the k -skeleton of Δ^i to $X(k)$. The positive simplices form a subSpace ${}^{\text{pos}}X$ of X , and we let ${}^{\text{pos}}X(k) := {}^{\text{pos}}X \cap X(k)$. Then ${}^{\text{pos}}X(k)$ is fibrant. If $X(0)$ is based, then

$$\pi_i({}^{\text{pos}}X(k), {}^{\text{pos}}X(k-1)) \xrightarrow{\cong} \pi_i(X(k), X(k-1))$$

for $i \geq k$, and $\pi_i({}^{\text{pos}}X(k), {}^{\text{pos}}X(k-1)) = *$ for $i < k$. *Example:* If $X(k) = *$ for $k \leq m$ and $X(k) = X$ for $k > m$, then ${}^{\text{pos}}X$ is the m -connected Postnikov cover of X . *Another example:* If $X(k)$ is $\text{TOP}^b(M \times \mathbb{R}^k) / \text{TOP}(M)$, then

$${}^{\text{pos}}X \simeq \widetilde{\text{TOP}}(M) / \text{TOP}(M).$$

See [WW1] for a more precise statement, and a proof.

1.4.3. Corollary [Ha]. *There exists a spectral sequence converging to the homotopy groups of $\widetilde{\text{TOP}}(M)/\text{TOP}(M)$, with $E_{pq}^1 = \pi_{q-1}(\text{C}(M \times I^p))$.*

Hatcher also described E_{pq}^2 for $p+n \gg q$. What he found is explained by the next theorem, which uses naturality of the pos-construction and the results of §1.3.

1.4.4. Theorem [WW1, Thm. C]. *There exists a homotopy commutative cartesian square of the form*

$$\begin{array}{ccc} \widetilde{\text{TOP}}(M)/\text{TOP}(M) \simeq \text{pos}\left(\bigcup_k \text{TOP}^b(M \times \mathbb{R}^k)/\text{TOP}(M)\right) & \longrightarrow & \Omega^\infty(\mathbf{H}^s(M)_{h\mathbb{Z}/2}) \\ \downarrow \cap & & \downarrow \\ \bigcup_k \text{TOP}(M \times \mathbb{R}^k)/\text{TOP}(M) & \xrightarrow{\Phi} & \Omega^\infty(\mathbf{H}(M)_{h\mathbb{Z}/2}) \end{array}$$

where $\mathbf{H}^s(M)$ is the 0-connected cover of $\mathbf{H}(M)$, the right-hand vertical arrow is induced by the canonical map $\mathbf{H}^s(M) \rightarrow \mathbf{H}(M)$, and Φ is the map from 1.3.7.

By 1.3.7, lower and hence upper horizontal arrow in 1.4.4 are j -connected for j in the topological concordance stable range of M .

1.5. h -STRUCTURES AND h -COBORDISMS

The goal here is to construct a *Whitehead torsion* map $w : \mathcal{S}(M) \rightarrow \mathcal{H}^\infty(M)$, and a simple version, $\mathcal{S}^s(M) \rightarrow (\mathcal{H}^\infty)^s(M) = \Omega^\infty \mathbf{H}^s(M)$, which makes the following diagram homotopy commutative (see 1.1.1):

$$(1.5.1) \quad \begin{array}{ccc} \widetilde{\text{TOP}}(M)/\text{TOP}(M) & \xrightarrow{\delta} & \mathcal{S}^s(M) \\ \downarrow 1.4.4 & & \downarrow w \\ \Omega^\infty(\mathbf{H}^s(M)_{h\mathbb{Z}/2}) & \xrightarrow{\text{transfer}} & \Omega^\infty \mathbf{H}^s(M). \end{array}$$

The map δ comes from $\widetilde{\text{TOP}}(M)/\text{TOP}(M) \rightarrow \mathcal{S}^s(M) \rightarrow \widetilde{\mathcal{S}}^s(M)$, a homotopy fiber sequence. In our description of w , we assume for simplicity that M^n is closed.

Let $Z \subset \mathcal{S}(M)$ be a finitely generated subSpace, that is to say, $|Z|$ is compact. Let $p : E(1) \rightarrow |Z|$ be the tautological bundle whose fiber over some vertex (N, f) , for example, is N . (Here N^n is closed and $f : N \rightarrow M$ is a homotopy equivalence.) Let $E(2) = M \times |Z|$. We have a canonical fiber homotopy equivalence $\lambda : E(1) \rightarrow E(2)$ over $|Z|$.

Let τ_1 and τ_2 be the vertical tangent bundles of $E(1)$ and $E(2)$, respectively. Choose $k \gg 0$, and a k -disk bundle ξ on $E(1)$ with associated euclidean bundle ξ^{cl} , and an isomorphism ι of euclidean bundles $\tau_1 \oplus \xi^{\text{cl}} \cong \lambda^*(\tau_2 \oplus \varepsilon^k)$. Let $E(1)^\xi$ be the total space of the disk bundle ξ ; this fibers over $|Z|$. Immersion theory [Gau] says that λ and ι together determine up to contractible choice a fiberwise codimension zero immersion, over $|Z|$, from $E(1)^\xi$ to $E(2) \times \mathbb{R}^k$. We can arrange that the image of this fiberwise immersion is contained in $E(2) \times \mathbb{B}^k$ where $\mathbb{B}^k \subset \mathbb{R}^k$ is the

open unit ball. Also, by choosing k sufficiently large and using general position arguments, we can arrange that the fiberwise immersion is a fiberwise embedding. In this situation, the closure of

$$E(2) \times \mathbb{D}^k \setminus \text{im}(E(1)^\xi)$$

is the total space of a fibered family of h -cobordisms over $|Z|$, with fixed base $M \times \mathbb{S}^{k-1}$. This family is classified by a map $Z \rightarrow \mathcal{H}(M \times \mathbb{S}^{k-1})$. Letting $k \rightarrow \infty$ we have

$$w_Z : Z \longrightarrow \text{hocolim}_k \mathcal{H}(M \times \mathbb{S}^{k-1}) \simeq \mathcal{H}^\infty(M),$$

a map well defined up to contractible choice. Finally view Z as a variable, use the above ideas to make a map w from the homotopy colimit of the various Z to $\mathcal{H}^\infty(M)$, and note that $\text{hocolim } Z \simeq \mathcal{S}(M)$. This completes the construction.

It is evident that w takes $\mathcal{S}^s(M)$, the moduli space of s -structures on M , to the space of s -cobordisms, $\Omega^\infty \mathbf{H}^s(M)$. Homotopy commutativity of (1.5.1) is less evident, but we omit the proof.

1.5.2. Corollary [BuLa2], [BuFil]. *In the topological concordance stable range for M , and localized at odd primes, there is a product decomposition*

$$\mathcal{S}^s(M) \simeq \widetilde{\mathcal{S}}^s(M) \times \widetilde{\text{TOP}}(M) / \text{TOP}(M).$$

Proof. The left-hand vertical map in (1.5.1) is a homotopy equivalence in the concordance stable range, and the lower horizontal map is a split monomorphism in the homotopy category, at odd primes. Consequently the homotopy fiber of

$$\mathcal{S}^s(M) \hookrightarrow \widetilde{\mathcal{S}}^s(M)$$

is a retract up to homotopy of $\mathcal{S}^s(M)$ (localized at odd primes, in the concordance stable range). \square

1.5.3. Remark. Suppose that M^n is compact with boundary. Let $Z \subset \mathcal{S}(M)$ be finitely generated. A modification of the construction above gives w_Z from Z to $\text{hocolim}_k \mathcal{H}(M \times \mathbb{S}^{k-1})$ and then $w : \mathcal{S}(M) \rightarrow \mathcal{H}^\infty(M)$.

Now let $\mathcal{T}_n(M)$ be the moduli Space of pairs (N, f) where N^n is a compact manifold and $f : N \rightarrow M$ is *any* homotopy equivalence, not subject to boundary conditions. Let $Z \subset \mathcal{T}_n(M)$ be finitely generated. Another modification of the construction described above gives w_Z from Z to $\text{hocolim}_k \mathcal{H}(\partial(M \times \mathbb{S}^{k-1}))$, and then $w : \mathcal{T}_n(M) \rightarrow \mathcal{H}^\infty(M)$ because again

$$\text{hocolim}_k \mathcal{H}(\partial(M \times \mathbb{S}^{k-1})) \simeq \mathcal{H}^\infty(M).$$

It follows that $w : \mathcal{S}(M) \rightarrow \mathcal{H}^\infty(M)$ extends to $w : \mathcal{T}_n(M) \rightarrow \mathcal{H}^\infty(M)$. Let $\mathbf{G}^{\natural}(M)$ be the space of *all* homotopy equivalences $M \rightarrow M$, not subject to boundary conditions. Then $\mathbf{G}(M_0) \simeq \mathbf{G}^{\natural}(M) \hookrightarrow \mathcal{T}_n(M)$, and we can think of w as a map from $\mathbf{G}(M_0)$ to $\mathcal{H}^\infty(M)$. This is the map promised directly after 1.2.4.

1.6. DIFFEOMORPHISMS

Suppose that M^n is a closed topological manifold, $n > 4$, with tangent (micro)bundle τ . Morlet [Mo1], [Mo2], [BuLa3], [KiSi] proves that the forgetful map from a suitably defined Space of smooth structures on M , denoted $V(M)$, to a suitably defined Space $V(\tau)$ of vector bundle structures on τ is a weak homotopy equivalence. Earlier Hirsch and Mazur [HiMa] had proved that the map in question induces a bijection on π_0 . The Space $V(\tau)$ is homotopy equivalent to the homotopy fiber of the inclusion map $\text{map}(M, BO(n)) \rightarrow \text{map}(M, B\text{TOP}(n))$ over $\hat{\tau}$.

The Space of smooth structures on M can be defined as $\coprod_N \text{TOP}(N)/\text{DIFF}(N)$, where N runs through a set of representatives of diffeomorphism classes of smooth manifolds homeomorphic to M . Therefore Morlet's theorem gives, in the case where M is smooth, a homotopy fiber sequence

$$(1.6.1) \quad \text{DIFF}(M) \rightarrow \text{TOP}(M) \xrightarrow{a} V(M).$$

The map a is obtained from an action of $\text{TOP}(M)$ on $V(M)$ by evaluating at the base point of $V(M)$. The homotopy fiber sequence (1.6.1) remains meaningful when M is smooth compact with boundary; in this case allow only vector bundle structures on τ extending the standard structure over ∂M .

Traditionally, 1.6.1 has been an excuse for neglecting $\text{DIFF}(M)$ in favor of $\text{TOP}(M)$. But concordance theory has changed that. See §3. It is therefore best to develop a theory of smooth automorphisms parallel to the theory of topological automorphisms where possible. For example, 1.2.1–3, 1.3.1–2, 1.3.5–8, 1.4.2–4 and 1.5.1–3 have smooth analogs; but 1.2.4 does not.

Notation: A subscript d will often be used to indicate smoothness, as in $\mathcal{H}_d(M)$ for a space of smooth h -cobordisms (when M is smooth).

2. L -theory and moduli spaces

The major theorems in this chapter are to be found in §2.3 and §2.5. Sections 2.1, 2.2 and 2.4 introduce concepts needed to state those theorems.

2.1. ASSEMBLY

2.1.1. Definitions. Fix a space Y . Let \mathcal{W}_Y be the category of spaces over Y . A morphism in \mathcal{W}_Y is a *weak homotopy equivalence* if the underlying map of spaces is a weak homotopy equivalence. A commutative square in \mathcal{W}_Y is *cartesian* if the underlying square of spaces is cartesian.

A functor \mathbf{J} from \mathcal{W}_Y to CW-spectra [A, III] is *homotopy invariant* if it takes weak homotopy equivalences to weak homotopy equivalences. It is *excisive* if, in addition, it takes cartesian squares to cartesian squares, takes \emptyset to a contractible spectrum, and satisfies a wedge axiom,

$$\vee_i \mathbf{J}(X_i \rightarrow Y) \xrightarrow{\cong} \mathbf{J}(\coprod_i X_i \rightarrow Y).$$

2.1.2. Proposition. *For every homotopy invariant functor \mathbf{J} from \mathcal{W}_Y to CW-spectra, there exist an excisive functor $\mathbf{J}^\%$ from \mathcal{W}_Y to CW-spectra and a natural transformation $\alpha_{\mathbf{J}} : \mathbf{J}^\% \rightarrow \mathbf{J}$ such that $\alpha_{\mathbf{J}} : \mathbf{J}^\%(X) \rightarrow \mathbf{J}(X)$ is a homotopy equivalence whenever X is a point (over Y).*

The natural transformation $\alpha_{\mathbf{J}}$ is essentially characterized by these properties; it is called the *assembly*. It is the best approximation (from the left) of \mathbf{J} by an excisive functor. We write $\mathbf{J}^\%(X)$ for the homotopy fiber of $\alpha_{\mathbf{J}} : \mathbf{J}^\%(X) \rightarrow \mathbf{J}(X)$, for any X in \mathcal{W}_Y .

