Instead of an Introduction

Our goal is to prove a result stated in the introduction to , under the name "pretheorem F". This is a theorem about L–theory and K–theory, about surgery theory and concordance theory, about automorphisms of a manifold and automorphisms of its tangent bundle.

We shall write Part I, Part II, and Part III to mean , , . Notation and conventions will be the same as in previous instalments, unless otherwise specified.

For our own sake as much as for the readers sake, we have decided to separate the philosophical points in the proof from the general nonsense and the merely well–educated mathematics. The philosophical points can be found in sections 1–4 (about Euler characteristics) and 5 (about the homology of discrete homeomorphism groups). Most of the results stated in these sections are either known, or plausible, or easy to prove; in any case most of the proofs are deferred to later sections (to the "well–educated" department, that is).

Assuming these results, and adding a fair amount of general nonsense, we then prove pretheorem F. This is done in sections . The remaining sections consist of proofs of results stated in sections 1–5.

About the philosophical points. Firstly, we observe and insist that A–theory is a functor "with Euler characteristics". That is, it is a functor

$$Y \longmapsto A(Y)$$

from spaces Y to infinite loop spaces; but it also comes equipped with a rule which to every finitely dominated Y associates a point

$$\langle Y\rangle \in A(Y).$$

This is the point determined by the retractive space $S^0 \times Y$ over Y. (It is understood that $A(Y)$ is the K–theory of finitely dominated retractive spaces over Y.) If Y is connected, then $\pi_0(A(Y)) \cong \mathbb{Z} \oplus \widetilde{K}_0(\mathbb{Z}\pi_1(Y))$, and the component of $\langle Y\rangle$ is given by the ordinary Euler characteristic in $\mathbb{Z}$ and the finiteness obstruction of Y in $\widetilde{K}_0(\mathbb{Z}\pi_1(Y))$. We shall not try to state axioms for $< \cdots >$, although this might be interesting; we merely think of it as a standard procedure for making algebra (algebraic K–theory) out of geometry (spaces). It has many variants.

This brings us to the second point. Spaces often come in continuous families, such as the fibers of a fiber bundle. In cases like this it is often easy to associate to each member of the family an appropriate Euler characteristic, but quite hard to exploit or even codify the continuity in the continuous family. This situation arises for example in Waldhausen's work on h–cobordism spaces and A–theory (see ,) and Waldhausen's answer was to add "more continuity" on the algebraic side to create a balance. However, Fiedorowicz and Steinberger have pointed out that one can also create a balance by leaving the algebraic side unchanged, but making the geometric side more discrete. Specifically, a result due to McDuff (see also Segal and Thurston) states that, for a closed manifold M , the inclusion map

$$B\delta TOP(M) \longrightarrow BTOP(M)$$

is a homology equivalence. (Here $TOP(M)$ is the homeomorphism group with the usual topology, and $\delta TOP(M)$ is the same group with the discrete topology.) Similarly,

$$B\delta\mathcal{C}(M) \longrightarrow B\mathcal{C}(M)$$

is a homology equivalence (where $\mathcal{C}(M)$ is the space of topological concordances). Obviously such a result can be very useful in making the passage from concordance theory to A–theory. Traditionally, results of this type are used in classifying foliations; for an account of this, see Lawson.

Last not least, we make heavy use of methods from controlled algebraic K–theory and controlled A–theory. We are of course forced to do so because ideas from the controlled theory are already quite prominent in Part I. But is is perhaps even more important that assembly maps in algebraic K–theory can often be described as "forget control maps", and thereby acquire a geometric meaning. The power of this idea is illustrated in a recent paper by Ranicki and Yamasaki. Many names are attached to it: Quinn seems to be the originator, Pedersen–Weibel and Anderson–Munkholm did much to clarify and formalize it, and Vogell, created the A–theory version which we use here. Actually, we also use some new ideas from Anderson–Connolly–Ferry–Pedersen.

1. A–Theory Euler Characteristics

1.1 The prototype. Let Y be a finitely dominated CW–space. With the traditional definition of $A(Y) = Ap(Y)$ as the K–theory of the category of finitely dominated retractive CW–spaces over Y, any finitely dominated retractive CW–space determines a point in $A(Y)$. Specifically, the retractive space

$$\{-1,+1\} \times Y \underset{s}{\overset{r}{\rightleftarrows}} Y$$

where r is the projection and s identifies Y with $\{1\} \times Y$, determines a point

$$\langle Y \rangle \in A(Y).$$

We call it the A*–theory Euler characteristic* of Y. It is perhaps more customary to use this expression for the connected component of $\langle Y \rangle$, but our interest is in the point $\langle Y \rangle$.

A homotopy equivalence

$$f : X \longrightarrow Y$$

between finitely dominated CW–spaces determines another homotopy equivalence

$$f_* : A(X) \longrightarrow A(Y).$$

It also determines a path $\langle f \rangle$ in $A(Y)$ from $f_*\langle X \rangle$ to $\langle Y \rangle$. Namely, $f_*\langle X \rangle$ is the point in $A(Y)$ corresponding to the retractive space

$$\{-1\} \times X \cup \{1\} \times Y \leftrightarrows Y$$

where the retraction is equal to the identity on $\{1\} \times Y$ and equal to f on $\{-1\} \times X$. Now f gives an h–equivalence from this retractive space over Y to

$$\{-1,+1\} \times Y \longrightarrow Y.$$

The h–equivalence determines a path $\langle f \rangle$ in $A(Y)$, by definition of $A(Y)$.

We could continue in this manner, looking e.g. at composable sequences of homotopy equivalences. Perhaps the best way to express the naturality properties of A–theory Euler characteristics is the following. Let $\mathcal{C}$ be a small category whose objects are finitely dominated CW–spaces and whose morphisms are homotopy equivalences (composition of morphisms being composition of maps). Form the homotopy inverse limit

$$A(\mathcal{C}) := \operatorname*{holim}_{Y} A(Y),$$

where Y runs over the objects of $\mathcal{C}$. Now A–theory Euler characteristics define a point

$$\langle \mathcal{C} \rangle \in A(\mathcal{C}).$$

(Use the very explicit description of homotopy inverse limits on p. of Bousfield–Kan.)

We shall refer to this type of naturality as *lax naturality*. We would say, then, that the A–theory Euler characteristic $\langle Y \rangle$ is lax natural with respect to homotopy equivalences.

1.2. The ENR–version. If Y is a compact ENR, with no specific CW–structure, then one can define $\langle Y \rangle$ as a point in the K–theory of retractive compact ENR's over Y. However, it is then also appropriate to write $Ah(Y)$ for this K–theory, because any compact ENR is homotopy equivalent to a finite CW–space. See Part III, 3.4 for details. To get from this model for $Ah(Y)$ to $Ap(Y) = A(Y)$, one can use the following construction. Let $\mathcal{C}_1$ be the category of retractive compact ENR's over Y, and let $\mathcal{C}_2$ be the category of finitely dominated retractive CW–spaces over Y (where the CW–structure is relative to Y). Then the (homotopy) pushout of

$$K(\mathcal{C}_1) \hookleftarrow K(\mathcal{C}_1 \cap \mathcal{C}_2) \hookrightarrow K(\mathcal{C}_2)$$

is a model for $Ap(Y)$ which contains $K(\mathcal{C}_1) = Ah(Y)$. Consequently, it is permitted to write

$$\langle Y \rangle \in A(Y)$$

also if Y is a compact ENR, with no specific CW–structure.

1.3. Euler characteristics and Poincaré duality. Suppose that Y is a finitely dominated CW–space and a Poincaré space of formal dimension n, with Spivak normal fibration γ. Together, γ and n determine an involution on the spectrum $\underline{\underline{A}}p(Y)$, and therefore on the space $A(Y) = Ap(Y) = Q(\underline{\underline{A}}p(Y))$. We ought to write

$$QH^{\blacktriangledown}(Z_2; \underline{\underline{A}}p(Y, \gamma, -n))$$

for the space of homotopy fixed point of this involution on $A(Y)$, but let us simply write

$$A(Y)^{SW}$$

in honour of Spanier and Whitehead. It is not very hard to lift

$$\langle Y \rangle \in A(Y)$$

to a point

$$\langle Y \rangle^{SW} \in A(Y)^{SW}$$

given section 3 of Part III. (In other words, the Poincaré duality of Y implies a Spanier–Whitehead self duality. This is essentially the symmetric construction of Mishchenko and Ranicki; see Part III, 3.6 and sequel.) Here $\langle Y\rangle^{SW}$ is lax natural with respect to homotopy equivalences between Poincaré spaces. Again, it is not absolutely necessary to insist on CW–structures; $\langle Y\rangle^{SW}$ is also defined for compact ENR's satisfying Poincaré duality.

More generally, a Poincaré pair $(Y,\partial Y)$ determines an Euler characteristic

$$\langle Y,\partial Y\rangle^{SW}\in A(Y,\partial Y)^{SW}$$

where $A(Y,\partial Y)$ is the K–theory of pairs of (finitely dominated) retractive spaces:

$$\begin{array}{ccc} X_1 & \subset & X_2 \\ s_1\uparrow\downarrow r_1 & & s_2\uparrow\downarrow r_2 \\ \partial Y & \subset & Y. \end{array}$$

(The inclusion $X_1 \amalg_{\partial Y} Y \hookrightarrow X_2$ is assumed to be a cofibration). In the notation of Part III, section 4, we have

$$A(Y,\partial Y)=Q(\underline{\underline{A}}p(Y,\partial Y,\gamma,-n))$$

where γ is the Spivak normal fibration of $(Y,\partial Y)$ and n is the formal dimension. This is heavy notation, but it specifies the involution, too (which depends on γ and n , while the underlying space or spectrum does not).