Remark. Assume that X and Y above are homotopy equivalent to CW-spaces. If $Y \simeq *$, then $\mathbf{J}^\%(X) \simeq X_+ \wedge \mathbf{J}(*)$ by a chain of natural homotopy equivalences. If $Y \not\simeq *$, build a quasi-fibered spectrum on Y with fiber $\mathbf{J}(y \hookrightarrow Y)$ over $y \in Y$. Pull it back to X using $X \rightarrow Y$. Collapse the zero section, a copy of X . The result is $\simeq \mathbf{J}^\%(X)$.

The assembly concept is due to Quinn, [Qun1], [Qun2], [Qun3]. See also [Lo]. For a proof of 2.1.2, see [WWa]. For applications to block s -structure Spaces we need the case where $Y = \mathbb{R}P^\infty$ and $\mathbf{J} = \mathbf{L}_\bullet^s$ is the L -theory functor $X \mapsto \mathbf{L}_\bullet^s(X)$ which associates to X the L -theory spectrum of $\mathbb{Z}\pi_1(X)$, say in the description of Ranicki [Ra2 §13]. Note the following technical points.

- \mathbf{L}_\bullet^s is the quadratic L -theory with decoration s which we make 0-connected (by force).
- Because of 2.1.1 we have no use for a base point in X . This makes it harder to say what $\mathbb{Z}\pi_1(X)$ should mean. For details see [WWa, 3.1], where $\mathbb{Z}\pi_1(X)$ or more precisely $\mathbb{Z}^w\pi_1(X)$ is a ringoid with involution depending on the double cover $w : X^\sim \rightarrow X$ induced by $X \rightarrow \mathbb{R}P^\infty$.

2.2. TANGENTIAL INVARIANTS

Geometric topology tradition requires that any classification of you-name-it structures on a manifold or Poincaré space [Kl] be accompanied by a classification of analogous structures on the normal bundle or Spivak normal fibration [Spi], [Ra3], [Br2] of the manifold or Poincaré space. We endorse this. However, we find tangent bundle language more convenient than normal bundle language. The constructions here in §2.2 will also be used in §3 and §4.

Terminology. When we speak of a *stable fiber homotopy equivalence* between euclidean bundles β and γ on a space X , we mean a fiber homotopy equivalence over X between the spherical fibrations associated with $\beta \oplus \varepsilon^k$ and $\gamma \oplus \varepsilon^k$, respectively, for some k .

Let M^n be a closed topological manifold with a choice [Kis], [Maz2] of euclidean tangent bundle τ . An *h-structure* on τ is a pair (ξ, ϕ) where ξ^n is a euclidean bundle on M and ϕ is a stable fiber homotopy equivalence from (the spherical fibration associated with) ξ to τ . The *h-structures* on τ and their isomorphisms

form a groupoid. Enlarge the groupoid to a simplicial groupoid by allowing families parametrized by Δ^k , and let

$$\mathcal{S}(\tau) := \text{diagonal nerve of the simplicial groupoid.}$$

Then $\mathcal{S}(\tau) \simeq \text{hofiber} [\text{map}(M, B\text{TOP}(n)) \hookrightarrow \text{map}(M, BG)]$ where $\hat{\tau}$ serves as base point in $\text{map}(M, B\text{TOP}(n))$ and $\text{map}(M, BG)$. There is a *tangential invariant map*, well defined up to contractible choice,

$$\nabla : \mathcal{S}(M) \longrightarrow \mathcal{S}(\tau).$$

Sketchy description: A homotopy equivalence $f : N \rightarrow M$ determines up to contractible choice a stable fiber homotopy equivalence $\psi : f^*\nu(M) \rightarrow \nu(N)$ because Spivak normal fibrations [Spi], [Ra], [Br3] are homotopy invariants. It then determines up to contractible choice a stable fiber homotopy equivalence $\psi^{\text{ad}} : \tau(N) \rightarrow f^*\tau(M)$. Now choose a euclidean bundle ξ^n on M and a stable fiber homotopy equivalence $\phi : \xi \rightarrow \tau(M)$ together with an isomorphism $j : f^*\xi \rightarrow \tau(N)$ and a homotopy $f^*\phi \simeq \psi^{\text{ad}} \cdot j$. This is a contractible choice. Let ∇ take (N, f) to (ξ, ϕ) . (This defines ∇ on the 0-skeleton; using the same ideas, complete the construction of ∇ by induction over skeletons.)

When $\partial M \neq \emptyset$, define $\mathcal{S}(\tau)$ in such a way that it is homotopy equivalent to the homotopy fiber of

$$\text{map}_{\text{rel}}(M, B\text{TOP}(n)) \xrightarrow{\subset} \text{map}_{\text{rel}}(M, BG)$$

where map_{rel} indicates maps from M which on ∂M agree with $\hat{\tau}$. Again $\hat{\tau}$ serves as base point everywhere.

When $\partial M \neq \emptyset$ and $\partial_+ M \subset \partial M$ is specified, with tangent bundle τ' of fiber dimension $n - 1$, we define $\mathcal{S}(\tau, \tau')$ in such a way that it is homotopy equivalent to the homotopy fiber of

$$\text{map}_{\text{rel}}((M, \partial_+ M), (B\text{TOP}(n), B\text{TOP}(n - 1))) \xrightarrow{\subset} \text{map}_{\text{rel}}(M, BG)$$

where map_{rel} indicates maps from M which on $\partial_- M$ agree with the classifying map for the tangent bundle of $\partial_- M$. The pair $(\hat{\tau}, \hat{\tau}')$ serves as base point. There is a *tangential invariant map* $\nabla : \mathcal{S}(M, \partial_+ M) \rightarrow \mathcal{S}(\tau, \tau')$ which fits into a homotopy commutative diagram where the rows are fiber homotopy sequences:

$$\begin{array}{ccccc} \mathcal{S}(M) & \xrightarrow{\subset} & \mathcal{S}(M, \partial_+ M) & \longrightarrow & \mathcal{S}(\partial_+ M) \\ \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla \\ \mathcal{S}(\tau) & \xrightarrow{\subset} & \mathcal{S}(\tau, \tau') & \longrightarrow & \mathcal{S}(\tau'). \end{array}$$

2.2.1. Illustration. Suppose that $(M, \partial_+ M) = (N \times I, N \times 1)$ where N^{n-1} is compact. Then $\mathfrak{S}(\tau, \tau')$ is homotopy equivalent to the homotopy fiber of

$$\mathrm{map}_{\mathrm{rel}}(N, \mathrm{BTOP}(n-1)) \hookrightarrow \mathrm{map}_{\mathrm{rel}}(N, \mathrm{BTOP}(n)).$$

Finally we need a moduli space $\tilde{\mathfrak{S}}(\tau)$ of *stable* h -structures on τ (assuming again that τ is the tangent bundle of a compact M). Define this as the moduli space of pairs (ξ, ϕ) where ξ^p is a euclidean bundle on M , of arbitrary fiber dimension p , and ϕ is a stable fiber homotopy equivalence $\xi \rightarrow \tau$, represented by an actual fiber homotopy equivalence between the spherical fibrations associated with $\xi \oplus \varepsilon^{k-p}$ and $\tau \oplus \varepsilon^{k-n}$ for some large k . Then

$$\tilde{\mathfrak{S}}(\tau) \simeq \mathrm{hofiber}[\mathrm{map}_{\mathrm{rel}}(M, \mathrm{BTOP}) \rightarrow \mathrm{map}_{\mathrm{rel}}(M, \mathrm{BG})]$$

which is homotopy equivalent to the space of based maps from $M/\partial M$ to G/TOP . Again there is a tangential invariant map— better known, and in this case more easily described, as the *normal invariant map* :

$$\nabla : \tilde{\mathfrak{S}}^s(M) \rightarrow \tilde{\mathfrak{S}}(\tau).$$

2.3. BLOCK s -STRUCTURES AND L -THEORY

2.3.1. Fundamental Theorem of Surgery (Browder–Novikov–Sullivan–Wall–Quinn–Ranicki). *For compact M^n with tangent bundle τ , where $n > 4$, there exists a homotopy commutative square of the form*

$$\begin{array}{ccc} \tilde{\mathfrak{S}}^s(M) & \xrightarrow{\nabla} & \tilde{\mathfrak{S}}(\tau) \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^{\infty+n}((\mathbf{L}_{\bullet}^s)_{\%}(M)) & \xrightarrow{\mathrm{forget}} & \Omega^{\infty+n}((\mathbf{L}_{\bullet}^s)_{\%}(M)). \end{array}$$

References: [Br3], [Br4], [Nov], [Rou1], [Su1], [Su2], [Wa1,§10], [ABK] for formulations without assembly, and [Ra4], [Qun1] for formulations with assembly.

Illustration. 2.3.1 gives $\tilde{\mathfrak{S}}^s(\mathbb{S}^n) \simeq \Omega^{\infty} \mathbf{L}_{\bullet}^s(*) \simeq \mathrm{G}/\mathrm{TOP}$ for $n > 4$. (Remember that $\mathbf{L}_{\bullet}^s(*)$ is 0-connected by definition, §2.1.) The honest structure space is

$$\mathfrak{S}(\mathbb{S}^n) \simeq \mathrm{G}(\mathbb{S}^n)/\mathrm{TOP}(\mathbb{S}^n)$$

(this uses the Poincaré conjecture). The inclusion $\mathfrak{S}(\mathbb{S}^n) \rightarrow \tilde{\mathfrak{S}}^s(\mathbb{S}^n)$ becomes the inclusion of $\mathrm{G}(\mathbb{S}^n)/\mathrm{TOP}(\mathbb{S}^n)$ in $\mathrm{hocolim}_k \mathrm{G}(\mathbb{S}^k)/\mathrm{TOP}(\mathbb{S}^k) \simeq \mathrm{G}/\mathrm{TOP}$; in particular, it is n -connected.

2.4. \mathbb{S}^1 -STABILIZATION

\mathbb{S}^1 -stabilization is a method for making new homotopy functors out of old ones. It was introduced in [Ra5] and applied in [AnPe], [HaMa] in a special case (the one we will need in §2.6 just below). It is motivated by the definition of the negative K -groups in [Ba].

Let \mathbf{J} be a homotopy functor from \mathcal{W}_Y (see 2.1.2) to CW-spectra. Let $\mathbb{S}^1(+)$ and $\mathbb{S}^1(-)$ be upper half and lower half of \mathbb{S}^1 , respectively. For X in \mathcal{W}_Y let $\sigma\mathbf{J}(X)$ be the homotopy pullback of

$$\frac{\mathbf{J}(X \times \mathbb{S}^1(+))}{\mathbf{J}(X \times *)} \longrightarrow \frac{\mathbf{J}(X \times \mathbb{S}^1)}{\mathbf{J}(X \times *)} \longleftarrow \frac{\mathbf{J}(X \times \mathbb{S}^1(-))}{\mathbf{J}(X \times *)}.$$

Note that $\sigma\mathbf{J}(X) \simeq \Omega[\mathbf{J}(X \times \mathbb{S}^1)/\mathbf{J}(X \times *)]$, and that $\sigma\mathbf{J}$ is a homotopy functor from \mathcal{W}_Y to CW-spectra. There are natural transformations

$$\mathbf{J}(X) \longrightarrow \frac{\mathbf{J}(X \times \mathbb{S}^0)}{\mathbf{J}(X \times *)} \longrightarrow \sigma\mathbf{J}(X),$$

the first induced by the inclusion $x \mapsto (x, -1)$ of X in $X \times \mathbb{S}^0$, and the second induced by the inclusions of \mathbb{S}^0 in $\mathbb{S}^1(-)$ and $\mathbb{S}^1(+)$. Let $\psi : \mathbf{J}(X) \rightarrow \sigma\mathbf{J}(X)$ be the composition. Finally let $\sigma^\infty\mathbf{J}(X)$ be the homotopy colimit of the $\sigma^k\mathbf{J}(X)$ for $k \geq 0$, using the maps $\psi : \sigma^{k-1}\mathbf{J}(X) \rightarrow \sigma(\sigma^{k-1}\mathbf{J}(X))$ to stabilize. We call $\sigma^\infty\mathbf{J}$ the \mathbb{S}^1 -stabilization of \mathbf{J} .

2.5. BOUNDED h -STRUCTURES AND L -THEORY

Let M^n be compact. A *bounded h -structure* on $M \times \mathbb{R}^k$ is a pair (N, f) where N^{n+k} is a manifold and $f : N \rightarrow M \times \mathbb{R}^k$ is a *bounded* homotopy equivalence restricting to a homeomorphism $\partial N \rightarrow \partial M \times \mathbb{R}^k$. (That is, there exist $c > 0$ and $g : M \times \mathbb{R}^k \rightarrow N$ and homotopies $h : fg \simeq \text{id}$, $j : gf \simeq \text{id}$ such that the sets $\{p_2 h_t(x) \mid t \in I\}$, $\{p_2 f j_t(y) \mid t \in I\}$ have diameter $< c$ for all $x \in M \times \mathbb{R}^k$ and $y \in N$; moreover h_t, j_t agree with the identity maps on $\partial M \times \mathbb{R}^k$ and ∂N , respectively.) References: [AnPe], [FePe].

A moduli Space $\mathcal{S}^b(M \times \mathbb{R}^k)$ of such bounded h -structures can be constructed in the usual way, as the diagonal nerve of a simplicial groupoid. There is a homotopy fiber sequence

$$\text{TOP}^b(M \times \mathbb{R}^k) \rightarrow \text{G}^b(M \times \mathbb{R}^k) \rightarrow \mathcal{S}^b(M \times \mathbb{R}^k)$$

where $\text{G}^b(M \times \mathbb{R}^k)$ is the Space of bounded homotopy automorphisms of $M \times \mathbb{R}^k$, relative to $\partial M \times \mathbb{R}^k$.

Again we need a *tangential invariant map* $\nabla : \mathcal{S}^b(M \times \mathbb{R}^k) \rightarrow \mathcal{S}(\tau \times \varepsilon^k)$ where $\tau = \tau(M)$ and $\tau \times \varepsilon^k$ plays the role of tangent bundle of $M \times \mathbb{R}^k$. Its definition resembles that of the tangential invariant maps in §2.2. Additional subtlety: one needs to know that *open* Poincaré spaces and *open* Poincaré pairs have Spivak normal fibrations which are invariants of proper homotopy type. See [Tay], [Mau], [FePe], [RaPe]. Note

$$\mathcal{S}(\tau \times \varepsilon^k) \simeq \text{hofiber}[\text{map}_{\text{rel}}(M, \text{BTOP}(n+k)) \rightarrow \text{map}_{\text{rel}}(M, \text{BG})]$$

where map_{rel} indicates maps which on ∂M agree with $\hat{\tau}$.

For a space X over $\mathbb{R}P^\infty$ let $\mathbf{L}_\bullet^{\langle -\infty \rangle}(X)$ be the 0-connected cover of $(\sigma^\infty \mathbf{L}_\bullet^s)(X)$, in the notation of §2.4.

2.5.1. Theorem. *For compact M^n with tangent bundle τ , where $n > 4$, there exists a homotopy commutative square of the form*

$$\begin{array}{ccc} \bigcup_k \mathcal{S}^b(M \times \mathbb{R}^k) & \xrightarrow{\nabla} & \bigcup_k \mathcal{S}(\tau \times \varepsilon^k) \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^{\infty+n}((\mathbf{L}_\bullet^{\langle -\infty \rangle})\% (M)) & \xrightarrow{\text{forget}} & \Omega^{\infty+n}((\mathbf{L}_\bullet^{\langle -\infty \rangle})\% (M)). \end{array}$$

Remark. $(\mathbf{L}_\bullet^{\langle -\infty \rangle})\% \simeq (\mathbf{L}_\bullet^s)\%$.

3. Algebraic K-theory and Moduli Spaces

3.1. ALGEBRAIC K-THEORY OF SPACES

Waldhausen’s homotopy functor \mathbf{A} from spaces to CW-spectra is a composition $\mathbf{K} \cdot \mathcal{R}$, where \mathcal{R} is a functor from spaces to *categories with cofibrations and weak equivalences* alias *Waldhausen categories*, and \mathbf{K} is a functor from Waldhausen categories to CW-spectra.

For a space X , let $\mathcal{R}(X)$ be the Waldhausen category of homotopy finite retractive spaces over X . The objects of $\mathcal{R}(X)$ are spaces Z equipped with maps

$$Z \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{i} \end{array} X \quad (ri = \text{id}_X)$$

subject to a finiteness condition. Namely, Z must be homotopy equivalent, relative to X , to a relative CW-space built from X by attaching a finite number of cells. A morphism in $\mathcal{R}(X)$ is a map relative to and over X . We call it a weak equivalence if it is a homotopy equivalence relative to X , and a cofibration if it has the homotopy extension property relative to X . See [Wah3, ch.2] for more information.

In [Wah3], [Wah1], Waldhausen associates with any Waldhausen category \mathcal{C} a spectrum $\mathbf{K}(\mathcal{C})$, generalizing Quillen’s construction [Qui] of the K -theory spectrum of an exact category. For us it is important that $\mathbf{K}(\mathcal{C})$ comes with a map reminiscent of “group completion”,

$$|w\mathcal{C}| \hookrightarrow \Omega^\infty \mathbf{K}(\mathcal{C})$$

where $w\mathcal{C}$ is the category of weak equivalences in \mathcal{C} and $|w\mathcal{C}|$ is its classifying space (geometric realization of the nerve).

3.2. ALGEBRAIC K -THEORY OF SPACES, AND h -COBORDISMS

Let M^n be compact, $n \geq 5$, with fundamental group(oid) π . The s -cobordism theorem due to Smale [Sm], [Mi2] in the simply connected smooth case and Barden–Mazur–Stallings [Ke] in the nonsimply connected smooth case states that $\pi_0\mathcal{H}_d(M)$ is isomorphic to the Whitehead group of π , that is, $K_1(\mathbb{Z}\pi)/\{\pm\pi^{\text{ab}}\}$. See [RoSa] for the PL version and [KiSi, Essay III] for the TOP version. Cerf [Ce] showed that $\pi_1\mathcal{H}_d(M)$ is trivial when M is smooth, simply connected and $n \geq 5$, and Rourke [Rou2] established the analogous statement in the PL category. In the early 70’s Hatcher and Wagoner [HaWa], working with a smooth but possibly nonsimply connected M , constructed a surjective homomorphism from $\pi_1\mathcal{H}_d(M)$ to a certain quotient of $K_2(\mathbb{Z}\pi)$, and they were able to describe the kernel of that homomorphism in terms of π and $\pi_2(M)$. See also [DIG]. These results follow from Waldhausen’s theorem 3.2.1, 3.2.2 below, which describes the homotopy types of $\mathcal{H}(M)$ and $\mathcal{H}_d(M)$ in a stable range, in algebraic K -theory terms. The size of the stable range is estimated by Igusa’s stability theorem, 1.3.4.

Remark. Note that the block s -cobordism space $\tilde{\mathfrak{S}}^s(M \times I, M \times 1)$ is not a very useful approximation to $\mathcal{H}^s(M)$, because it is contractible. Hence surgery theory as in §2 does not elucidate the homotopy type of $\mathcal{H}^s(M)$.

For compact M^n write $\mathcal{H}(\tau(M)) := \mathfrak{S}(\tau(M \times I), \tau(M \times 1))$ so that there is a tangential invariant map $\nabla : \mathcal{H}(M) \rightarrow \mathcal{H}(\tau(M))$. See 2.2.1. There is a stabilization map from $\mathcal{H}(\tau(M))$ to $\mathcal{H}(\tau(M \times I))$, analogous to the stabilization map $\mathcal{H}(M) \rightarrow \mathcal{H}(M \times I)$. We let $\tau = \tau(M)$ and $\mathcal{H}^\infty(\tau) = \text{hocolim}_k \mathcal{H}(\tau(M \times I^k))$ and obtain, since ∇ commutes with stabilization,

$$\nabla : \mathcal{H}^\infty(M) \rightarrow \mathcal{H}^\infty(\tau).$$

The following result is essentially contained in [Wah2].

3.2.1. Theorem (Waldhausen). *There exists a homotopy commutative square*

$$\begin{array}{ccc} \mathcal{H}^\infty(M) & \xrightarrow{\nabla} & \mathcal{H}^\infty(\tau) \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^\infty(\mathbf{A}^\% (M)) & \xrightarrow{\text{forget}} & \Omega^\infty(\mathbf{A}^\% (\tau)). \end{array}$$

3.2.2. Remark. Suppose that M is smooth. Then $\mathcal{H}^\infty(M)$ and $\mathcal{H}^\infty(\tau)$ have smooth analogues $\mathcal{H}_d^\infty(M)$ and $\mathcal{H}_d^\infty(\tau)$, and there is a homotopy commutative cartesian square

$$\begin{array}{ccc} \mathcal{H}_d^\infty(M) & \xrightarrow{\nabla} & \mathcal{H}_d^\infty(\tau) \\ \downarrow & & \downarrow \\ \mathcal{H}^\infty(M) & \xrightarrow{\nabla} & \mathcal{H}^\infty(\tau) \end{array}$$

with forgetful vertical arrows. It is not hard to see that $\mathcal{H}_d^\infty(\tau) \simeq \Omega^\infty \Sigma^\infty(M_+)$ and that $\nabla : \mathcal{H}_d^\infty(M) \rightarrow \mathcal{H}_d^\infty(\tau)$ is nullhomotopic. In this way 3.2.1 implies

$$(3.2.3) \quad \mathcal{H}_d^\infty(M) \times \Omega^{\infty+1} \Sigma^\infty(M_+) \simeq \Omega^{\infty+1}(\mathbf{A}(M))$$

which is better known than 3.2.1. Conversely, 3.2.1 is a formal consequence of (3.2.3), if we add the information that (3.2.3) comes from a spectrum level splitting, $\mathbf{H}_d^h(M) \vee \Omega \Sigma^\infty(M_+) \simeq \Omega \mathbf{A}(M)$ or equivalently $\mathbf{Whd}(M) \vee \Sigma^\infty(M_+) \simeq \mathbf{A}(M)$, where $\mathbf{Whd}(M)$ is the delooping of $\mathbf{H}_d^h(M)$ and $\mathbf{H}_d^h(M)$ is the (-1) -connected cover of $\mathbf{H}_d(M)$. We prefer formulation 3.2.1 because of its amazing similarity with 2.3.1 and 2.5.1.

References: 3.2.3 is stated in [Wah2]. It is reduced in [Wah4] to the spectrum level analog of the left-hand column in (3.2.1),

$$(3.2.4) \quad \mathbf{H}^h(M) \simeq \mathbf{A}_\% (M) ,$$

where $\mathbf{H}^h(M)$ denotes the (-1) -connected cover of $\mathbf{H}(M)$.) For the proof of (3.2.4), see [Wah3, §3] and the preprints [WaVo1] and [WaVo2]. The papers [Stb] and [Cha5] contain results closely related to [WaVo1] and [WaVo2], respectively. A very rough but helpful guide to this vast circle of ideas is [Wah5]. A more detailed guide by us is in preparation [WWg].

4. Mixing L -theory and algebraic K -theory of spaces

Introduction. We now have a large amount of indirect knowledge about the s -structure Space $\mathcal{S}^s(M)$ for a compact M . Namely, from the definitions there is a homotopy fiber sequence

$$\widetilde{\text{TOP}}(M)/\text{TOP}(M) \longrightarrow \mathcal{S}^s(M) \longrightarrow \widetilde{\mathcal{S}}^s(M).$$

In 2.3.1 we have an expression for $\widetilde{\mathcal{S}}^s(M)$ in terms of L -theory. In the concordance stable range, we also have the expression 1.3.7 for

$$\widetilde{\text{TOP}}(M)/\text{TOP}(M)$$

in terms of stabilized concordance theory. But 3.2.1 expresses stabilized concordance theory through the algebraic K -theory of spaces. Therefore, in the concordance stable range, $\mathcal{S}^s(M)$ must be a concoction of L -theory and algebraic K -theory of spaces. It remains to find out what concoction exactly. This problem was previously addressed by Hsiang–Sharpe (roughly speaking, using only the Postnikov 2-coskeleton of the algebraic K -theory of spaces), by Burghilea–Fiedorowicz (rationally), by Burghilea–Lashof (at odd primes), by Fiedorowicz–Schwänzl–Vogt (at odd primes); see references [HsiSha], [BuFi1], [BuFi2], [FiSVo1], [FiSVo2], [FiSVo3], [BuLa2]. In addition, the literature contains many results about $\mathcal{S}^s(M)$ or $\mathcal{S}(M)$, or the differentiable analogs, for specific M ; see §6 for a selection and further references.

Our analysis, Thm. 4.2.1 and remark 4.2.3 below, is based on the following idea. Using 3.2.1, Poincaré duality notions, and a more algebraic description of w in (1.5.1), we find that w lifts to an *equivariant Whitehead torsion* map

$$w^\sharp : \mathcal{S}^s(M) \longrightarrow \Omega^\infty(\mathbf{H}^s(M)^{h\mathbb{Z}/2}) \simeq \Omega^\infty((\mathbf{A}_{\%}^s(M)^{h\mathbb{Z}/2}))$$

where $(-)^{h\mathbb{Z}/2}$ indicates homotopy fixed points for a certain action of $\mathbb{Z}/2$. This refinement of w fits into a homotopy commutative diagram refining (1.5.1),

$$\begin{array}{ccccc} \widetilde{\text{TOP}}(M)/\text{TOP}(M) & \longrightarrow & \mathcal{S}^s(M) & \longrightarrow & \widetilde{\mathcal{S}}^s(M) \\ \downarrow & & \downarrow w^\sharp & & \downarrow \\ \Omega^\infty((\mathbf{A}_{\%}^s(M)_{h\mathbb{Z}/2})) & \xrightarrow{\text{norm}} & \Omega^\infty((\mathbf{A}_{\%}^s(M)^{h\mathbb{Z}/2})) & \longrightarrow & \Omega^\infty((\mathbf{A}_{\%}^s(M)^{th\mathbb{Z}/2})), \end{array}$$

where $\mathbf{A}_{\%}^s(M)^{th\mathbb{Z}/2}$ is the mapping cone of a *norm map* $\mathbf{A}_{\%}^s(M)_{h\mathbb{Z}/2} \rightarrow \mathbf{A}_{\%}^s(M)^{h\mathbb{Z}/2}$ which we will say more about below. For the right-hand vertical arrow in the diagram, we have an algebraic description. Note that the left-hand vertical arrow is highly connected and the rows are homotopy fiber sequences. Curiously, w^\sharp does not have an easy analog in the smooth category. See however §4.3.