Something new and moderately interesting happens in the relative case. Namely, the spectrum with involution

$$\underline{\underline{A}}p(Y,\partial Y,\gamma,-n)$$

contains a subspectrum with involution

$$\underline{\underline{A}}p(\partial Y,\partial Y,\gamma,-n).$$

So the quotient spectrum

$$\underline{\underline{A}}p(Y,\partial Y,\gamma,-n)/\underline{\underline{A}}p(\partial Y,\partial Y,\gamma,-n)$$

comes with a compatible involution; but according to proposition 4.1 of Part III, this quotient spectrum is homotopy equivalent to

$$(\underline{\underline{A}}p(Y)\vee\underline{\underline{A}}p(\partial Y))/(\underline{\underline{A}}p(\partial Y)\vee\underline{\underline{A}}p(\partial Y))\simeq\underline{\underline{A}}p(Y)/\underline{\underline{A}}p(\partial Y).$$

We can therefore write

$$A(Y)/A(\partial Y)$$

for its infinite loop space, and

$$(A(Y)/A(\partial Y))^{SW}$$

for the space of homotopy fixed points of the involution. Summarizing, there is an interesting map

$$A(Y,\partial Y)^{SW} \longrightarrow (A(Y)/A(\partial Y))^{SW}.$$

1.4. Convention. The spectra with Z_2–action that we consider here lose none of their desirable properties if we make the actions semi–free by taking smash product with $(EZ_2)_+$. For example, the desirable property of the Z_2–spectrum $\underline{\underline{A}}p(Y,\gamma,-n)$ in 1.3 is mainly that it comes with a forgetful map to $\underline{\underline{A}}p(Y)$ which is a homotopy equivalence.

Therefore, where Z_2–actions on spectra occur, we shall assume that they are semi–free.

1.5. Products. For CW–spaces X and Y , we have the external products in A–theory,

$$\underline{\underline{A}}p(X) \wedge \underline{\underline{A}}p(Y) \longrightarrow \underline{\underline{A}}p(X \times Y)$$

(see the digression following Prop. 1.5.3 in Waldhausen). We can also equip the three spectra involved with involutions, as in Part III, section 4 and Part II, section 5, by choosing spherical fibrations β on X , and γ on Y , and assigning formal dimensions to them: $k \in \mathbb{Z}$ (for β), and $m \in \mathbb{Z}$ (for γ). (See remark 4.7 in Part III.) Then $X \times Y$ carries $\beta \times \gamma$, of formal dimension $k + m$. The external product map above then becomes a Z_2–map of spectra.

Suppose further that Y is a Poincaré space of formal dimension n , that $n = -m$ and that γ is the Spivak normal fibration of Y . Then the Euler characteristic $\langle Y \rangle^{SW}$ is a Z_2-map

$$(EZ_2)_+ \longrightarrow \underline{\underline{A}}p(Y).$$

Plugging this into the external product, we have a Z_2–map

$$\underline{\underline{A}}p(X) \wedge (EZ_2)_+ \longrightarrow \underline{\underline{A}}p(X \times Y).$$

By convention 1.4, we may drop the EZ_2 and get a Z_2–map

$$\underline{\underline{A}}p(X) \longrightarrow \underline{\underline{A}}p(X \times Y)$$

(*external product with* $\langle Y\rangle^{SW}$). Of course, the spherical fibration γ on X and the integer k are still there, but notationally suppressed.

More generally, for a Poincaré pair $(Y,\partial Y)$ we would get a Z_2–map

$$\underline{\underline{A}}p(X) \longrightarrow \underline{\underline{A}}p(X\times Y, X\times\partial Y)$$

(external product with $\langle Y,\partial Y\rangle^{SW}$.) The assumptions on X are as before. Finally, we can compose this with the Z_2–map

$$\underline{\underline{A}}p(X\times Y, X\times\partial Y) \longrightarrow \underline{\underline{A}}p(X\times Y)/\underline{\underline{A}}p(X\times\partial Y)$$

described at the end of 1.3. The result is a Z_2–map

$$\underline{\underline{A}}p(X) \longrightarrow \underline{\underline{A}}p(X\times Y)/\underline{\underline{A}}p(X\times\partial Y)$$

(still *external product with* $\langle Y,\partial Y\rangle^{SW}$).

In the special case where $(Y,\partial Y) = (D^1,S^0)$, the target of this map becomes $S^1\wedge\underline{\underline{A}}p(X)$, and we call it SW*–suspension*:

$$\underline{\underline{A}}p(X) \longrightarrow S^1\wedge\underline{\underline{A}}p(X).$$

It is still a Z_2–map (and the involutions still depend on γ and k).

1.6. Analysis of the SW–suspension. Let $\underline{\underline{F}}$ be the homotopy fiber of the SW–suspension

$$\underline{\underline{A}}p(X) \longrightarrow S^1\wedge\underline{\underline{A}}p(X).$$

Denote by τ the canonical involution on $\underline{\underline{F}}$ (of course, this depends on γ and k , too).

Claim: The forgetful map

$$z : \underline{\underline{F}} \longrightarrow \underline{\underline{A}}p(X)$$

has a homotopy right inverse $u : \underline{\underline{A}}p(X) \longrightarrow \underline{\underline{F}}$ (zu $\simeq$ id) such that

$$u\vee\tau u : \underline{\underline{A}}p(X)\vee\underline{\underline{A}}p(X) \longrightarrow \underline{\underline{F}}$$

is a homotopy equivalence.

Proof: See

In 1.5 and 1.6, the CW–structures are unnecessary if the spaces involved are ENR's.

2. Microcharacteristics

2.1. Notation. For a space Y, preferrably an ENR, we write

$$\underset{=}{A}^{\%}(Y) \quad \text{instead of} \quad Y_+ \wedge \underset{=}{A}p(*)$$

(as in Part III, section 2 and Introduction). We also write

$$A^{\%}(Y) \quad \text{to mean} \quad Q(\underset{=}{A}^{\%}(Y)),$$

$$A^{\%}(Y)/A^{\%}(X) \quad \text{to mean} \quad Q(\underset{=}{A}^{\%}(Y)/\underset{=}{A}^{\%}(X))$$

if X is a subspace (preferrably also an ENR) of Y. Recall from Part III, section 2 the assembly maps

$$\underset{=}{A}^{\%}(Y) \longrightarrow \underset{=}{A}p(Y),$$

inducing

$$A^{\%}(Y) \longrightarrow A(Y).$$

2.2. Microcharacteristics and cell–like maps. For a compact ENR, say Y, the A–theory Euler characteristic

$$\langle Y \rangle \in A(Y)$$

has a canonical lift to $A^{\%}(Y)$:

$$\ll Y \gg \in A^{\%}(Y),$$

which we call the *microcharacteristic* of Y. "Lift" means that the assembly map

$$\alpha : A^{\%}(Y) \longrightarrow A(Y)$$

sends $\ll Y \gg$ to $\langle Y \rangle$, strictly. The price to be paid for this refinement is a loss of naturality: the microcharacteristic is lax natural only for *cell–like maps* between compact ENR's.

(A map $f : Y_1 \longrightarrow Y_2$ between ENR's, compact or not, is *cell–like* if it is proper and if, for every open $U \subset Y_2$, the appropriate restriction of f is a homotopy equivalence from $f^{-1}(U)$ to Y. This is not the original definition, but it is the least technical one. See the foundational papers by Lacher. For the history of this notion, before and after Lacher's work: see Brown, Siebenmann.)

We shall write $\ll f \gg$ for the path in $A^{\%}(Y_2)$ from $f_* \ll Y_1 \gg$ to $\ll Y_2 \gg$ determined by a cell–like map

$$f : Y_1 \longrightarrow Y_2$$

between compact ENR's. Compare 1.1.

Strangely enough, we will never really use lax naturality of the microcharacteristic for cell–like maps; we will only use lax naturality for homeomorphisms. But it does not cost anything extra to establish it for cell–like maps.

2.3. Local microcharacteristics. Let Y be an ENR, possibly noncompact, and let $K \subset Y$ be a compact subset. In this situation there is a *local microcharacteristic*

$$\ll Y \gg_K \in A^{\%}(Y)/A^{\%}(Y-K)$$

with the following properties.

(i) If Y is compact and $K = Y$, then

$$\ll Y \gg_K = \ll Y \gg \ \text{ in } A^{\%}(Y)/A^{\%}(\emptyset) \cong A^{\%}(Y).$$

(ii) The local microcharacteristic is lax natural for maps of the form

$$f : (Y', Y' - K') \longrightarrow (Y, Y - K)$$

(K' compact in Y', and K compact in Y), where f is cell–like near K. (This means that K has an open neighbourhood U in Y such that the appropriate restriction of f is a cell–like map from $f^{-1}(U)$ to U). We write $\ll f \gg$ for the path determined by f (in $A^{\%}(Y)/A^{\%}(Y-K)$, from $f_* \ll Y' \gg_{K'}$, to $\ll Y \gg_K$).

2.4. Reminders. A homotopy invariant functor F from ENR's with spherical fibration still has an assembly transformation

$$\alpha : F^{\%}(Y, \gamma) \longrightarrow F(Y, \gamma).$$

Here $F^{\%}$ is homotopy invariant, strongly excisive, and the assembly map is a homotopy equivalence if Y is a point. (These properties essentially characterize $F^{\%}$ and α; see Part III, section 2, esp. 2.3. (ii).) Further, if F is a functor to spectra with Z_2–action, then so is $F^{\%}$; this follows from the construction of $F^{\%}$, also Part III, section 2. Applying this to the functor

$$(Y, \gamma) \longmapsto \underline{\underline{A}}p(Y, \gamma, k) \qquad \text{(for fixed } k\text{)}$$

we find that γ and k determine an involution not only on $\underline{\underline{A}}p(Y)$ and on $A(Y)$, but also on $\underline{\underline{A}}^{\%}(Y)$ and on $A^{\%}(Y)$ (notation of 2.1). Assembly respects the involutions.