This work is still in progress. Currently available: [WW1], [WW2], [WWa], [WWx], [WWd], [WWp], [We1]. The paper [DWW] is closely related and uses identical technology.

4.1. LA-THEORY

We describe a functor \mathbf{LA}_\bullet^h from $\mathcal{W}_{BG} \times \mathbb{N}$ to CW-spectra. Here \mathcal{W}_{BG} is the category of spaces over BG (alternatively, spaces equipped with a stable spherical fibration) and \mathbb{N} is regarded as a category with exactly one morphism $m \rightarrow n$ if $m \leq n$, and no morphism from m to n if $m > n$. For fixed n , the functor $\mathbf{LA}_\bullet^h(-, n)$ is a homotopy functor. It is a composition $\mathbf{F}_n \cdot \mathcal{R}^\infty$, where \mathcal{R}^∞ is a functor from \mathcal{W}_{BG} to the category of Waldhausen categories with Spanier–Whitehead (SW) product [WWd], and \mathbf{F}_n is a functor from certain Waldhausen categories with SW product to CW-spectra.

Let X be a space over BG . In the Waldhausen category $\mathcal{R}(X)$ of 3.1, we have notions of mapping cylinder, mapping cone and suspension Σ_X . Let $\mathcal{R}^\infty(X)$ be the colimit of the direct system of categories

$$\mathcal{R}(X) \xrightarrow{\Sigma_X} \mathcal{R}(X) \xrightarrow{\Sigma_X} \mathcal{R}(X) \xrightarrow{\Sigma_X} \dots$$

Again, $\mathcal{R}^\infty(X)$ is a Waldhausen category. We will need additional structure on $\mathcal{R}^\infty(X)$ in the shape of an *SW product* which depends on the reference map from X to BG . The SW product is a functor $(Z_1, Z_2) \mapsto Z_1 \odot Z_2$ from $\mathcal{R}^\infty(X) \times \mathcal{R}^\infty(X)$ to based spaces. Its main properties are:

- *symmetry*, that is, $Z_1 \odot Z_2 \cong Z_2 \odot Z_1$ by a natural involutory homeomorphism;

- *bilinearity*, that is, for fixed Z_2 the functor $Z_1 \mapsto Z_1 \odot Z_2$ takes the zero object to a contractible space and takes pushout squares where the horizontal arrows are cofibrations to cartesian squares;
- *w-invariance*, that is, a weak equivalence $Z_1 \rightarrow Z'_1$ induces a weak homotopy equivalence $Z_1 \odot Z_2 \rightarrow Z'_1 \odot Z_2$ for any Z_2 .

Modulo technicalities, the definition of $Z_1 \odot Z_2$ for Z_1, Z_2 in $\mathcal{R}(X) \subset \mathcal{R}^\infty(X)$ is as follows. Let γ be the spherical fibration on X pulled back from $B\mathbb{G}$; we can assume that it comes with a distinguished section and has fibers $\simeq \mathbb{S}^k$ for some k . Convert the retraction maps $Z_1 \rightarrow X$ and $Z_2 \rightarrow X$ into fibrations, with total spaces Z_1^\sim and Z_2^\sim ; form the fiberwise smash product (over X) of Z_1^\sim, Z_2^\sim and the total space of γ ; collapse the zero section to a point; finally apply $\Omega^{\infty+k}\Sigma^\infty$.

Imitating [SpaW], [Spa,§8 ex.F] we say that $\eta \in Z' \odot Z$ is a *duality* if it has certain nondegeneracy properties. For every Z in $\mathcal{R}(X)$ there exists a Z' and $\eta \in Z' \odot Z$ which is a duality; the pair (Z', η) is determined up to contractible choice by Z , and we can say that Z' is the *dual* of Z . Modulo technicalities, an involution on the K -theory spectrum $\mathbf{K}(\mathcal{R}^\infty(X))$ results, induced by $Z \mapsto Z'$. The inclusion $\mathcal{R}(X) \rightarrow \mathcal{R}^\infty(X)$ induces a homotopy equivalence of the K -theory spectra, so that we are talking about an involution on $\mathbf{A}(X)$. For all details, we refer to [WWd]. The involution on $\mathbf{A}(X)$ was first constructed in [Vo]. See also [KVWW2].

More generally, suppose that \mathcal{B} is any Waldhausen category with SW product, satisfying the axioms of [WWd,§2] which assure existence and essential uniqueness of SW duals. Then the K -theory spectrum $\mathbf{K}(\mathcal{B})$ has a preferred involution. In this setup it is also possible to define spectra $\mathbf{L}_\bullet(\mathcal{B})$ (quadratic L -theory), $\mathbf{L}^\bullet(\mathcal{B})$ (symmetric L -theory), a forgetful map from quadratic to symmetric L -theory, and a map

$$\Xi : \mathbf{L}^\bullet(\mathcal{B}) \longrightarrow \mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$$

where $\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$ is the mapping cone of the *norm map* [AdCD], [GrMa], [WW2, 2.4],

$$\mathbf{K}(\mathcal{B})_{h\mathbb{Z}/2} \longrightarrow \mathbf{K}(\mathcal{B})^{h\mathbb{Z}/2},$$

from the homotopy orbit spectrum to the homotopy fixed point spectrum of the involution on $\mathbf{K}(\mathcal{B})$. The norm map refines the transfer $\mathbf{K}(\mathcal{B})_{h\mathbb{Z}/2} \rightarrow \mathbf{K}(\mathcal{B})$.

Our constructions of $\mathbf{L}_\bullet(\mathcal{B})$ and $\mathbf{L}^\bullet(\mathcal{B})$ are bordism-theoretic and follow [Ra2] very closely, except that with a view to the applications here we need 0-connected versions. In particular, if $\mathcal{B} = \mathcal{R}^\infty(X)$, then $\mathbf{L}_\bullet(\mathcal{B})$ is homotopy equivalent to the 0-connected cover of the quadratic L -theory spectrum (decoration h) of the ring(oid) with involution $\mathbb{Z}\pi_1(X)$. Regarding Ξ , we offer the following explanations. Let $\mathcal{B}^*[i]$ be the category of covariant functors from the poset of nonempty faces of Δ^i to \mathcal{B} . Then $\mathcal{B}^*[i]$ inherits from \mathcal{B} the structure of a Waldhausen category with SW product, with weak equivalences and cofibrations defined coordinatewise. The axioms of [WWd,§2] are still satisfied. Modulo technicalities, there is a duality involution on $|\mathcal{B}^*[i]|$ for each $i \geq 0$, and an inclusion of simplicial spaces:

$$\begin{array}{ccc}
i & \mapsto & |w\mathcal{B}^*[i]|^{h\mathbb{Z}/2} \\
& & \downarrow \\
i & \mapsto & K(\mathcal{B}^*[i])^{h\mathbb{Z}/2}.
\end{array}$$

The geometric realizations of these simplicial spaces turn out to be Ω^∞ of $\mathbf{L}^\bullet(\mathcal{B})$ and $\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2}$ respectively. (Recognition is easy in the first case, harder in the second case.) The inclusion map of geometric realizations is Ω^∞ of Ξ , by definition.

We come to the description of $\mathbf{F}_n(\mathcal{B})$, promised at the beginning of this section. Let $\mathbb{S}_1^n = \mathbb{R}^n \cup \infty$ with the involution $z \mapsto -z$ for $z \in \mathbb{R}^n$. This has fixed point set $\{0, \infty\} \cong \mathbb{S}^0$. Let $\mathbf{K}(\mathcal{B}, n) := \mathbb{S}_1^n \wedge \mathbf{K}(\mathcal{B})$ with the diagonal involution. The inclusion of $\mathbf{K}(\mathcal{B}) \cong \mathbf{K}(\mathcal{B}, 0)$ in $\mathbf{K}(\mathcal{B}, n)$ induces a homotopy equivalence of Tate spectra,

$$\mathbf{K}(\mathcal{B})^{th\mathbb{Z}/2} \xrightarrow{t^n} \mathbf{K}(\mathcal{B}, n)^{th\mathbb{Z}/2}$$

(proof by induction on n). Write $?$ for the forgetful map from quadratic L -theory to symmetric L -theory, and let $\mathbf{F}_n(\mathcal{B})$ be the homotopy pullback of

$$\begin{array}{ccc}
& & \mathbf{K}(\mathcal{B}, n)^{h\mathbb{Z}/2} \\
& & \downarrow \cap \\
\mathbf{L}_\bullet(\mathcal{B}) & \xrightarrow{t^n \cdot \Xi \cdot ?} & \mathbf{K}(\mathcal{B}, n)^{th\mathbb{Z}/2}.
\end{array}$$

4.1.1. Summary. $\mathbf{LA}_\bullet^h(X, n)$ is a spectrum defined for any space X over BG and any $n \geq 0$. It is the homotopy pullback of a diagram

$$\begin{array}{ccc}
& & \mathbf{A}(X, n)^{h\mathbb{Z}/2} \\
& & \downarrow \cap \\
\mathbf{L}_\bullet^h(X) & \xrightarrow{t^n \cdot \Xi \cdot ?} & \mathbf{A}(X, n)^{th\mathbb{Z}/2}
\end{array}$$

in which $\mathbf{L}_\bullet^h(X)$ denotes the 0-connected L -theory spectrum of $\mathbb{Z}\pi_1(X)$ with decoration h , and $\mathbf{A}(X, n) = \mathbb{S}_1^n \wedge \mathbf{A}(X)$. Hence there are homotopy fiber sequences

$$\begin{aligned}
(4.1.2) \quad & \mathbf{A}(X, n)_{h\mathbb{Z}/2} \rightarrow \mathbf{LA}_\bullet^h(X, n) \rightarrow \mathbf{L}_\bullet^h(X), \\
& \mathbf{LA}_\bullet^h(X, n-1) \rightarrow \mathbf{LA}_\bullet^h(X, n) \xrightarrow{v} \mathbf{A}(X, n).
\end{aligned}$$

4.2. LA -THEORY AND h -STRUCTURE SPACES

In the following theorem, we mean by $(\mathbf{LA}_\bullet^h)^\%(-, n)$ and $(\mathbf{LA}_\bullet^h)^\%(-, n)$ domain and homotopy fiber, respectively, of the assembly transformation for the homotopy functor $\mathbf{LA}_\bullet^h(-, n)$ on \mathcal{W}_{BG} . The manifold M becomes an object in \mathcal{W}_{BG} by means of the classifying map for $\nu(M)$.

4.2.1. Theorem. *For compact M^n there exists a homotopy commutative square with highly connected (see 4.2.2) vertical arrows*

$$\begin{array}{ccc} \mathcal{S}(M) & \xrightarrow{\nabla} & \mathcal{S}(\tau) \\ \downarrow & & \downarrow \\ \Omega^{\infty+n}(\mathbf{LA}_{\bullet}^h)_{\%}(M, n) & \xrightarrow{\text{forget}} & \Omega^{\infty+n}(\mathbf{LA}_{\bullet}^h)_{\%}(M, n). \end{array}$$

4.2.2. Details. The right-hand vertical arrow in 4.2.1 is $(j+2)$ -connected if j is in the smooth concordance stable range for the disk \mathbb{D}^n and $j \leq n-2$. The left-hand vertical arrow in 4.2.1 induces a bijection on π_0 . Each component of $\mathcal{S}(M)$ determines a homeomorphism class of manifolds N homotopy equivalent to M . If j is in the topological concordance stable range for N , then the left-hand vertical arrow in 4.2.1 restricted to that component (and the corresponding component of the codomain) is $(j+1)$ -connected.

4.2.3. Remark. There is an s -decorated version of 4.2.1, in which the Space of s -structures $\mathcal{S}^s(M)$ replaces $\mathcal{S}(M)$ and \mathbf{LA}_{\bullet}^s replaces \mathbf{LA}_{\bullet}^h . To define \mathbf{LA}_{\bullet}^s use L -theory and algebraic K -theory of spaces with an s -decoration in §4.1. The homotopy groups of $\mathbf{LA}_{\bullet}^s(X, n)$ differ from those of $\mathbf{LA}_{\bullet}^h(X, n)$ only in dimensions $\leq n$.

A $\langle -\infty \rangle$ -decorated version of 4.2.1 exists, but does not give anything new since the homotopy groups of

$$\mathbf{LA}_{\bullet}^{\langle -\infty \rangle}(-, n)$$

differ from those of $\mathbf{LA}_{\bullet}^h(-, n)$ only in dimensions $< n$. Nevertheless, it is good to have this in mind when making the comparison with 2.5.1 (next remark).

4.2.4. Remark. Theorems 4.2.1 and 2.5.1 are compatible: the commutative squares in 4.2.1 and 2.5.1 are opposite faces of a commutative cube. Of the 12 arrows in the cube, the four not mentioned in 4.2.1 or 2.5.1 are inclusion maps (top) and forgetful maps (bottom). Also, the s -version of 4.2.1 is compatible with 2.3.1 in the same sense.

4.2.5. Remark. The left-hand column of the diagram in 4.2.1 matches the left-hand column of the diagram in 3.2.1. In detail: there exists a commutative diagram with highly connected vertical arrows

$$\begin{array}{ccc} \mathcal{H}^{\infty}(M) & \xleftarrow{w} & \mathcal{S}(M) \\ \downarrow & & \downarrow \\ \Omega^{\infty}(\mathbf{A}_{\%}(M)) & \xleftarrow{v_{\%}} & \Omega^{\infty+n}(\mathbf{LA}_{\bullet}^h)_{\%}(M, n) \end{array}$$

where w is the Whitehead torsion map of 1.5 and $v_{\%}$ is induced by v of (4.1.2).

4.3. SPECIAL FEATURES OF THE SMOOTH CASE

In this section we assume that M^n is smooth. We use the notation of 4.2.1 and 2.5.1. By 4.2.4, there is a commutative diagram

$$(4.3.1) \quad \begin{array}{ccc} \mathcal{S}(\tau) & \xrightarrow{\subset} & \bigcup_k \mathcal{S}(\tau \oplus \varepsilon^k) \\ \downarrow & & \downarrow \\ \Omega^{\infty+n}(\mathbf{L}\mathbf{A}_{\bullet}^h)\% (M, n) & \longrightarrow & \Omega^{\infty+n}(\mathbf{L}_{\bullet}^h)\% (M, n). \end{array}$$

We can write the resulting map of horizontal homotopy fibers in the form

$$(4.3.2) \quad \begin{array}{c} u\mathcal{S}(\tau) \\ \downarrow \\ \Omega^{\infty+n}(\mathbf{A}\% (M, n)_{h\mathbb{Z}/2}) \end{array}$$

where the prefix u indicates *unstable* structures. It is highly connected ; details as in 4.2.2. Our goal here is to describe a smooth analog of (4.3.2).

4.3.3. Proposition. *Let X be a space over BO . Then the map $\Sigma^{\infty}(X_+) \rightarrow \mathbf{A}(X)$ of 3.2.2 has a canonical refinement to a $\mathbb{Z}/2$ -map.*

Remarks, Notation. As in §4.1, we work in “naive” stable $\mathbb{Z}/2$ -homotopy theory. That is, whenever we see a $\mathbb{Z}/2$ -map $\mathbf{Y}' \rightarrow \mathbf{Y}$ which is an ordinary homotopy equivalence, we are allowed to replace \mathbf{Y} by \mathbf{Y}' .

The involution on $\mathbf{A}(X)$ that we use here is determined by $X \rightarrow BO \hookrightarrow BG$ as in §4.1. The involution on $\Sigma^{\infty}(X_+)$ that we have in mind is as follows. For simplicity, assume that X is a compact CW-space ; then the reference map $X \rightarrow BO$ factors through $BO(k)$ for some k . Let γ^k be the vector bundle on X pulled back from $BO(k)$, and let η^{ℓ} be a complementary vector bundle on X , so that $\gamma \oplus \eta$ is trivialized. Now, to see the involution, replace $\Sigma^{\infty}(X_+)$ by the homotopy equivalent

$$\Omega_!^{\ell} \Omega^k \Sigma^{\infty} \Sigma^{\gamma} \Sigma_!^{\eta}(X_+)$$

where $\Sigma^{\eta}(X_+)$ and $\Sigma^{\gamma} \Sigma^{\eta}(X_+)$ are the Thom spaces of η and $\gamma \oplus \eta$, respectively. Subscripts ! indicate that $\mathbb{Z}/2$ acts on loop or suspension coordinates by scalar multiplication with -1 . Compare §4.1. — We abbreviate

$$\Sigma^{\infty}(X_+, n) := \mathbb{S}_!^n \wedge \Sigma^{\infty}(X_+).$$

While the map $\Sigma^{\infty}(X_+) \rightarrow \mathbf{A}(X)$ of 3.2.2 can be described in algebraic K -theory terms, including the algebraic K -theory of finite sets over X , our proof of 4.3.3 is not entirely K -theoretic. It uses 3.2.1 to re-interpret the involution on $\mathbf{A}(X)$ geometrically.

4.3.4. Theorem. *There is a commutative diagram with highly connected vertical arrows (and lower row resulting from 4.3.3)*

$$\begin{array}{ccc} u\mathcal{S}(\tau) & \longleftarrow & u\mathcal{S}_d(\tau) \\ \downarrow & & \downarrow \\ \Omega^{\infty+n}(\mathbf{A}^{\%}(M, n)_{h\mathbb{Z}/2}) & \longleftarrow & \Omega^{\infty+n}((\Sigma^{\infty}(X_+, n))_{h\mathbb{Z}/2}). \end{array}$$

Remarks. The right-hand column of the diagram in 4.3.4 is $(n-2)$ -connected. It is essentially an old construction due to Toda and James ; see [Jm] for references.

Something should be said about compatibility between 4.3.4 and 4.2.5, but we will leave it unsaid.

4.3.5. Remark. In calculations involving 4.3.4 the concept of *stabilization* is often useful. Stabilization is a way to make new homotopy functors on \mathcal{W}_Y (spaces over Y) from old ones. Idea: Given a homotopy functor \mathbf{J} from \mathcal{W}_Y to spectra, and X in \mathcal{W}_Y , let $s\mathbf{J}(X)$ be the homotopy colimit of the $\Omega^n(\mathbf{J}(X \times \mathbb{S}^n)/\mathbf{J}(X))$ for $n \geq 0$. There is a natural transformation $\mathbf{J}(X) \rightarrow s\mathbf{J}(X)$ induced by $x \mapsto (x, -1) \in X \times \mathbb{S}^0$ for $x \in X$. The formalities are much as in §2.4, even though the result is quite different. The main examples for us are these:

- Take $\mathbf{J}(X) = \mathbf{A}(X)$ for X in \mathcal{W}_{BO} . Then $s\mathbf{A}(X) \simeq \Sigma^{\infty}(\Lambda X_+)$ where ΛX is the free loop space. (See [Go1] for details.) Hence $(s\mathbf{A})^{\%}(X) \simeq \Sigma^{\infty}(X_+)$.
- Take $\mathbf{J}(X) = \mathbf{L}_{\bullet}^h(X)$ for X in \mathcal{W}_{BO} . Then $s\mathbf{L}_{\bullet}^h(X)$ is contractible for all X by the π - π -theorem. Hence $(s\mathbf{L}_{\bullet}^h)^{\%}(X)$ is also contractible.

One can use these facts to split the lower row in 4.3.4, up to homotopy. See 6.5 for another application.

5. Geometric structures on fibrations

5.1. BLOCK BUNDLE STRUCTURES

Here we address the following question. Given a fibration p on a Space B whose fibers are Poincaré duality spaces of formal dimension n , can we find a block bundle p_0 on B with closed manifold fibers, fiber homotopy equivalent to p ? For earlier work on this problem, see [Qun4], [Qun1]. We combine this with ideas from [Ra4] and [Ra2].

The *block s -structure Space* $\tilde{\mathcal{S}}^s(X)$ of a simple Poincaré duality space X of formal dimension n (alias *finite* Poincaré space, [Wa1, §2]) is defined literally as in the case of a closed manifold. Any simple homotopy equivalence $M^n \rightarrow X$, where M^n is a closed manifold, induces a homotopy equivalence

$$\tilde{\mathcal{S}}^s(M) \rightarrow \tilde{\mathcal{S}}^s(X)$$

and $\tilde{\mathcal{S}}^s(M)$ was described in L -theoretic terms in 2.3.1. But such a homotopy equivalence $M \rightarrow X$ might not exist, and even if it does, we might want to see an

L -theoretic description of the block s -structure Space of X which does not use a choice of homotopy equivalence $M \rightarrow X$.

Ranicki [Ra4], [Ra2, §17] associates to a simple Poincaré duality space X of formal dimension n its *total surgery obstruction*, a point

$$\partial\sigma^*(X) \in \Omega^{\infty+n-1}((\mathbf{L}_\bullet^s)_\%(X)).$$

The element has certain naturality properties. For example, a homotopy equivalence $g : X \rightarrow Y$ determines a path in $\Omega^{\infty+n-1}((\mathbf{L}_\bullet^s)_\%(X))$ from $g_*\partial\sigma^*(X)$ to $\partial\sigma^*(Y)$; more later.

5.1.1. Theorem. $\tilde{\mathfrak{S}}^s(X)$ is (naturally) homotopy equivalent to the space of paths from $\partial\sigma^*(X)$ to the base point in $\Omega^{\infty+n-1}((\mathbf{L}_\bullet^s)_\%(X))$.

Note that any choice of (base) point in the space of paths in 5.1.1 leads to an identification of it with $\Omega^{\infty+n}((\mathbf{L}_\bullet^s)_\%(X))$, up to homotopy equivalence. In this way, we recover the result

$$\tilde{\mathfrak{S}}^s(M) \simeq \Omega^{\infty+n}((\mathbf{L}_\bullet^s)_\%(M)).$$

The advantages of 5.1.1 become clearer when it is applied to *families*, that is, fibrations $p : E(p) \rightarrow B$ whose fibers are Poincaré spaces of formal dimension n . (One must pay attention to simple homotopy types, so we assume that B is connected and p is classified by a map $B \rightarrow BG^s(X)$ for some simple Poincaré duality space X as above.) Given such a $p : E(p) \rightarrow B$, we obtain an associated fibration $q : E(q) \rightarrow B$ with fiber $\Omega^{\infty+n-1}((\mathbf{L}_\bullet^s)_\%(p^{-1}(b)))$ over $b \in B$, and a section $\partial\sigma^*(p)$ of q selecting the total surgery obstruction $\partial\sigma^*(p^{-1}(b))$ in $q^{-1}(b)$, for $b \in B$. The fibers of q are infinite loop spaces, so we also have a zero section. (Technical point: For these constructions it is convenient to assume that B is a simplicial complex, and to apply a suitable $(n+2)$ -ad version of 5.1.1 to $E(p)|_\sigma$ for each n -simplex σ in B .) We say that p admits a *block bundle structure* if the classifying map $B \rightarrow BG^s(X)$ lifts to a map

$$B \rightarrow \widetilde{BTOP}(M)$$

for some closed manifold M equipped with a simple homotopy equivalence to X .

5.1.2. Corollary. *The fibration $p : E(p) \rightarrow B$ with Poincaré duality space fibers admits a block bundle structure if and only if $\partial\sigma^*(p)$ is vertically nullhomotopic.*

In the case $B = BG^s(X)$ we can add the following:

5.1.3. Corollary. *Let $p : E(p) \rightarrow BG^s(X)$ be the canonical fibration with fibers $\simeq X$. There is a cartesian square*

$$\begin{array}{ccc} \amalg_M \widetilde{BTOP}(M) & \longrightarrow & BG^s(X) \\ \downarrow & & \downarrow \text{total surgery obstruction section} \\ BG^s(X) & \xrightarrow{\text{zero section}} & E(q) \end{array}$$

where M runs through a maximal set of pairwise non-homeomorphic closed n -manifolds in the simple homotopy type of X .