2.5. Microduality. For a closed (topological) manifold M, the microcharacteristic $\ll M \gg$ in $A^{\%}(M)$ has a canonical lift

$$\ll M \gg^{SW} \in A^{\%}(M)^{SW}.$$

Here $A^{\%}(M)^{SW}$ denotes the homotopy fixed point space of the involution on $A^{\%}(M)$ determined by γ, the Spivak normal fibration of M, and $k = -\dim(M)$.

It is essential here that M be a closed manifold, not just a Poincaré space. Right now it is hard to explain why this is essential; we will be in a better position in the next section (see). Suffice it to say that Poincaré duality is too crude a concept here because it is homotopy invariant. What we need is a self–duality (one might call it cell–like Poincaré duality) which is to ordinary Poincaré duality as cell–like maps are to ordinary homotopy equivalences: more hereditary. Closed manifolds have this cell–like Poincaré duality.

Unsurprisingly, $\ll M \gg^{SW}$ is lax natural for cell–like maps between closed manifolds.

2.6. Local microduality. For a topological manifold M (not necessarily compact but without boundary) and a compact $K \subset M$, the local microcharacteristic $\ll M \gg_K$ has a canonical lift

$$\ll M \gg_K^{SW} \in (A^{\%}(M)/A^{\%}(M-K))^{SW}.$$

This has the expected properties:

(i) If M is compact, and $K = M$, then

$$\ll M \gg_K^{SW} = \ll M \gg^{SW} \in A^{\%}(M)^{SW}$$

(ii) $\ll M \gg_K^{SW}$ is lax natural for maps of the form

$$f : (M', M'-K') \longrightarrow (M, M-K)$$

(K' compact in M' and K compact in M) where f is cell–like near K.

2.7. Products. Given (X, γ, k) and (Y, γ, m) as in 1.5, we have the external product

$$\underline{\underline{A}}^{\%}(X, \beta, k) \wedge \underline{\underline{A}}^{\%}(Y, \gamma, m) \longrightarrow \underline{\underline{A}}^{\%}(X \times Y, \beta \times \gamma, k+m)$$

or just

$$\underset{=}{A}^{\%}(X) \wedge \underset{=}{A}^{\%}(Y) \longrightarrow \underset{=}{A}^{\%}(X \times Y).$$

It is a Z_2–map between spectra with Z_2–action. If X and Y are ENR's, say, then we need not insist on CW–structures.

If $Y = M$ is a closed manifold, γ is its Spivak normal fibration, and $m = -\dim(M)$, then we have the external product with $\ll M \gg^{SW}$, which is a Z_2–map

$$\underset{=}{A}^{\%}(X) \longrightarrow \underset{=}{A}^{\%}(X \times M)$$

(data β, k on X and $\beta \times \gamma, k - \dim(M)$ on $X \times M$ are understood). Compare 1.5. More generally, if M is a topological manifold, not necessarily compact but without boundary, then we have the external product with $\ll M \gg^{SW}_K$; this is a Z_2–map

$$\underset{=}{A}^{\%}(X) \longrightarrow \underset{=}{A}^{\%}(X \times M)/A^{\%}(X \times (M - K)).$$

In the very special case $M = \mathbb{R}$ and $K = \{0\}$, its target is canonically and obviously homotopy equivalent to $S^1 \wedge \underset{=}{A}^{\%}(X)$, and we call the map SW–*suspension*:

$$\underset{=}{A}^{\%}(X) \longrightarrow S^1 \wedge \underset{=}{A}^{\%}(X).$$

Compare 1.5. This is still a Z_2–map.

2.7. Analysis of the SW–suspension. (Compare 1.6.) Let $\underset{=}{F}^{\%}$ be the homotopy fiber of the SW–suspension

$$\underset{=}{A}^{\%}(X) \longrightarrow S^1 \wedge \underset{=}{A}^{\%}(X).$$

Denote by τ the canonical involution on $\underset{=}{F}^{\%}$ (of course, this depends on β and k).

Claim: The forgetful map

$$z : \underset{=}{F}^{\%} \longrightarrow \underset{=}{A}^{\%}(X)$$

has a homotopy right inverse $u : \underset{=}{A}^{\%}(X) \longrightarrow \underset{=}{F}^{\%}$ (zu $\simeq$ id) such that

$$u \vee \tau u : \underset{=}{A}^{\%}(X) \vee \underset{=}{A}^{\%}(X) \longrightarrow \underset{=}{F}^{\%}$$

is a homotopy equivalence.

Proof: See

2.8. Remarks. It is stated in 2.2 that the microcharacteristic

$$\ll Y \gg \in A^{\%}(Y)$$

of a compact ENR maps to the ordinary characteristic

$$\langle Y \rangle \in A(Y)$$

under the assembly map $A^{\%}(Y) \longrightarrow A(Y)$. There is of course a more precise statement. Let $\mathcal{C}$ be a small category whose objects are compact ENE 's, and whose morphisms are cell–like maps (composition of morphisms being composition of maps). Then

$$\langle \mathcal{C} \rangle \in A(\mathcal{C})$$

is defined according to 1.1, and for similar reasons

$$\ll \mathcal{C} \gg \in A^{\%}(\mathcal{C}) = \operatorname*{holim}_{Y \text{ in } \mathcal{C}} A^{\%}(Y)$$

is defined. The assembly map from $A^{\%}(\mathcal{C})$ to $A(\mathcal{C})$ maps $\ll \mathcal{C} \gg$ to $\langle \mathcal{C} \rangle$.

A similar remark applies in the situation of 2.6.

2.9. Microcharacteristics in the noncompact case. Let E be an ENR , not necessarily compact. For each compact $K \subset E$, we have the local microcharacteristic

$$\ll E \gg_K \in A^{\%}(E)/A^{\%}(E-K),$$

and lax naturlity implies that the various $\ll E \gg_K$ are the "coordinates" of a point

$$\ll E \gg \in \operatorname*{holim}_{K} A^{\%}(E)/A^{\%}(E-K).$$

We can call this the microcharacteristic of E ; when E is compact, it reduces to that in 2.2.

Similarly, if E is a manifold (possibly noncompact), we have

$$\ll E \gg^{SW} \in \operatorname*{holim}_{K} A^{\%}(E)^{SW}/A^{\%}(E-K)^{SW},$$

a mild generalization of 2.5 which uses 2.6.

These microcharacteristics are still lax natural for cell–like maps.

3. Control spaces and controlled characteristics

3.1. Control spaces. (Compare Anderson, Connolly, Ferry, Pedersen.) A *control space* is a pair consisting of a compact Hausdorff space $\overline{E}$ and an open dense subset E of $\overline{E}$. Here we also assume that E is an ENR and that $\overline{E}$ is metrizable.

A *morphism* between control spaces, say from

$$(E \subset \overline{E}) \quad \text{to} \quad (E' \subset \overline{E}'),$$

is a continuous map $g : \overline{E} \to \overline{E}'$ such that $g^{-1}(E') = E$.

3.2. Retractive spaces over a control space. (Compare Vogell,). A retractive CW–space over the control space $(E \subset \overline{E})$ is a retractrive CW–space over E,

$$V \underset{s}{\overset{r}{\rightleftarrows}} E$$

(with a CW–structure relative to E only), satisfying a condition on the size of cells Z in $V - s(E)$. Namely, for any $x \in \overline{E} - E$ and any neighbourhood U of x in $\overline{E}$, there should exist a smaller neighbourhood W of x in $\overline{E}$ such that

$$r(Z) \cap W \neq \emptyset \quad \text{implies} \quad r(Z) \subset U$$

for any cell Z in $V - s(E)$.

We will have to deal with different kinds of morphisms between such retractive CW–spaces.

a) A *retractive map* from

$$V \underset{s}{\overset{r}{\rightleftarrows}} E \quad \text{to} \quad V' \underset{s'}{\overset{r'}{\rightleftarrows}} E$$

is a map $f : V \to V'$ satisfying $r = r'f$ and $s' = fs$.

b) A *controlled map* from

$$V \underset{s}{\overset{r}{\rightleftarrows}} E \quad \text{to} \quad V' \underset{s}{\overset{r'}{\rightleftarrows}} E$$

is a map $f : V \to V'$ satisfying the control condition: Given $x \in \overline{E} - E$, and a neighbourhood U of x in $\overline{E}$, there exists a smaller neighbourhood

W of x in $\overline{E}$ such that $r(y) \in W$ implies $r'(f(y)) \in U$, for all y in V. Apart from that, we require that $fs = s'$.

Note that a retractive map is a controlled map. Note also that it is straightforward to define controlled homotopies between controlled maps. So there is a homotopy category $\mathcal{H}$ of retractive CW–spaces over $(E \subset \overline{E})$, as above, and controlled maps. A retractive map as in a) is a *controlled* h*–equivalence* if it becomes an isomorphism in $\mathcal{H}$. A retractive map as in a) is a *cofibration* if it is isomorphic to the inclusion of a retractive CW–subspace. A retractive CW–space

$$V \underset{s}{\overset{r}{\rightleftarrows}} E$$

as above is *proper* if r is proper. It is *properly dominated* if, in the homotopy category $\mathcal{H}$, it is a retract of a proper object.

We see that the properly dominated retractive CW–spaces over $(E \subset \overline{E})$ form a category with cofibrations and weak equivalences, the weak equivalences being the controlled h–equivalences (and the morphisms in the category being the retractive maps as in a)). Its K–theory will be denoted by

$$A(E \subset \overline{E}) \quad \text{or} \quad Ap(E \subset \overline{E}).$$

This is an infinite loop space, and we write

$$\underline{\underline{A}}p(E \subset \overline{E})$$

for the corresponding spectrum.