A few words on how $\partial\sigma^*(X)$ is constructed: Ranicki creates a homotopy functor \mathbf{VL}_s^\bullet on spaces over $\mathbb{R}P^\infty$, and a natural transformation $\mathbf{L}_s^\bullet \rightarrow \mathbf{VL}_s^\bullet$ with the property that

$$\begin{array}{ccc} (\mathbf{L}_s^\bullet)^\%(X) & \longrightarrow & (\mathbf{VL}_s^\bullet)^\%(X) \\ \downarrow \text{assembly} & & \downarrow \text{assembly} \\ \mathbf{L}_s^\bullet(X) & \longrightarrow & \mathbf{VL}_s^\bullet(X) \end{array}$$

is cartesian for any X over $\mathbb{R}P^\infty$. (The functor \mathbf{VL}_s^\bullet we have in mind is defined in [Ra, §15], and we should really call it $\mathbf{VL}_s^\bullet\langle 1/2 \rangle$ to conform with Ranicki's notation.) Any finite Poincaré duality space X of formal dimension n determines an element $\sigma^*(X) \in \Omega^{\infty+n}(\mathbf{VL}_s^\bullet(X))$, the *visible symmetric signature* of X . The image of $\sigma^*(X)$ under the boundary map

$$\Omega^{\infty+n}(\mathbf{VL}_s^\bullet(X)) \longrightarrow \Omega^{\infty+n-1}((\mathbf{VL}_s^\bullet)^\%(X)) \simeq \Omega^{\infty+n-1}((\mathbf{L}_s^\bullet)^\%(X))$$

is the total surgery obstruction $\partial\sigma^*(X)$. Therefore: X is simple homotopy equivalent to a closed n -manifold if and only if the component of $\sigma^*(X)$ is in the image of the assembly homomorphism,

$$\pi_n(\mathbf{VL}_s^\bullet)^\%(X) \rightarrow \pi_n \mathbf{VL}_s^\bullet(X).$$

The functor \mathbf{VL}_s^\bullet has a *ring structure*, that is, for X_1 and X_2 over $\mathbb{R}P^\infty$ there is a multiplication

$$\mu : \mathbf{VL}_s^\bullet(X_1) \wedge \mathbf{VL}_s^\bullet(X_2) \longrightarrow \mathbf{VL}_s^\bullet(X_1 \times X_2),$$

with a unit in $\mathbf{VL}_s^\bullet(*)$. The visible symmetric signature is multiplicative:

$$\mu(\sigma^*(X_1), \sigma^*(X_2)) = \sigma^*(X_1 \times X_2),$$

up to a canonical path, for Poincaré duality spaces X_1 and X_2 . This property makes σ^* useful (more useful than $\partial\sigma^*$ alone) in dealing with products, say, in giving a description along the lines of 5.1.1 of the product map

$$\tilde{\mathfrak{S}}^s(X_1) \times \tilde{\mathfrak{S}}^s(X_2) \rightarrow \tilde{\mathfrak{S}}^s(X_1 \times X_2).$$

5.2. FIBER BUNDLE STRUCTURES

Here the guiding question is: Given a fibration p over some space B , with fibers homotopy equivalent to finitely dominated CW-spaces, does there exist a bundle p_0 on B with compact manifolds as fibers, fiber homotopy equivalent to p ? We do not assume that the fibers of p satisfy Poincaré duality. We do not ask that the fibers of p_0 be closed and we do not care what dimension they have. See [DWW] for all details.

Let Z be a compact CW–space, equipped with a euclidean bundle ξ . Let $\mathcal{T}_n(Z, \xi)$ be the moduli Space of pairs (M, f, j) where M^n is a compact manifold with boundary, $f : M \rightarrow Z$ is a homotopy equivalence, and j is a stable isomorphism $f^*\xi \rightarrow \tau(M)$. Let $\mathcal{T}(Z, \xi)$ be the colimit of the $\mathcal{T}_n(Z, \xi)$ under stabilization (product with I).

Any choice of vertex (M, f, j) in $\mathcal{T}(Z, \xi)$ leads to a homotopy equivalence from $\mathcal{T}(M, \tau)$ to $\mathcal{T}(Z, \xi)$. There is a fiber homotopy sequence

$$\bigcup_k \text{TOP}(M \times I^k, \partial(M \times I^k)) \longrightarrow \bigcup_k \text{G}_{\text{tan}}(M_0 \times \mathbb{R}^k) \longrightarrow \mathcal{T}(M, \tau)$$

where $M_0 = M \setminus \partial M$, and $\text{G}_{\text{tan}}(\dots)$ refers to homotopy automorphisms f of $M_0 \times \mathbb{R}^k$ covered by isomorphisms $\tau(M_0 \times \mathbb{R}^k) \rightarrow f^*(\tau(M_0 \times \mathbb{R}^k))$. By 1.2.1 we may write $\bigcup_k \text{TOP}(M_0 \times \mathbb{R}^k)$ instead of $\bigcup_k \text{G}_{\text{tan}}(M_0 \times \mathbb{R}^k)$. Now 1.2.3 implies

$$\Omega\mathcal{T}(M, \tau) \simeq \Omega\mathcal{H}^\infty(M)$$

and suggests $\mathcal{T}(M, \tau) \simeq \mathcal{H}^\infty(M)$. This is easily confirmed with the methods of §1.2. Summarizing these observations: any choice of vertex (M, f, j) in $\mathcal{T}(Z, \xi)$ leads to a homotopy equivalence $\mathcal{T}(Z, \xi) \rightarrow \mathcal{H}^\infty(M)$. Furthermore, $\mathcal{H}^\infty(M) \simeq \Omega^\infty \mathbf{A}_\% (M) \simeq \Omega^\infty \mathbf{A}_\% (Z)$ by 3.2.1.

Again, we might want to see a description of $\mathcal{T}(Z, \xi)$ in terms of $\mathbf{A}(Z)$ which does not depend on a choice of base point in $\mathcal{T}(Z, \xi)$. To get such a description we proceed very much as in §5.1, by associating to Z a *characteristic*

$$\chi(Z) \in \Omega^\infty \mathbf{A}(Z),$$

analogous to Ranicki's $\sigma^*(X) \in \Omega^{\infty+n} \mathbf{VL}_s^\bullet(X)$ for a simple Poincaré duality space X of formal dimension n . The element $\chi(Z)$ is the image of the object/vertex

$$\mathbb{S}^0 \times Z \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{i} \end{array} Z \quad (r(x, z) = z, i(z) = (1, z))$$

in $\mathcal{R}(Z)$ under the inclusion $|w\mathcal{R}(Z)| \hookrightarrow \Omega^\infty \mathbf{A}(Z)$ mentioned in §3.1. If Z is connected, then the component of $\chi(Z)$ in $\pi_0 \Omega^\infty \mathbf{A}(Z) \cong \mathbb{Z}$ is the Euler characteristic of Z .

5.2.1. Theorem. $\mathcal{T}(Z, \xi)$ is (naturally) homotopy equivalent to the homotopy fiber of the assembly map $\Omega^\infty(\mathbf{A}_\%(Z)) \rightarrow \Omega^\infty \mathbf{A}(Z)$ over the point $\chi(Z)$.

Note that the A –theoretic expression for $\mathcal{T}(Z, \xi)$ does not depend on ξ . Again, *naturality* in 5.2.1 is a license to apply the statement to families. Let $p : E \rightarrow B$ be a fibration where the fibers E_b are homotopy equivalent to compact CW–spaces. Let $\Omega^\infty \mathbf{A}(p)$ and $\Omega^\infty \mathbf{A}_\%(p)$ be the associated fibrations on B with fibers $\Omega^\infty \mathbf{A}(E_b)$ and $\Omega^\infty \mathbf{A}_\%(E_b)$, respectively, over $b \in B$. The rule $b \mapsto \chi(E_b)$ defines a section of $\Omega^\infty \mathbf{A}_\%(p)$ which we call $\chi(p)$. See [DWW] for explanations regarding the *continuity* of this construction. Assembly gives a map over B from the total space of $\Omega^\infty \mathbf{A}_\%(p)$ to that of $\Omega^\infty \mathbf{A}(p)$.

5.2.2. Corollary. *The fibration $p : E \rightarrow B$ is fiber homotopy equivalent to a bundle with compact manifolds as fibers if and only if the section $\chi(p)$ of $\Omega^\infty \mathbf{A}(p)$ lifts (after a vertical homotopy) to a section of $\Omega^\infty \mathbf{A}^\%(p)$.*

We leave it to the reader to state an analog of 5.1.3, and turn instead to the smooth case. Suppose that ξ is a *vector bundle* over Z . There is then a smooth variant $\mathcal{T}_d(Z, \xi)$ of $\mathcal{T}(Z, \xi)$. Any choice of base vertex (M, f, ξ) in $\mathcal{T}_d(Z, \xi)$ leads to a homotopy equivalence

$$\mathcal{T}(Z, \xi) \simeq \mathcal{H}_d^\infty(M).$$

Remember now (3.2.2) that $\mathcal{H}_d^\infty(M)$ can be A -theoretically described as the homotopy fiber of some map $\eta : \Omega^\infty \Sigma^\infty(M_+) \rightarrow \Omega^\infty \mathbf{A}(M)$. The map η is, in homotopy invariant terms, Ω^∞ of the composition

$$M_+ \wedge \mathbf{S}^0 \xrightarrow{\text{id} \wedge u} M_+ \wedge \mathbf{A}(\ast) \simeq \mathbf{A}^\%(M) \longrightarrow \mathbf{A}(M)$$

where $u : \mathbf{S}^0 \rightarrow \mathbf{A}(\ast)$ is the unit of the ring spectrum $\mathbf{A}(\ast)$.

5.2.3. Theorem. *$\mathcal{T}_d(Z, \xi)$ is (naturally) homotopy equivalent to the homotopy fiber of $\eta : \Omega^\infty \Sigma^\infty(Z_+) \rightarrow \Omega^\infty \mathbf{A}(Z)$ over the point $\chi(Z)$.*

Returning to the notation and hypotheses of 5.2.2, we are compelled to introduce yet another fibration $\Omega^\infty \Sigma_+^\infty p$ on B , with fiber $\Omega^\infty \Sigma^\infty(E_b)_+$ over $b \in B$.

5.2.4. Corollary. *The fibration $p : E(p) \rightarrow B$ is fiber homotopy equivalent to a bundle with smooth compact manifolds as fibers if and only if the section $\chi(p)$ of $\Omega^\infty \mathbf{A}(p)$ lifts (after a vertical homotopy) to a section of $\Omega^\infty \Sigma_+^\infty p$.*

Remarks. In 5.2.4, *bundle with smooth compact manifolds as fibers* means, say in the case where B is connected, a fiber bundle with fibers $\cong M$ where M is smooth compact, and structure group $\text{DIFF}(M, \partial M)$.

Corollary 5.2.4 is closely related to the Riemann–Roch theorem of [BiLo], see also [DWW], which we discuss in §6.7.

5.2.5. Corollary. *Let $p : E \rightarrow B$ be a fibration with fibers homotopy equivalent to compact CW-spaces. If Y is any compact connected CW-space of Euler characteristic 0, then the composition $pq : Y \times E \rightarrow B$ (where $q : Y \times E \rightarrow E$ is the projection) is fiber homotopy equivalent to a bundle with smooth compact manifolds as fibers.*

Proof. A suitable product formula implies that $\chi(pq)$ is vertically homotopic to $\chi(Y) \times \chi(p)$. We saw earlier that the component of $\chi(Y)$ in $\pi_0 \mathbf{A}(Y) \cong \mathbb{Z}$ is the Euler characteristic of Y . \square

Statements 5.2.1–5 can easily be generalized to the case where Z is a *finitely dominated* CW-space. But it is then necessary to use a variant $\mathbf{A}^p(Z)$ of $\mathbf{A}(Z)$ with a larger π_0 isomorphic to $K_0(\mathbb{Z}\pi_1(X))$. Then $\chi(Z) \in \Omega^\infty \mathbf{A}^p(Z)$ is defined,

and 5.2.1 and 5.2.3 remain correct as stated. In this more general formulation, 5.2.1 includes Wall's theory [Wa2] of the finiteness obstruction.

Cassoon and Gottlieb [CaGo] established 5.2.5 in the case $Y = (\mathbb{S}^1)^n$, with a large n depending on $p : E \rightarrow B$.

6. Examples and Calculations

6.1. Smooth automorphisms of disks. The smooth version of (1.2.2) gives a homotopy fiber sequence $\mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow \mathrm{DIFF}(\mathbb{R}^n) \rightarrow \mathcal{H}_d(\mathbb{S}^{n-1})$ where \mathcal{H}_d is a space of differentiable h -cobordisms. The composition of group homomorphisms $\mathrm{O}(n) \hookrightarrow \mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow \mathrm{DIFF}(\mathbb{R}^n)$ is a homotopy equivalence, so that

$$\mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \simeq \mathrm{O}(n) \times \Omega \mathcal{H}_d(\mathbb{S}^{n-1}).$$

By 1.3.4 and 3.2.2, there is a map from $\mathcal{H}_d(\mathbb{S}^{n-1})$ to $\Omega^{\infty+1} \mathbf{Whd}(\mathbb{S}^{n-1})$ which is an isomorphism on π_j for $j \leq (n-4)/3$. Further, $\mathbf{Whd}(\mathbb{S}^{n-1}) \rightarrow \mathbf{Whd}(\ast)$ is approximately $2n$ -connected [Wah2] and the rational homotopy groups of $\mathbf{Whd}(\ast)$ in dimensions > 0 are those of $\mathbf{K}(\mathbb{Q})$, which are known [Bo]. Therefore:

$$\pi_j \mathrm{DIFF}(\mathbb{D}^n, \mathbb{S}^{n-1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & \text{if } 0 < j \leq (n-7)/3 \text{ and } 4 \mid j+1 \\ 0 & \text{if } 0 < j < (n-7)/3 \text{ and } 4 \nmid j+1. \end{cases}$$

For a calculation of the rational homotopy groups of $\mathrm{DIFF}(\mathbb{D}^n)$ in the concordance stable range, following Farrell and Hsiang [FaHs], we note $\mathrm{DIFF}(\mathbb{D}^n) \simeq \Omega \mathcal{S}_d(\mathbb{D}^n)$ and use the smooth version of 1.5.2, which gives (at odd primes and in the concordance stable range)

$$\mathcal{S}_d(\mathbb{D}^n) \simeq \widetilde{\mathcal{S}}_d(\mathbb{D}^n) \times \widetilde{\mathrm{DIFF}}(\mathbb{D}^n) / \mathrm{DIFF}(\mathbb{D}^n) \simeq \widetilde{\mathcal{S}}_d(\mathbb{D}^n) \times \Omega^\infty (\mathbf{H}_d(\mathbb{D}^n)_{h\mathbb{Z}/2}).$$

Here $\widetilde{\mathcal{S}}_d(\mathbb{D}^n) \simeq \Omega^n (\mathrm{TOP} / \mathrm{O})$, which is rationally trivial, and $\mathbf{H}_d(\mathbb{D}^n) \simeq \Omega \mathbf{Whd}(\ast)$, so 3.2.2 and [Bo] give

$$\pi_j \mathbf{H}_d(\mathbb{D}^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } 4 \mid j \text{ and } j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The canonical involution on $\mathbf{H}_d(\mathbb{D}^n)$ acts trivially on these rationalized homotopy groups if n is odd, and nontrivially if n is even. Therefore, if $0 < j \leq (n-4)/3$, then [FaHs]

$$\pi_j \mathrm{DIFF}(\mathbb{D}^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & n \text{ odd and } 4 \mid j+1 \\ 0 & \text{otherwise.} \end{cases}$$

Beware that Farrell and Hsiang write $\mathrm{DIFF}(\mathbb{D}^n, \partial)$ for our $\mathrm{DIFF}(\mathbb{D}^n)$.

6.2. Smooth automorphisms of spherical space forms. Let M^n be smooth closed orientable, with universal cover $\simeq \mathbb{S}^n$, where $n \geq 5$. Hsiang and Jahren [HsiJ] calculate $\pi_* \text{DIFF}(M) \otimes \mathbb{Q}$ in the smooth concordance stable range, assuming that n is odd. They begin with the observation that $\pi_j G^s(M)$ is finite for all j . Therefore

$$\pi_i \text{DIFF}(M) \otimes \mathbb{Q} \cong \pi_{i+1} \mathcal{S}_d^s(M) \otimes \mathbb{Q}$$

for $i > 0$. By the smooth versions of 1.5.2 and 1.4.4 we have a splitting

$$\mathcal{S}_d^s(M) \simeq \tilde{\mathcal{S}}_d^s(M) \times \Omega^\infty(\mathbf{H}_d^s(M)_{h\mathbb{Z}/2})$$

at odd primes, in the concordance stable range. Therefore, rationally,

$$\pi_j \text{DIFF}(M) \cong \pi_{j+1} \mathcal{S}_d^s(M) \cong \pi_{j+n+2} \tilde{\mathbf{L}}_\bullet^h(M) \oplus \pi_{j+1} \mathbf{L}_\bullet^h(*) \oplus \pi_{j+1}^- \mathbf{H}_d^s(M)$$

where $\tilde{\mathbf{L}}_\bullet^h$ is the reduced L -theory and $\pi_{j+1}^- \mathbf{H}_d^s(M)$ is the fixed subgroup of $\mathbb{Z}/2$ acting on $\pi_{j+1} \mathbf{H}_d^s(M)$. This is the Hsiang–Jahren result. Rationally, the multesignature homomorphisms on $\pi_* \mathbf{L}_\bullet^h(M)$ are isomorphisms [Wa3]. Rationally, $\pi_*(\mathbf{H}_d^s(M)) \cong \pi_{*+1} \mathbf{K}(\mathbb{Q}\pi_1(M))$ for $0 < * < n-1$. The calculation of $\pi_* \mathbf{K}(\mathbb{Q}\pi) \otimes \mathbb{Q}$ for a finite group π can often be accomplished with [Bo], certainly in the case where $\pi_1(M)$ is commutative.

6.3. Automorphisms of negatively curved manifolds. Let M^n be smooth, closed, connected, with a Riemannian metric of sectional curvature < 0 . In the course of their proof of the Borel conjecture for such M , Farrell and Jones [FaJo1], [FaJo5] show that $\alpha : (\mathbf{L}_\bullet)^\%(M) \rightarrow \mathbf{L}_\bullet(M)$ is a homotopy equivalence (with a 4-periodic definition of $\mathbf{L}_\bullet(M)$, decoration s or h). With our 0-connected definition of $\mathbf{L}_\bullet(M)$, it is still true that

$$\begin{aligned} \Omega^{\infty+n}((\mathbf{L}_\bullet)^\%(M)) &\simeq * \\ \Omega^{\infty+n}((\mathbf{L}\mathbf{A}_\bullet^h)^\%(M, n)) &\simeq \Omega^{\infty+n}(\mathbf{A}_\%(M, n)_{h\mathbb{Z}/2}). \end{aligned}$$

For a simple closed geodesic T in M , let T^\sharp be the “desingularization” of T , so that $T^\sharp \cong \mathbb{S}^1$. Farrell and Jones also show that the map

$$\bigvee_T \mathbf{A}_\%(T^\sharp) \rightarrow \mathbf{A}_\%(M)$$

induced by $T^\sharp \rightarrow T \hookrightarrow M$ for all simple closed geodesics T in M is a homotopy equivalence [FaJo3]; see also [FaJo4], [FaJo2] for extensions. Now there is a *fundamental theorem* in the algebraic K -theory of spaces: The assembly from $\mathbb{S}^1 \wedge \mathbf{A}(*) \simeq \mathbf{A}^\%(S^1)$ to $\mathbf{A}(S^1)$ is a split monomorphism in the homotopy category [KVWW1], [KVWW2] and its mapping cone splits up to homotopy into two copies of a spectrum $\mathbf{Nil}_\mathbf{A}(*)$. Under any of the involutions constructed by the method of §4.1, these two copies are interchanged. Therefore $\Omega^{\infty+n}((\mathbf{L}\mathbf{A}_\bullet)^\%(M, n)) \simeq \Omega^{\infty+1}(\bigvee_T \mathbf{Nil}_\mathbf{A}(*))$ and we get from 4.2.1 a map

$$\mathcal{S}(M) \rightarrow \Omega^{\infty+1}(\bigvee_T \mathbf{Nil}_\mathbf{A}(*))$$

which is approximately $(n/3)$ -connected (see 4.2.2). It is known that $\mathbf{Nil}_{\mathbf{A}}(*)$ is rationally trivial and 1-connected, but $\pi_2 \mathbf{Nil}_{\mathbf{A}}(*) \neq 0$ [HaWa],[Wah1]. From [Dun],[BHM], one has a homological algebra description of $\mathbf{Nil}_{\mathbf{A}}(*)$, as explained in [Ma1, 4.5] and [Ma2, §5]. But the homological algebra is over the ring spectrum \mathbf{S}^0 and it is not considered easy.— From the fiber sequence $\mathrm{TOP}(M) \rightarrow \mathbf{G}(M) \rightarrow \mathcal{S}(M)$ we get $\pi_j \mathrm{TOP}(M) \cong \bigoplus_T \pi_{j+2} \mathbf{Nil}_{\mathbf{A}}(*)$ if $1 < j \leq (n-7)/3$, and an exact sequence

$$\begin{array}{ccc} \bigoplus_T \pi_3 \mathbf{Nil}_{\mathbf{A}}(*) & \twoheadrightarrow & \pi_1 \mathrm{TOP}(M) \rightarrow \mathrm{center}(\pi_1(M)) \\ & & \downarrow \\ & & \bigoplus_T \pi_2 \mathbf{Nil}_{\mathbf{A}}(*) \rightarrow \pi_0 \mathrm{TOP}(M) \twoheadrightarrow \mathrm{Out}(\pi_1(M)). \end{array}$$

6.4. The h -structure space of \mathbb{S}^n , for $n \geq 5$. By 4.2.1 and 4.2.2 there is a commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{S}^n) & \longrightarrow & \Omega^{\infty+n}((\mathbf{LA}_{\bullet})_{\%}(\mathbb{S}^n, n)) \\ \downarrow \cap & & \downarrow \\ \tilde{\mathcal{S}}(\mathbb{S}^n) & \xrightarrow{\cong} & \Omega^{\infty+n}((\mathbf{L}_{\bullet}^s)_{\%}(\mathbb{S}^n)), \end{array}$$

where the top horizontal arrow is highly connected. Here $\Omega^n(\mathbf{L}_{\bullet}^s)_{\%}(\mathbb{S}^n)$ simplifies to $\mathbf{L}_{\bullet}(*),$ and $\Omega^n(\mathbf{LA}_{\bullet})_{\%}(\mathbb{S}^n, n)$ has an analogous simplifying map (not a homotopy equivalence, but highly connected) to $\mathbf{LA}_{\bullet}(*, n).$ Summarizing: there is a homotopy commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{S}^n) & \longrightarrow & \Omega^{\infty} \mathbf{LA}_{\bullet}(*, n) \\ \downarrow \cap & & \downarrow \text{forget} \\ \tilde{\mathcal{S}}(\mathbb{S}^n) & \xrightarrow{\cong} & \Omega^{\infty} \mathbf{L}_{\bullet}(*). \end{array}$$

Recall the homotopy fiber sequence $\mathbf{LA}_{\bullet}(*, n) \rightarrow \mathbf{L}_{\bullet}(*) \xrightarrow{\beta} \mathbb{S}^1 \wedge (\mathbf{A}(*, n))_{h\mathbb{Z}/2}$ from (4.1.2). Calculations [WWp] using connective K -theory \mathbf{bo} as a substitute for $\mathbf{A}(*),$ via $\mathbf{A}(*) \rightarrow \mathbf{K}(\mathbb{Z}) \rightarrow \mathbf{bo},$ show that β detects all elements in $\pi_{n+q} \mathbf{L}_{\bullet}(*)$ whose signature is not divisible by $2a_q$ if 4 divides $n+q,$ and by $4a_q$ if 4 divides both $n+q$ and $q.$ Here $a_q = 1, 2, 4, 4, 8, 8, 8, 8$ for $q = 1, 2, \dots, 8$ and $a_{q+8} = 16a_q;$ the numbers a_q are important in the theory of Clifford modules [ABS]. Consequently, if 4 divides $n+q,$ then the image of the inclusion-induced homomorphism

$$\pi_{n+q} \mathcal{S}(\mathbb{S}^n) \longrightarrow \pi_{n+q} \tilde{\mathcal{S}}(\mathbb{S}^n) \cong 8\mathbb{Z}$$

is contained in $2a_q\mathbb{Z}$ if 4 does not divide $q,$ and in $4a_q\mathbb{Z}$ if 4 does divide $q.$ Note the similarity of this statement with [At, 3.3], [LM, IV.2.7], [Tho, Thm.14].

6.5. Obstructions to unblocking smooth block automorphisms. One of the main points of §4.2 and the introduction to §4 is a homotopy commutative diagram, for compact M^n ,

$$(6.5.1) \quad \begin{array}{ccc} \Omega^{\infty+n+1}(\mathbf{L}_\bullet^s)_\% (M) & \longrightarrow & \Omega^{\infty+n}(\mathbf{A}_\%^s(M, n)_{h\mathbb{Z}/2}) \\ \downarrow \simeq & & \uparrow \\ \Omega\widetilde{\mathcal{S}}^s(M) & \longrightarrow & \widetilde{\text{TOP}}(M)/\text{TOP}(M) \end{array}$$

in which the upper row is the connecting map from the first of the two homotopy fiber sequences in (4.1.2),

$$(6.5.2) \quad \Omega\mathbf{L}_\bullet^s(M) \rightarrow \mathbf{A}^s(M, n)_{h\mathbb{Z}/2} ,$$

with $\%$ and decoration s and $\Omega^{\infty+n}$ inflicted. Modulo the identification $\mathbf{A}_\%^s(M) \simeq \mathbf{H}^s(M)$ (the s -decorated version of 3.2.2), the right-hand column of (6.5.1) is the highly connected map which we found at the end of §1.4 using purely geometric methods. The lower row of (6.5.1) is the connecting map from the homotopy fiber sequence

$$\widetilde{\text{TOP}}(M)/\text{TOP}(M) \rightarrow \mathcal{S}^s(M) \rightarrow \widetilde{\mathcal{S}}^s(M) .$$