A morphism between control spaces, say

$$g : (E \subset \overline{E}) \longrightarrow (E' \subset \overline{E}'),$$

induces maps

$$g_* : \underline{\underline{A}}p(E \subset \overline{E}) \longrightarrow \underline{\underline{A}}p(E' \subset \overline{E}')$$

in the usual way.

3.3. The ENR version. As a variation on 3.2, one could allow retractive ENR's

$$V \underset{s}{\overset{r}{\rightleftarrows}} E$$

over E, where $E \subset \overline{E}$ is a control space and where r is proper. Retractive maps, cofibrations and weak equivalences ($=$ h–equivalences) are defined much as before. The result is a category $\mathcal{C}$ with cofibrations and weak equivalences. It contains the category of proper retractive CW–spaces over $(E \subset \overline{E})$ as an exact subcategory, and the inclusion of the subcategory induces a homotopy equivalence of the K–theories (see). We therefore write

$$\underline{\underline{A}}h(E \subset \overline{E}) \quad \text{and} \quad Ah(E \subset \overline{E})$$

for the K–theory spectrum and K–theory space of $\mathcal{C}$. These can be mapped to $\underline{\underline{A}}p(E \subset \overline{E})$ and $Ap(E \subset \overline{E})$, respectively, by a pushout construction as in 1.2.

3.4. Controlled Euler characteristics. For a control space $(E \subset \overline{E})$, where E is an ENR , we have the element

$$\langle E \subset \overline{E} \rangle \in A(E \subset \overline{E})$$

determined by the retractive ENR

$$S^0 \times E \underset{s}{\overset{r}{\rightleftarrows}} E,$$

where r is the projection and s identifies E with $\{1\} \times E$. We call $\langle E \subset \overline{E} \rangle$ the controlled A–theory Euler characteristic of $(E \subset \overline{E})$.

A morphism

$$g : (E \subset \overline{E}) \longrightarrow (E' \subset \overline{E}')$$

between control spaces is a *mollifier* if it restricts to a homeomorphism from E to E' . It is a *homotopy mollifier* if the map

$$\{-1\} \times E \cup \{+1\} \times E' \longrightarrow \{-1, +1\} \times E'$$

given by $(-1, e) \longmapsto (-1, g(e))$ and $(+1, e') \longmapsto (+1, e')$, is an h–equivalence of retractive ENR's over the control space $(E' \subset \overline{E}')$. (A mollifier is a homotopy mollifier.) It follows immediately from the definition that the controlled characteristic $\langle E \subset \overline{E} \rangle$ is lax natural for homotopy mollifiers.

3.5. Geometric chain algebra without control. This and the following subsection (on geometric chain algebra with control) are taken from Quinn, with some ideas from Anderson–Connolly–Ferry–Pedersen. We need geometric chain algebra (with control) e.g. in order to be able to say when a control space $(E \subset \overline{E})$ is a Poincaré control space.

Let R be a ring and let Y be a space. For $x, y \in Y$ let $P(x, y)$ be the free R–module generated by all paths $\lambda : [0, 1] \to Y$ from x to y . Let $Q(x, y)$ be the free R–module generated by all path homotopies $h : [0, 1] \times [0, 1] \to Y$ from

x to y . (That is, $h(0,t)=x$ and $h(1,t)=y$ for all t .) There are obvious face maps d_0, d_1 from $Q(x,y)$ to $P(x,y)$, and we let

$$\hom(x,y) := P(x,y)/\mathrm{im}(d_0 - d_1)$$

(which is also the free R–module generated by the path classes from x to y).

A geometric module on Y is a free R–module $R[S]$ with a map from the basis $f : S \longrightarrow Y$. Given two geometric modules on Y , say $R[S_0]$ and $R[S_1]$ with labelling maps $f_i : S_i \longrightarrow Y$ $(i = 0,1)$, a geometric morphism from the first to the second is a matrix $A = (a_{uv})$ with one entry

$$a_{uv} \in \hom(f_0(u), f_1(v))$$

for each $(u,v) \in S_0 \times S_1$. Each row is supposed to have finitely many nonzero entries only, i.e. for each u there exist only finitely many v with $a_{uv} \neq 0$.

Composition of geometric morphisms is by matrix multiplication, using the fact that there is a composition law

$$\hom(y,z) \times \hom(x,y) \longmapsto \hom(x,z)$$

(for $x,y,z \in Y$). The composition law is by concatenation of paths, and it is bilinear. See Part III, section 1, esp. 1.1.

Note that each $\hom(x,y)$ comes with a homomorphism to R given by $\Sigma m_j[\lambda_j] \mapsto \Sigma m_j$. Applying this to the entries of a matrix A as above, we see that a geometric morphism between geometric modules on Y determines an ordinary R–module homomorphism between the underlying free R–modules.

For geometric modules C_0, C_1 on Y , we let $\hom_Y(C_0, C_1)$ be the abelian group of geometric morphisms from C_0 to C_1 . If R is commutative, which we assume from now on, then there is a geometric tensor product $C_0 \otimes_Y C_1$ with reasonable properties: Let S_i be the basis for C_i , with labelling map $f_i : S_i \to Y$ (for $i = 0,1$). Let $C_0 \otimes_Y C_1$ constist of all matrices $A = (a_{uv})$ with

$$a_{uv} \in \hom(f_0(u), f_1(v)),$$

such that each row *and each column* of the matrix has finitely many nonzero entries only. It is not hard to see that $C_0 \otimes_Y C_1$ is a covariant functor in the second variable, C_1 (use matrix multiplication). But it is also easy to see that

$$C_0 \otimes_Y C_1 \cong C_1 \otimes_Y C_0,$$

naturally, so it must be a covariant functor in the first variable, too. (The isomorphism is given by matrix transposition; the entries also have to be transposed, i.e. paths must be inverted.)

The *dual* of a geometric module on Y with *finite* basis is the module itself. The dual of a geometric morphism $A : C_0 \longrightarrow C_1$ between geometric modules on Y with finite basis is the transpose, $A^t : C_1 \longrightarrow C_0$. (Transpose the matrix A, and transpose the entries by inverting paths.)

These notions $(\hom_Y, \otimes_Y$, and duals) are extended to chain complexes of geometric modules on Y in the usual way. Given two such geometric chain complexes C, D on Y, one has

$$\hom_Y(D^{-*}, D) \cong C \otimes_Y D$$

provided C is finite dimensional with finite basis in each dimension. C^{-*} denotes the dual of C.

Example: The geometric singular chain complex of Y. In the ordinary singular chain complex $S(Y)$, with R–coefficients, the chain modules $S_n(Y)$ can be made into geometric modules on Y by labelling each singular n–simplex with the image in Y of its barycenter. Similarly, the boundary maps $S_n(Y) \longrightarrow S_{n-1}(Y)$ can be made into geometric morphisms on Y. See for details. (Note that Quinn's terminology is slightly different from ours: our geometric morphisms are equivalence classes of Quinn's geometric morphisms.) We write $SGC(Y)$ for this geometric chain complex on Y (singular geometric chain complex, say).

Example: The geometric Alexander–Whitney diagonal.

The Alexander–Whitney–Eilenberg–Zilber diagonal chain map

$$\Delta : S(Y) \longrightarrow S(Y) \otimes_R S(Y)$$

has an obvious refinement or lift to

$$\Delta : S(Y) \longrightarrow SGC(Y) \otimes_Y SGC(Y).$$

3.6. Geometric chain algebra with control. We repeat the exercise (3.5), adding control. Let R be a commutative ring, and let $(E \subset \overline{E})$ be a control space. For $x, y \in E$ let $P(x, y)$ be the free R–module generated by all paths in E from x to y. Let $Q(x, y)$ be the free R–module generated by all path homotopies $h : [0, 1] \times [0, 1] \to E$ from x to y. There are obvious face maps $d_0, d_1 : Q(x, y) \to P(x, y)$.

The *track* of an element $\Sigma m_j \lambda_j$ in $P(x, y)$ is the union of the images of the λ_j with coefficient $m_j \neq 0$. The track of an element in $Q(x, y)$ is defined similarly.

A *geometric module* on $(E \subset \overline{E})$ is a free R–module $R[S]$ together with a map from the basis $f : S \to E$. Given two such geometric modules $R[S_0]$, $R[S_1]$ with labelling maps $f_i : S_i \longrightarrow E \quad (i = 0, 1)$, a *geometric pre–morphism* from the first to the second is a matrix

$$A = (a_{uv})$$

with one entry $a_{uv} \in P(f_0(u), f_1(v))$ for each $(u,v) \in S_0 \times S_1$. Requirements on A:

(i) Each row of A has finitely many nonzero entries only.

(ii) The tracks of the entries of A are controlled. This means that for every $z \in \overline{E} - E$ and neighbourhood U of z in $\overline{E}$, there exists a smaller neighbourhood W of z in $\overline{E}$ such that

$$\operatorname{track}(a_{uv}) \cap W \neq \emptyset \qquad \text{implies} \qquad \operatorname{track}(a_{uv}) \subset U$$

for all entries a_{uv}.

Two such matrices $A = (a_{uv})$, $A' = (a'_{uv})$ define the same *geometric morphism* if there exists a third matrix $B = (b_{uv})$ with one entry $b_{uv} \in Q(f_0(u), f_1(v))$ for each $(u,v) \in S_0 \times S_1$, such that $d_0 B = A$ and $d_1 B = A'$. Here d_0, d_1 are applied to each entry of B. Requirements on B:

(i) Each row of B has finitely many nonzero entries only.

(ii) The tracks of the entries of B are controlled. This means that, for every $z \in \overline{E} - E$, etc. .

Composition of geometric morphisms is by matrix multiplication. As in the case without control a geometric morphism between geometric modules on $(E \subset \overline{E})$ determines an ordinary R–module homomorphism between the underlying free R–modules.

For geometric modules C_0, C_1 on $(E \subset \overline{E})$, we let $\hom_{E \subset \overline{E}}(C_0, C_1)$ consist of the geometric morphisms from C_0 to C_1. The tensor product $C_0 \otimes_{E \subset \overline{E}} C_1$ is defined as a subgroup of $\hom_{E \subset \overline{E}}(C_0, C_1)$. It consists of all geometric morphisms from C_0 to C_1 having a matrix representative A such that each row *and each column* of A has finitely many nonzero entries only. (Note: the matrix entries are linear combinations of paths, not path classes.) There is a natural isomorphism

$$C_0 \otimes_{E \subset \overline{E}} C_1 \longrightarrow C_1 \otimes_{E \subset \overline{E}} C_0$$

given by matrix transposition (which includes transposition of entries by path inversion). The tensor product is covariant in both variables.

Let C be a geometric module on $(E \subset \overline{E})$ with basis S and labelling map $f : S \to E$. Call C *locally finitely generated* if $f^{-1}(K)$ is finite for every compact $K \subset E$. For a locally finitely generated C, the dual of C is defined to be C itself. The dual of a geometric morphism

$$[A] : C_0 \longrightarrow C_1$$

on $E \subset \overline{E}$, where C_0 and C_1 are locally finitely generated, is the transpose

$$[A^t] : C_1 \longrightarrow C_0.$$

(Here A is a geometric pre–morphism, in matrix form, representing the geometric morphism, and as usual transposition includes transposition of entries. One has to check that A^t satisfies the relevant finiteness conditions.)

The notions $\hom_{E\subset\overline{E}}, \otimes_{E\subset\overline{E}}$ and duality are extended to chain complexes of geometric modules on $E \subset \overline{E}$ in the usual way. Given two such geometric chain complexes C, D on $E \subset \overline{E}$, one has

$$\hom_{E\subset\overline{E}}(C^{-*}, D) \cong C \otimes_{E\subset\overline{E}} D$$

provided C is finite dimensional with locally finite basis in each dimension. C^{-*} denotes the dual of C.

Example: The geometric singular chain complex of $E \subset \overline{E}$. Since $\overline{E}$ is metrizable (3.1), there exists an $\overline{E}$–controlled covering $\mathcal{V}$ of E: a covering of E by open sets such that, for every $x \in \overline{E} - E$ and every neighbourhood U of x in $\overline{E}$, there exists a smaller neighbourhood W of x in $\overline{E}$ such that

$$V \cap W \neq \emptyset \qquad \text{implies} \qquad V \subset U$$

for all V in $\mathcal{V}$. (See Hu, on "canonical coverings".) Let $S(E;\mathcal{V})$ be the singular chain complex of E (with coefficients in R) based on $\mathcal{V}$–small singular simplices (i.e. singular simplices with image contained in some $V \in \mathcal{V}$). It is well known that $S(E;\mathcal{V})$ is chain homotopy equivalent to $S(E)$. More important here is the fact that the chain modules $S_n(E;\mathcal{V})$ become geometric modules on $E \subset \overline{E}$ if we label each singular simplex with the image in E of its barycenter; and that the boundary homomorphisms $S_n(E;\mathcal{V}) \longrightarrow S_{n-1}(E;\mathcal{V})$ can be lifted to geometric morphisms on $E \subset \overline{E}$. Denote the resulting geometric chain complex on $E \subset \overline{E}$ by

$$SGC(E \subset \overline{E}; \mathcal{V}).$$

If $\mathcal{V}'$ is another $\overline{E}$–controlled covering of E, then so is $\mathcal{V}\cup\mathcal{V}'$, and the inclusions

$$SGC(E \subset \overline{E}; \mathcal{V}) \hookrightarrow SGC(E \subset \overline{E}; \mathcal{V}\cup\mathcal{V}') \hookleftarrow SGC(E \subset \overline{E}; \mathcal{V}')$$

are geometric chain homotopy equivalences. In this sense, the choice of controlled covering is irrelevant. Further, $SGC(E \subset \overline{E}; \mathcal{V})$ is usually not finite dimensional and locally finitely generated in each dimension, but it is always geometrically chain homotopy equivalent to another geometric chain complex on $E \subset \overline{E}$ which has these desirable properties.

Example: The geometric Alexander–Whitney diagonal chain map.

The Alexander–Whitney diagonal

$$\Delta : S(E;\mathcal{V}) \longrightarrow S(E;\mathcal{V}) \otimes_R S(E;\mathcal{V})$$

has an obvious refinement/extension/lift to a chain map

$$\Delta : S\hat{}(E;\mathcal{V}) \longrightarrow SGC(E \subset \overline{E};\mathcal{V}) \otimes SGC(E \subset \overline{E};\mathcal{V})$$

where $\otimes$ means $\otimes_{E\subset\overline{E}}$ and $S\hat{}(E;\mathcal{V})$ is the *locally finite* version of $S(E;\mathcal{V})$. (An element in $S\hat{_n}(E;\mathcal{V})$ is a formal linear combination $\Sigma m_j\sigma_j$, where $m_j \in R$, where the σ_j are $\mathcal{V}$–small singular simplices in E , and where for each compact $K \subset E$ there exist only finitely many σ_j with nonzero coefficient m_j which meet K .)

Note that the inclusion

$$S\hat{}(E;\mathcal{V}) \longrightarrow S\hat{}(E)$$

is a chain homotopy equivalence (where $S\hat{}(E)$ is the chain complex of locally finite singular chains on E), and that $S\hat{}(E)$ only depends on E , not on the inclusion $E \subset \overline{E}$. Up to chain homotopy therefore, both source and target of the refined Alexander–Whitney diagonal are independent of the controlled covering $\mathcal{V}$, and we will allow ourselves to write

$$\Delta : S\hat{}(E) \longrightarrow SGC(E \subset \overline{E}) \otimes SGC(E \subset \overline{E}).$$

3.7. Poincaré control spaces. From now on take R to be $\mathbb{Z}$. A control space $(E \subset \overline{E})$ is an *orientable Poincaré control space* of formal dimension n if there exists a class $[\omega] \in H_n(S\hat{}(E))$ whose image

$$\Delta_*[\omega] \in H_n(SGC(E \subset \overline{E}) \otimes SGC(E \subset \overline{E}))$$

under the Alexander–Whitney diagonal is *nondegenerate*. Nondegeneracy means the following. Choose a geometric chain complex D on $(E \subset \overline{E})$, finite dimensional and with locally finite basis in each dimension, and a geometric chain homotopy equivalence from D to $SGC(E \subset \overline{E})$. Then we can say that $\Delta_*[\omega]$ lives in

$$H_n(D \otimes D) \longrightarrow H_n(\hom_{E\subset\overline{E}}(D^{-*}, D)),$$

so it determines a geometric chain map from $\Sigma^N D^{-*}$ to D , up to chain homotopy equivalence. If this chain map is a chain homotopy equivalence, call $\Delta_*[\omega]$ nondegenerate.

A control space $(E \subset \overline{E})$ is a *Poincaré control space* of formal dimension n if there exists a morphism of control spaces

$$g : (E' \subset \overline{E}') \longrightarrow (E \subset \overline{E})$$

where $(E' \subset \overline{E}')$ is an *orientable* Poincaré control space of formal dimension n , and g restricts to a homeomorphism

$$\overline{E}' - E' \longrightarrow \overline{E} - E$$

and to a double covering

$$E' \longrightarrow E.$$

EXAMPLE: Any control space of the form $(E \subset \overline{E})$, where E is an $n-$dimensional manifold, is a Poincaré control space of formal dimension n. This is rather easy to verify if E can be triangulated (see e.g. the argument in Wall, p. 24, but use controlled triangulations). For the general case, see

EXAMPLE: If $g : (E \subset \overline{E}) \longrightarrow (E' \subset \overline{E}')$ is a morphism between control spaces which is a homotopy mollifier (3.4) and if $(E \subset \overline{E})$ is Poincaré, then so is $(E' \subset \overline{E}')$. Therefore, in order to decide whether a control space $(E \subset \overline{E})$ is Poincaré, one should first ask whether the control space $(E \subset E \cup \{\infty\})$ is Poincaré (where $E \cup \{\infty\}$ is the one–point compactification).

DEFINITION: If E is an ENR such that $(E \subset E \cup .\{\infty\})$ is a Poincaré control space of formal dimension n, then we say that E satisfies *proper* Poincaré $n-$duality.

In this situation, E has a *Spivak normal fibration.* In the orientabel case, this is an orientable quasifibration

$$\gamma : L \longrightarrow E$$

with fibers homotopy equivalent to S^k for some k, and with a specific section $s : E \to L$. We can assume that γ is pulled back from BF_k, where F_k is the topological monoid of base–point preserving homotopy equivalences $S^k \to S^k$. Then L is locally compact and γ is proper. Being "the" Spivak normal fibration, γ comes with a based map

$$\eta : S^{n+k} \longrightarrow (L \cup \{\infty\})/(E \cup \{\infty\})$$

whose class in

$$H_{n+k}(S\hat{\ }(L)/S\hat{\ }(E)) \cong H_n(S\hat{\ }(E))$$

is a fundamental class for the Poincaré control space $(E \subset E \cup \{\infty\})$. Note that we have used a version of the Thom isomorphism for homology made from locally finite singular chains.

In the nonorientable case, γ and η have much the same properties as before, but γ is nonorientable and $H_n(S\hat{\ }(E))$ must be replaced by a twisted version. We omit the details; see .