One of the main points of §4.3 is that much of (6.5.1) has a smooth analog, in the shape of a homotopy commutative diagram

$$(6.5.3) \quad \begin{array}{ccc} \Omega^{\infty+n+2}\mathbf{L}_\bullet^s(M) & \longrightarrow & \Omega^{\infty+n+1}(\mathbf{Whd}^s(M, n)_{h\mathbb{Z}/2}) \\ \downarrow & & \uparrow \\ \Omega\widetilde{\mathcal{S}}_d^s(M) & \longrightarrow & \widetilde{\text{DIFF}}(M)/\text{DIFF}(M) \end{array}$$

defined for smooth compact M . Here $\mathbf{Whd}^s(M)$ is the mapping cone of the map $\Sigma^\infty(M_+) \rightarrow \mathbf{A}(M)$ discussed in 4.3.3, and $\mathbf{Whd}^s(M, n) := \mathbb{S}_1^n \wedge \mathbf{Whd}^s(M)$. The upper row in (6.5.3) is $\Omega^{\infty+n+1}$ of (6.5.2) composed with the projection $\mathbf{A}^s(M, n)_{h\mathbb{Z}/2} \rightarrow \mathbf{Whd}^s(M, n)_{h\mathbb{Z}/2}$. Again, modulo an identification of $\mathbf{Whd}^s(M)$ with $\mathbf{H}_d^s(M)$, coming from (3.2.3), the right-hand column of (6.5.3) is a purely geometric construction going back to (the smooth version of) §1.4. It is highly connected. The left-hand column of (6.5.3) is not a homotopy equivalence, which makes the analogy a little imperfect. We arrive at (6.5.3) by first making full use of 4.2.1 and 4.2.4 to produce a *framed* version of (6.5.1), with upper left-hand vertex $\Omega^{\infty+n+2}\mathbf{L}_\bullet^s(M)$, upper right-hand vertex $\Omega^{\infty+n+1}(\mathbf{A}(M, n)_{h\mathbb{Z}/2})$, and lower left-hand vertex equal to Ω of the homotopy fiber of

$$\nabla : \widetilde{\mathcal{S}}^s \rightarrow \widetilde{\mathcal{S}}(\tau) .$$

Now assume that the classifying map $M \rightarrow BO$ for the stable tangent bundle factors up to homotopy through an aspherical space. (For example, this is the case if M is stably framed.) Stabilization arguments as in 4.3.5 show that then

$\mathbf{Whd}^s(M, n)_{h\mathbb{Z}/2}$ splits off $\mathbf{A}^s(M, n)_{h\mathbb{Z}/2}$, up to homotopy. Also, the connecting map (6.5.2) factors through the summand $\mathbf{Whd}^s(M, n)_{h\mathbb{Z}/2}$, up to homotopy. It follows that elements in $L_k^s(\mathbb{Z}\pi)$ detected under (6.5.2) are also detected under

$$(6.5.4) \quad L_k^s(\mathbb{Z}\pi) \longrightarrow \pi_{k-2}(\widetilde{\text{DIFF}}(M)/\text{DIFF}(M)),$$

the homomorphism coming from (6.5.3). To exhibit such elements in $L_k^s(\mathbb{Z}\pi)$, we suppose in addition that M is orientable and $\pi = \pi_1(M)$ is finite. Then we have the multisignature homomorphisms

$$(6.5.5) \quad L_k^s(\mathbb{Z}\pi) \rightarrow L_k^p(\mathbb{R}\pi) \cong \bigoplus_V L_k^p(E_V)$$

where the direct sum extends over a maximal set of pairwise non-isomorphic irreducible real representations V of π , and E_V is the endomorphism ring of V , isomorphic to \mathbb{R} , \mathbb{C} or \mathbb{H} equipped with a standard conjugation involution [Le], [Wa3]. It is known that $L_k^p(E_V) \cong \mathbb{Z}$ if $4|k$, and also $L_k^p(E_V) \cong \mathbb{Z}$ if $2|k$ and $E_V \cong \mathbb{C}$; otherwise $L_k^p(E_V) = 0$. A calculation similar to that mentioned in 6.4 shows that an element in $L_k^s(\mathbb{Z}\pi) = \pi_k(\mathbf{L}_\bullet^s(M))$ will map nontrivially under the homomorphism associated with (6.5.2) if the V -component of its multisignature is not divisible by

$$\begin{aligned} 2a_{k-n} & \quad (\text{assuming } 4|k \text{ and } E_V \cong \mathbb{R}) \\ 2a_{k-n}^c & \quad (\text{assuming } 2|k \text{ and } E_V \cong \mathbb{C}) \\ a_{k-n+4}/4 & \quad (\text{assuming } 4|k \text{ and } E_V \cong \mathbb{H}) \end{aligned}$$

where $a_q^c = 1$ if $q = 1, 2$ and $a_{q+2}^c = 2a_q$ for $q > 2$. Now the multisignature homomorphisms (6.5.5) are of course periodic in k with period 4, and are rational isomorphisms [Wa1]. Therefore many elements in $L_*^s(\mathbb{Z}\pi)$ are indeed detected by (6.5.2) and therefore also by (6.5.4).

6.6. Gromoll filtration. The Gromoll filtration of an element $x \in \pi_i(\text{TOP}/\text{O})$ is the largest number $j = j(x)$ such that x comes from $\pi_i(\text{TOP}(i-j)/\text{O}(i-j))$. By smoothing theory and the Alexander trick [Br5]

$$\begin{aligned} \widetilde{\text{DIFF}}(\mathbb{D}^{i-j}) & \simeq \Omega^{i-j+1}(\text{TOP}/\text{O}) \\ \text{DIFF}(\mathbb{D}^{i-j}) & \simeq \Omega^{i-j+1}(\text{TOP}(i-j)/\text{O}(i-j)). \end{aligned}$$

Therefore: $j = j(x)$ is the largest number such that x lifts from $\pi_{j-1}\widetilde{\text{DIFF}}(\mathbb{D}^{i-j})$ to $\pi_{j-1}\text{DIFF}(\mathbb{D}^{i-j})$. This is the original definition of [Grom]; see also [Hit].

To obtain upper bounds on $j(x)$ in special cases, we use 6.5, with $k = i + 1$ and $n = i - j$ and $M = \mathbb{D}^n$. If $4|k$, then the (multi)signature homomorphism $L_k(1) \rightarrow \mathbb{Z}$ has image $8\mathbb{Z}$ and the estimates in 6.5 can be improved slightly as follows. An element in $L_k(\mathbb{Z})$ will be detected by (6.5.4), case $M = \mathbb{D}^n$, if its signature is not divisible by $2a_{k-n}$ ($4a_{k-n}$ in the case where $4|n$). Therefore: if 4 divides $i + 1$ and $x \in \pi_i(\text{TOP}/\text{O})$ has Gromoll filtration $\geq j$, and is the image of $\bar{x} \in \pi_{i+1}(\text{G}/\text{TOP}) \cong L_{i+1}(\mathbb{Z})$, then the signature of \bar{x} is divisible by $2a_{j+1}$ ($4a_{j+1}$ in the case where 4 divides $j + 1$).

6.7. Riemann–Roch for smooth fiber bundles. Let $p : E \rightarrow B$ be a fiber bundle with fibers $\cong M$ and structure group $\text{DIFF}(M, \partial M)$. Let R be a ring, and let V be a bundle (with discrete structure group) of f.g. left proj. R -modules on E . This determines $[V] : E \rightarrow \Omega^\infty \mathbf{K}(R)$. Let V_i be the bundle on B with fiber $H_i(p^{-1}(b); V)$ over b . We assume that the fibers of V_i are projective; then each V_i determines $[V_i] : B \rightarrow K(R)$. Then the following Riemann–Roch formula holds:

$$(6.7.1) \quad \text{tr}^*[V] = \sum (-1)^i [V_i] \in [B, \Omega^\infty \mathbf{K}(R)]$$

where $\text{tr} : \Sigma^\infty B_+ \rightarrow \Sigma^\infty E_+$ is the Becker–Gottlieb–Dold transfer [BeGo], [Do], [DoP], a stable map determined by p . Both sides of (6.7.1) have meaning when $p : E \rightarrow B$ is a fibration whose fibers are homotopy equivalent to compact CW-spaces. However, (6.7.1) does not hold in this generality. It can fail for a fiber bundle with compact (and even closed) *topological* manifolds as fibers.

Formula (6.7.1) is a distant corollary of 5.2.4. Namely, both (6.7.1) and 5.2.4 are ways of saying that certain generalized Euler characteristics of a smooth compact M lift canonically to $\Omega^\infty \Sigma^\infty(M_+)$. For the proof of (6.7.1), see [DWW]. Earlier, Bismut and Lott [BiLo] had proved by analytic methods that (6.7.1) holds in the case $R = \mathbb{C}$ after certain characteristic classes are applied to both sides of the equation.

6.8. Obstructions to finding block bundle structures. Let $p : E \rightarrow B$ be a fibration with connected base whose fibers are oriented Poincaré duality spaces of formal dimension $2k$. Let $E^\sim \rightarrow E$ be a normal covering with translation group π . With these data we can associate a map

$$(6.8.1) \quad B \longrightarrow \Omega^\infty \mathbf{bhm}_\pi(k)$$

where $\mathbf{bhm}_\pi(k)$ is the (topological, connective) K -theory of f.g. projective $\mathbb{R}\pi$ -modules with nondegenerate $(-1)^k$ -hermitian form [Wa1], [Wa4]. The map (6.8.1) stably classifies the hermitian bundle on B with fiber $H^k(E_x^\sim; \mathbb{R})$ over $x \in B$. There is a *hyperbolic map* [Wa4] from \mathbf{bo}_π , the (topological, connective) K -theory of f.g. projective $\mathbb{R}\pi$ -modules, to $\mathbf{bhm}_\pi(k)$. Let $\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi$ be its mapping cone. The map

$$(6.8.2) \quad B \longrightarrow \Omega^\infty(\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi)$$

obtained by composing (6.8.1) with Ω^∞ of $\mathbf{bhm}_\pi(k) \rightarrow \mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi$ is the *family multisignature* of p . Now suppose that p admits a block bundle structure; see §5.1. Then (6.8.2) factors *rationally* as

$$B \longrightarrow \Omega^\infty(\mathbf{bhm}_e(k)/\mathbf{bo}_e) \xrightarrow{\otimes \mathbb{R}\pi} \Omega^\infty(\mathbf{bhm}_\pi(k)/\mathbf{bo}_\pi)$$

where e is the trivial group. The case $B = *$ appears in [Wa1, §13B]. The general statement can be proved by expressing the rationalized family multisignature in terms of the family visible symmetric signature of §5.1.

6.9. Relative calculations of h -structure Spaces. Using §1.6 and 4.2.1 one obtains, in the case where M is smooth, a diagram

$$(6.9.1) \quad \mathcal{S}_d(M^n) \xrightarrow{\nabla} \mathcal{S}_d(\tau) \xrightarrow{\lambda} \Omega^{\infty+n} \mathbf{LA}_\bullet^h(M, n)$$

which is a homotopy fiber sequence in the concordance stable range; more precisely, the composite map in (6.9.1) is trivial and the resulting map from $\mathcal{S}_d(M)$ to the homotopy fiber of λ is $(n-1)/3$ -connected. This formulation has some relative variants which are attractive because relative LA -theory is often easier to describe than absolute LA -theory, and the estimates available for the relative concordance stable range are often better than those for the absolute concordance stable range.

Illustration: Let M be smooth, compact, connected, with connected boundary, and suppose the inclusion $\partial M \rightarrow M$ induces an isomorphism of fundamental groups. Let $M_0 = M \setminus \partial M$. Define $\mathcal{S}_d(M_0)$ as the moduli space of pairs (N, f) where $f : N \rightarrow M_0$ is a proper homotopy equivalence of smooth manifolds without boundary. Let $\mathcal{S}_d(\tau_0)$ be the corresponding moduli space of n -dimensional vector bundles on M_0 equipped with a stable fiber homotopy equivalence to $\tau_0 := \tau|_{M_0}$. Using some controlled L -theory and algebraic K -theory of spaces, one obtains the following variation on (6.9.1): a diagram

$$(6.9.2) \quad \mathcal{S}_d(M_0) \xrightarrow{\nabla} \mathcal{S}_d(\tau_0) \longrightarrow \Omega^{\infty+n} (\mathbf{LA}_\bullet^h(M, n) / \mathbf{LA}_\bullet^h(\partial M, n))$$

which is a homotopy fiber sequence in the $\leq (n-1)/3$ range. Our hypothesis on fundamental groups implies $\mathbf{LA}_\bullet^h(M, n) / \mathbf{LA}_\bullet^h(\partial M, n) \simeq (\mathbb{S}^n \wedge (\mathbf{A}(M) / \mathbf{A}(\partial M)))_{h\mathbb{Z}/2}$ and [BHM], [Go3] imply

$$\mathbf{A}(M) / \mathbf{A}(\partial M) \simeq \mathbf{TC}(M) / \mathbf{TC}(\partial M)$$

where \mathbf{TC} is the *topological cyclic homology* viewed as a spectrum-valued functor. See also [Ma1], [Ma2], [HeMa]. So (6.9.2) simplifies to

$$\mathcal{S}_d(M_0) \xrightarrow{\nabla} \mathcal{S}_d(\tau_0) \longrightarrow \Omega^{\infty+n} ((\mathbb{S}^n \wedge (\mathbf{TC}(M) / \mathbf{TC}(\partial M)))_{h\mathbb{Z}/2}).$$

This is still a homotopy fiber sequence in the $\leq (n-1)/3$ range. Unpublished results of T.Goodwillie and G.Meng indicate that it is a fibration sequence in the $\leq n-c$ range for some constant c , maybe $c=7$, provided $\partial M \rightarrow M$ is 2-connected.

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