The pair consisting of γ and η is unique up to contractible choice, in a stable sense. See .

3.8. Controlled Euler characteristics and Poincaré duality. Suppose that $(E \subset \overline{E})$ is a Poincaré control space of formal dimension n, and let γ be the Spivak normal fibration on E. The pair $(\gamma, -n)$ determines, via a suitable controlled version of Spanier–Whitehead duality, an involution on $\underline{\underline{A}}p(E \subset \overline{E})$ and therefore on $A(E \subset \overline{E})$. This has nothing to do with the Poincaré assumption on $(E \subset \overline{E})$. Write $A(E \subset \overline{E})^{SW}$ for the space of homotopy fixed points of this involution on $A(E \subset \overline{E})$. The controlled Euler characteristic

$$\langle E \subset \overline{E} \rangle \ \in A(E \subset \overline{E})$$

has a canonical lift to an element

$$\langle E \subset \overline{E} \rangle^{SW} \in A(E \subset \overline{E})^{SW}.$$

(This has a lot to do with the Poincaré assumption on $(E \subset \overline{E})$.) The refined characteristic $\langle E \subset \overline{E} \rangle^{SW}$ is still lax natural with respect to homotopy mollifiers (see 3.4 and the second example in 3.7).

3.9. Products. (Compare 1.5, 2.6.) For control spaces $(E \subset \overline{E})$ and $(F \subset \overline{F})$, we have an external product

$$\underline{\underline{A}}p(E \subset \overline{E}) \wedge \underline{\underline{A}}p(F \subset \overline{F}) \longrightarrow \underline{\underline{A}}p(E \times F \subset \overline{E} \times \overline{F}).$$

We can also equip the three spectra involved with involutions, by choosing spherical fibrations β on E , and γ on F , and assigning formal dimensions to them: k and m . Then $E \times F$ carries $\beta \times \gamma$, of formal dimension $k+m$. The external product map above then becomes a Z_2–map of spectra.

If $(F \subset \overline{F})$ is a Poincaré control space, $-m$ is its formal dimension, and γ is its Spivak normal fibration, then we have the external product with $\langle F \subset \overline{F} \rangle^{SW}$, which is a Z_2–map between spectra,

$$\underline{\underline{A}}p(E \subset \overline{E}) \longrightarrow \underline{\underline{A}}p(E \times F \subset \overline{E} \times \overline{F})$$

or a Z_2–map between infinite loop spaces

$$A(E \subset \overline{E}) \longrightarrow A(E \times F \subset \overline{E} \times \overline{F}).$$

In the special case $F = (-1,1)$ and $\overline{F} = [-1,1]$, we call the map SW–suspension:

$$A(E \subset \overline{E}) \longrightarrow A(E \times (-1,1) \subset \overline{E} \times [-1,1]).$$

3.10. Analysis of the SW–suspension. (Compare 1.6, 2.7.) Let Y be the homotopy fiber of the SW–suspension

$$A(E \subset \overline{E}) \longrightarrow A(E \times (-1,1) \subset \overline{E} \times [-1,1]).$$

Denote by τ the canonical involution on Y . (This depends on β and k .)

Claim: The forgetful map

$$z : Y \longrightarrow A(E \subset \overline{E})$$

has a homotopy right inverse $u : A(E \subset \overline{E}) \longrightarrow Y$ such that the map

$$A(E \subset \overline{E}) \times A(E \subset \overline{E}) \longrightarrow Y$$

$$(p, q) \longmapsto u(p) + \tau(u(q))$$

is a homotopy equivalence.

REMARK: The map u in the claim respects infinite loop space structures, and so the claim implies

$$A(E \subset \overline{E}) \simeq \Omega A(E \times (-1, 1) \subset \overline{E} \times [-1, 1])$$

as infinite loop spaces. It does not follow that

$$S^1 \wedge \underline{\underline{A}}p(E \subset \overline{E}) \simeq \underline{\underline{A}}p(E \times (-1, 1) \subset \overline{E} \times [-1, 1])$$

because π_0 of the spectrum on the right can be nonzero. This is the only reason for not stating the claim in spectrum language.

PROOF OF CLAIM: See

3.11. Cell–like Poincaré duality. Let Y be a compact ENR . One could say that Y satisfies cell–like Poincaré n–duality if the control space

$$(Y \times (0, 1) \subset \text{ Cone}(Y))$$

is a Poincaré control space of formal dimension $n+1$. (The cone on Y is a quotient of $Y \times [0, 1]$.) The conjunctive, “one could say”, is appropriate here because it may very well turn out that Y has this property if and only if it is a homology manifold. At any rate, if Y is a closed manifold, then $Y \times (0, 1)$ is a manifold, so Y satisfies cell–like Poincaré duality by the first example in 3.7. Compare 2.5.

Related to this is the following observation: A map $f : Y_1 \longrightarrow Y_2$ between compact ENR ’s is cell–like if and only if the map between control spaces

$$(Y_1 \times (0, 1) \subset \text{ Cone}(Y_1)) \longrightarrow (Y_2 \times (0, 1) \subset \text{ Cone}(Y_2))$$

induced by f is a homotopy mollifier. See 3.4.

4. Microstudies vs. control

Subsection 3.11 indicates that there is some overlap between the microtheory of section 2, and the controlled theory of section 3. Here we investigate this overlap. Since assembly maps were fundamental in the microtheory, we shall have to generalize them to the controlled setting.

4.1. Assembly maps in the controlled setting. Let $(E \subset \overline{E})$ be a control space. The assembly map we are about to describe has the form

$$\alpha : \operatorname*{holim}_K \underline{\underline{A}}^{\%}(E)/\underline{\underline{A}}^{\%}(E-K) \longrightarrow \underline{\underline{A}}p(E \subset \overline{E})$$

where the homotopy inverse limit is taken over all compact subsets $K \subset E$. In the case where not only E , but also $\overline{E}$ and $\overline{E}-E$ and therefore $\overline{E}/(\overline{E}-E) \cong E \cup \{\infty\}$ are ENR 's, we need only consider those K for which the inclusion

$$\overline{E}-E \longrightarrow \overline{E}-K$$

is a homotopy equivalence. Then the inverse system is essentially constant, and the assembly map takes the form

$$: \underline{\underline{A}}^{\%}(E \cup \{\infty\})/\underline{\underline{A}}^{\%}(\{\infty\}) \longrightarrow \underline{\underline{A}}p(E \subset \overline{E}).$$

In any case, the source of the assembly map depends on E only, not on the compactification $E \subset \overline{E}$. It is nevertheless convenient and systematic to write $\underline{\underline{A}}^{\%}(E \subset \overline{E})$ for it. If E is a compact ENR , then $\underline{\underline{A}}^{\%}(E \subset E)$ is defined and not much different from $\underline{\underline{A}}^{\%}(E)$.

An elementary description of this assembly map (in the controlled setting) can be given along the lines of Part III, section 2.

Fix a compact (metrizable) space D . Let $\mathcal{C}$ be the category of control spaces $(E \subset \overline{E})$ such that $\overline{E}-E$ is contained in D . Morphisms in $\mathcal{C}$ are morphisms between control spaces

$$g : (E_1 \subset \overline{E}_1) \longrightarrow (E_2 \subset \overline{E}_2)$$

compatible with the inclusion $\overline{E}_i - E_i \subset D \quad (i = 1,2)$. In the category $\mathcal{C}$, it makes sense to talk of homotopies: a homotopy is a continuous one–parameter family g_t of morphisms, $0 \le t \le 1$. So there is a notion of homotopy equivalence. The category also has coproducts. More generally, if g_1 and g_2 are morphisms in $\mathcal{C}$ with the same domain, and g_1 is injective, then the categorical pushout of g_1 and g_2 exists in $\mathcal{C}$. (This uses the lemma in Part III, 3.4.)

Let F be a functor from $\mathcal{C}$ to spectra. We make the following reasonable assumption.

(C1) F takes homotopy equivalences to homotopy equivalences, and preserves finite coproducts up to homotopy equivalence.

In addition, we make a strange assumption.

(C2) For an object $(E \subset \overline{E})$ in $\mathcal{C}$ where E is discrete, the canonical map (see comment below)

$$F(E \subset \overline{E}) \longrightarrow \prod_{x \in E} F(\{x\})$$

is a homotopy equivalence.

(Agreement: We write $F(K)$ to mean $F(K \subset K)$ if K is a compact ENR. Comment: The canonical map in (C2) is not so obvious: for any $x \in E$, we have the map from $F(E \subset \overline{E})$ to the cofiber of

$$F(E - \{x\} \subset \overline{E} - \{x\}) \longrightarrow F(E \subset \overline{E})$$

which by (C1) is homotopy equivalent to $F(\{x\})$.

The following condition (excision) may or may not be satisfied by F.

(C3) F takes a pushout diagram

$$\begin{array}{ccc} (E_0 \subset \overline{E}_0) & \hookrightarrow & (E_2 \subset \overline{E}_2) \\ \downarrow & & \downarrow \\ (E_1 \subset \overline{E}_1) & \hookrightarrow & (E_3 \subset \overline{E}_3) \end{array}$$

(where all morphisms are injective) to a homotopy pushout diagram of spectra.

Examples: The functor $(E \subset \overline{E}) \longmapsto \underset{=}{Ap}(E \subset \overline{E})$ satisfies C1 and C2. The functor $(E \subset \overline{E}) \longmapsto \underset{=}{A}^{\%}(E \subset \overline{E})$ defined earlier in this subsection satisfies C1, C2 and C3.

Observation. If F satisfies C1, C2, C3 above, then

$$F(E \subset \overline{E}) \simeq \operatorname*{holim}_K \ [F(\text{point}) \wedge (E/E - K)]$$

by a natural chain of homotopy equivalences. (K denotes compact subsets of E, and the slant $/$ denotes a mapping cone.)

Details are given below.

Natural construction. Given a functor F from $\mathcal{C}$ to spectra which satisfies C1 and C2, we construct another functor $F^{\%}$ from $\mathcal{C}$ to spectra which satisfies C1, C2, C3; we then produce a natural transformation

$$\alpha : F^{\%}(E \subset \overline{E}) \longrightarrow F(E \subset \overline{E})$$

which is a homotopy equivalence for $E = \overline{E} =$ point .

4.2. Controlled assembly: the technicalities. To simplify the task, we replace the category $\mathcal{C}$ in 4.1 by a more convenient category $\mathcal{C}'$. An object of $\mathcal{C}'$ is an object $(E \subset \overline{E})$ of $\mathcal{C}$ together with a homeomorphism $E \cong |Y|$, where Y is a finite dimensional incomplete simplicial set $(= \Delta\text{–set})$. We require that the resulting CW–structure on E be controlled, as in 3.2. (For any $x \in \overline{E} - E$ and any neighbourhood U of x in $\overline{E}$, there exists a smaller neighbourhood W of x in $\overline{E}$ such that any cell of E which meets W is contained in U .) We write informally E_n to mean Y_n , so that

$$E \cong (\coprod_n E_n \times \Delta^n)/\sim .$$

A morphism in $\mathcal{C}'$, from $(E_1 \subset \overline{E}_1)$ to $(E_2 \subset \overline{E}_2)$, is a morphism in $\mathcal{C}$ which restricts to a simplicial map from E_1 to E_2 . Such a morphism is a *homotopy equivalence* (by definition) if it becomes a homotopy equivalence in $\mathcal{C}$. Conditions C_1 , C_2 , C_4 in 4.1 make sense for functors from $\mathcal{C}'$ to spectra.

It is not hard to see that the classification of homotopy invariant functors from $\mathcal{C}$ to spectra (up to natural homotopy equivalence) is the same as the classification of homotopy invariant functors from $\mathcal{C}'$ to spectra (up to natural homotopy equivalence). It is not necessary to use the results of Chapman on the homotopy type of ENR 's; it is quite enough to remember that an ENR is a retract of something triangulable. In particular, given a homotopy invariant functor F' from $\mathcal{C}'$ to spectra, define a homotopy invariant functor F from $\mathcal{C}$ to spectra by the formula

$$F(c) = \operatorname*{hocolim}_{c' \to c} F'(c')$$

where the hocolim is taken over the category whose objects are morphisms $c' \to c$ in $\mathcal{C}$, with c' in $\mathcal{C}'$. F is not strictly an extension of F' , but up to natural homotopy equivalence it is. In the same weak sense, F is the unique extension of F' (from $\mathcal{C}'$ to $\mathcal{C}$).

Now for the "natural construction" in 4.1: We may replace $\mathcal{C}$ by $\mathcal{C}'$. For $(E \subset \overline{E})$ in $\mathcal{C}'$, we let

$$F^{\%}(E \subset \overline{E}) = \Big[\coprod_n F(E_n \times \Delta^n \subset \overline{E_n \times \Delta^n})\Big] / \sim$$

where $\overline{E_n \times \Delta^n}$ is the closure of $E_n \times \Delta^n$ in the pullback of

$$\begin{array}{ccc} & & (E_n \times \Delta^n) \cup \{\infty\} \\ & & \Big\downarrow \text{characteristic map} \\ \overline{E} & \longrightarrow & E \cup \{\infty\} \end{array}$$

(and $\cdots \cup \{\infty\}$ denotes one–point compactifications). Note that the inclusion $(E_n \times \{b_n\{\subset \overline{E_n \times \Delta^n}) \longrightarrow (E_n \times \Delta^n \subset \overline{E_n \times \Delta^n})$ is a homotopy equivalence (b_n is the barycenter of Δ^n). Therefore, applying C1 and especially C2 in 4.1, we have a chain of natural homotopy equivalences

$$F^{\%}(E \subset \overline{E}) \simeq \left[\amalg_n \prod_{E_n} F(\text{point}) \right] \sim .$$

This shows that $F^{\%}$ satisfies C3. It also shows that the homotopy groups of $F^{\%}(E \subset \overline{E})$ can be computed from a spectral sequence whose E^2–term is some homology of E – based on locally finite singular chains, that is. It follows that $F^{\%}$ satisfies C1 also. It is obvious that $F^{\%}$ satisfies C_2 . Having checked that much, we define the assembly map

$$\alpha : F^{\%}(E \subset \overline{E}) \longrightarrow F(E \subset \overline{E})$$

to be the map induced by the characteristic maps from $E_n \times \Delta^n$ to E . Clearly α is a homotopy equivalence (in fact, an isomorphism) if E is a point. This completes the “natural construction” in 4.1.

Suppose next that F (from $\mathcal{C}'$ to spectra) satisfies C1, C2 and C3. Then the assembly map

$$F^{\%}(E \subset \overline{E}) \longrightarrow F(E \subset \overline{E})$$

is a homotopy equivalence for all $(E \subset \overline{E})$ in $\mathcal{C}'$. This is easily proved by induction on the dimension of E . On the other hand, we saw that there is a chain of natural homotopy equivalences

$$F^{\%}(E \subset \overline{E}) \simeq \left[\amalg_n \prod_{E_n} F(\text{point}) \right] / \sim .$$

It follows that F is determined up to a chain of natural homotopy equivalences by its value on a point. But the functor F_0 given by

$$(E \subset \overline{E}) \longmapsto \operatorname*{holim}_K \; [F(\text{point}) \wedge (E/E - K)]$$

also satisfies C1, C2, C3, and moreover $F_0(\text{point}) \cong F(\text{point})$. It follows that $F \simeq F_0$ by a chain of natural homotopy equivalences. This proves the observation 4.1.

4.3. Compatibility between microcharacteristics and controlled characteristics. For a control space $(E \subset \overline{E})$, we have the microcharacteristic

$$\ll E \gg \in \operatorname*{holim}_K A^{\%}(E)/A^{\%}(E - K) \quad = A^{\%}(E \subset \overline{E})$$

of 2.9, and the controlled characteristic

$$\langle E \subset \overline{E} \rangle^{SW} \in A(E \subset \overline{E})$$

of 3.4, and an assembly map

$$A^{\%}(E \subset \overline{E}) \longrightarrow A(E \subset \overline{E})$$

(of 4.1, here as a map between infinite loop spaces). One would expect that $\ll E \gg$ is mapped to $\langle E \subset \overline{E} \rangle$ under assembly. This is true, and there is a more precise statement (along the line of 2.8) which is left to the reader. Note that both $\ll E \gg$ and $\langle E \subset \overline{E} \rangle$ are lax natural with respect to morphisms

$$g : (E \subset \overline{E}) \longrightarrow (E' \subset \overline{E}')$$

restricting to a cell–like map $E \to E'$.

Further, if E above is a manifold, then $(E \subset \overline{E})$ is a Poincaré control space, and we have the microchracteristic

$$\ll E \gg^{SW} \in A^{\%}(E \subset \overline{E})^{SW}$$

as well as the controlled characteristic

$$\langle E \subset \overline{E} \rangle^{SW} \in A(E \subset E)^{SW}.$$

See 2.6, 2.9 and 3.8. Again, $\ll E \gg^{SW}$ is mapped to $\langle E \subset \overline{E} \rangle^{SW}$ under assembly. Again, there is a more precise statement along the lines of 2.8.

5. Homology of discrete homeomorphism groups

5.1. The manifold case. For a compact manifold M with boundary, we have the homeomorphism group $TOP(M)$, a simplicial group, and its 0–skeleton $\delta TOP(M)$, a discrete group.

THEOREM (McDuff). The inclusion

$$B\delta TOP(M) \longrightarrow BTOP(M)$$

of classifying spaces is a homology equivalence (i.e., its homotopy fiber has trivial integer homology).

Slightly more generally, suppose that $C \subset M$ is closed, and let $TOP(M,C)$ be the (simplicial) group of all homeomorphisms $f : M \to M$ which are the identity near C. Then the inclusion

$$B\delta TOP(M,C) \longrightarrow BTOP(M,C)$$

is a homology equivalence. This is also proved in . In practive, C is a codimension zero submanifold (compact, with boundary) of ∂M.

5.2. The controlled case. Let $(E \subset \overline{E})$ be a control space. Denote by $TOP(E \subset \overline{E})$ the simplicial group of automorphisms of $(E \subset \overline{E})$, and by $\delta TOP(E \subset \overline{E})$ its 0–skeleton. Assume that

(i) E is an n–manifold without boundary;

(ii) $\overline{E} - E$ is a k–manifold without boundary, $n > k$;

(iii) there exists a fiber bundle p over $\overline{E} - E$, with closed manifold fibers, such that the mapping cylinder $Z(p)$ of p is homeomorphic (rel $\overline{E} - E$) to a neighbourhood of $\overline{E} - E$ in $\overline{E}$.

In effect this means that $\overline{E}$ is a stratified space, in a rather restrictive sense, with two strata E and $\overline{E} - E$. Compare Siebenmann.

THEOREM. In this situation, the inclusion

$$B\delta TOP(E \subset \overline{E}) \longrightarrow BTOP(E \subset \overline{E})$$

is a homology equivalence.

The proof is similar to that of the Theorem in 5.1. See . As a useful corollary, we obtain a “discrete” description (up to homology equivalence) of $BTOP^b(M \times \mathbb{R}^i)$, where M is a closed topological manifold and the superscript b means bounded homeomorphisms as in Part I.

Identify $M \times \mathbb{R}^i$ with the complement of S^{i-1} in the join $M * S^{i-1}$. A homeomorphism $f : M \times \mathbb{R}^i \longrightarrow M \times \mathbb{R}^i$ is *controlled* if it extends to a homeomorphism $M * S^{i-1} \longrightarrow M * S^{i-1}$, equal to the identity on S^{i-1}. Denoting the simplicial group of these controlled homeomorphisms by $TOP^c(M\times\mathbb{R}^i)$, we have an inclusion $TOP^b(M \times \mathbb{R}^i) \longrightarrow TOP^c(M \times \mathbb{R}^i)$.

Observation 1. The inclusion

$$u : TOP^b(M \times \mathbb{R}^i) \longrightarrow TOP^c(M \times \mathbb{R}^i)$$

is a homotopy equivalence.

Observation 2. The inclusion of classifying spaces

$$e : B\delta TOP^c(M \times \mathbb{R}^i) \longrightarrow BTOP^c(M \times \mathbb{R}^i)$$

is a homology equivalence.

The first of these observations is well known. Sketch proof: Let X be a compact subset of the geometric realization of $TOP^c(M\times\mathbb{R}^i)$. Construct a diffeomorphism

$$g : [0, +\infty[\longrightarrow [0, +\infty[$$

with $g'(t) \leq 1$ for all $t \geq 0$, such that conjugation by

$$\widehat{g} : M \times \mathbb{R}^i \longrightarrow M \times \mathbb{R}^i \ ; \ (y, z) \longmapsto (y, g(\|z\|) \cdot z)$$

sends X into the image of u. (Conjugation by $\widehat{g}$ is the mp $f \longmapsto \widehat{g}f\widehat{g}^{-1}$ from $TOP^c(M \times \mathbb{R}^i)$ to itself.) Since g is diffeotopic to the identity, conjugation by $\widehat{g}$ is homotopic to the identity. Moreover, a homotopy h_t can be chosen in such a way that each h_t maps the image of u to itself. Taking X to be the image of a continuous map from a sphere or a disk, we see that u induces isomorphisms on homotopy groups.

Proof of observation 2 (modulo the theorems in 5.1 and 5.2): In the commutative diagram

$$\begin{array}{ccc}
B\delta TOP^c(M \times \mathbb{R}^i) & \overset{e}{\hookrightarrow} & BTOP^c(M \times \mathbb{R}^i) \\
\downarrow & & \downarrow \\
B\delta TOP(M \times \mathbb{R}^i \subset M * S^{i-1}) & \hookrightarrow & BTOP(M \times \mathbb{R}^i \subset M * S^{i-1}) \\
\downarrow \text{restriction} & & \downarrow \text{restriction} \\
B\delta TOP(S^{i-1}) & \hookrightarrow & BTOP(S^{i-1})
\end{array}$$

the columns are fibrations up to homotopy, because the restriction homomorphisms

$$TOP(M \times \mathbb{R}^i \subset M * S^{i-1}) \longrightarrow TOP(S^{i-1}),$$
$$\delta TOP(M \times \mathbb{R}^i \subset M * S^{i-1}) \longrightarrow \delta TOP(S^{i-1})$$

are onto with kernel $TOP^c(M \times \mathbb{R}^i)$ and $\delta TOP^c(M \times \mathbb{R}^i)$, respectively. The middle and lower horizontal arrows in the diagram are homology equivalences.

5.3. The microbundle case. Let $B\Gamma_n^0$ be Haefliger's classifying space for foliations of codimension n ; explanations follow. There is a well known forgetful map

$$B\Gamma_n^0 \longrightarrow BTOP_n$$

where $BTOP_n$ classifies n–dimensional topological microbundles.

> Theorem. This map
>
> $$B\Gamma_n^0 \longrightarrow BTOP_n$$
>
> is a homotopy equivalence.

This is also proved in MacDuff , and attributed to Mather–Thurston. (McDuff writes Γ_0^n in Theorem 1.1 of , but Thurston writes Γ_n^0 in .)

Here is a description of $B\Gamma_n^0$, and of the map in the theorem. Let $\mathcal{C}$ be the simplicial category whose objects in degree k are maps $f : \Delta^k \longrightarrow M$, where M is some n–manifold without boundary. A morphism from $f : \Delta^k \longrightarrow M$ to $g : \Delta^k \longrightarrow N$ is a continuous family of embeddings $e_t : M \longrightarrow N$, for $t \in \Delta^k$, such that $e_t(f(t)) = g(t)$ for all $t \in \Delta^k$. The nerve $B\mathcal{C}$ (a bisimplicial set, or rather a bisimplicial class) classifies topological microbundles of dimension n , i.e.

$$B\mathcal{C} \simeq BTOP_n;$$

this is obvious from the definition of a microbundle. Let

$$\delta\mathcal{C} \subset \mathcal{C}$$

be the simplicial subcategory defined as follows: All objects of $\mathcal{C}$ belong to $\delta\mathcal{C}$; a morphism

$$(e_t)_{t\in\Delta^k} : (f : \Delta^k \to M) \longrightarrow (g : \Delta^k \to N)$$

as above belongs to $\delta\mathcal{C}$ if and only if e_t is independent of t . Then

$$B\delta\mathcal{C} = B\Gamma_n^0$$

(say by definition), and the inclusion map

$$B\delta\mathcal{C} \longrightarrow B\mathcal{C}$$

becomes the map in the Theorem. In view of the theorems in 5.1 and 5.2, it is not surprising that the map is a homology equivalence; but it is even a homotopy equivalence.

5.4. Remark on tangent microbundles. Let M be an n–manifold without boundary, and let

$$u : M \longrightarrow BTOP_n$$

classify the tangent microbundle of M . It follows from the theorem in 5.3 that u lifts to

$$\overline{u} : M \longrightarrow B\Gamma^0_n.$$

This can be seen directly. Any singular simplex $f : \Delta^k \longrightarrow M$ is at the same time an object of $\mathcal{C}$ and of $\delta\mathcal{C}$ in 5.3. Therefore the singular simplicial set SM is contained in the simplicial class of objects of $\delta\mathcal{C}$, which is contained in the bisimplicial class $B\delta\mathcal{C}$, which is contained in $B\mathcal{C}$. The composition

$$SM \hookrightarrow B\delta\mathcal{C} \hookrightarrow B\mathcal{C}$$

classifies the tangent bundle of M .

In more geometric terms: Maps from M^n to $B\delta\mathcal{C}$ classify *foliated* n–microbundles on M , i.e. n–microbundles where a neighbourhood of the zero section is equipped with a topological foliation with leaves transverse to the fibers and of codimension n . It is clear that the tangent microbundle of M is foliated in this sense.

6. Poincaré duality by scanning

The theme of this simple–minded section is an unpractical description of Poincaré duality for manifolds. Let M be a closed manifold, and let $\underline{\underline{E}}$ be a spectrum. For $y \in M$, let

$$\begin{aligned} M/\neg y &= \{y\} \times \text{ cofiber of } (M - \{y\} \hookrightarrow M) \\ &\cong \text{ cofiber of } (M - \{y\} \hookrightarrow M). \end{aligned}$$

The inclusions of M_+ in the various $M/\neg y$ induce

$$Q(M_+ \wedge \underline{\underline{E}}) \hookrightarrow Q((M/\neg y) \wedge \underline{\underline{E}}) \qquad (Q = \Omega^\infty).$$

So, letting y vary, we have a "scanning map"

$$\rho_M : Q(M_+ \wedge \underline{\underline{E}}) \longrightarrow \Gamma(p_M)$$

where $\Gamma(p_M)$ is the space of sections of the fibration p_M over M with point inverses

$$p_M^{-1}(y) = Q((M/\neg y) \wedge \underline{\underline{E}}).$$

One wants to prove that ρ_M is a homotopy equivalence – this is a form of Poincaré duality. What makes it unpractical is that is seems to work for closed manifolds only, not for Poincaré spaces in general. Obviously though, the domain $Q(M_+ \wedge \underline{\underline{E}})$ of ρ_M does not use any manifold structure. Less obviously, p_M can be defined for a Poincaré space M (of formal dimension n) to be the fibration with point inverses

$$p_M^{-1}(y) = \Omega^k Q(\tau_y \wedge \underline{\underline{E}})$$

where $\tau_y \simeq S^{n+k}$ is the fiber over y of the Spivak tangent bundle (a spherical fibration with distinguished section, stably inverse to the Spivak normal fibration). In this situation, we can ask whether ρ_M is as homotopy invariant as its domain and codomain. The answer is: yes.

6.1. Tangent bundle and normal fibration. Fixing M^n, we let $\tau : T \longrightarrow M$ be the fiber bundle with fibers

$$\tau_y = \tau^{-1}(y) = M/\neg y \simeq S^n.$$

We can think of τ as the spherical tangent bundle of M.

Let ν be a Spivak normal fibration for M (with fibers $\nu_y \simeq S^i$); so ν comes with a map

$$\eta : S^{n+i+1} \longrightarrow Th(M, \nu) = \text{Thom space of } \nu$$

such that

$$\text{Thom class } \cap [\eta] = \text{fundamental class } [M].$$

The Thom space $Th(M,\nu)$ is the mapping cone of ν . For each $y \in M$, there is an obvious inclusion

$$\Sigma\nu_y \longrightarrow Th(M,\nu)$$

where the Σ in this case denotes an unreduced suspension.

LEMMA. For each $y \in M$, the composition f_y given by

$$\begin{array}{ccccc} S^{n+i+1} & \xrightarrow{\eta} & TH(M,\nu) & \xrightarrow{\text{diagonal}} & M_+ \wedge TH(M,\nu) \\ & & & & \downarrow \\ & & & & (M/\neg y) \wedge (Th(M,\nu)/\Sigma\nu_y) \end{array}$$

is nullhomotopic.

Proof (and explanation). The last map in the composition is simply an inclusion, if we define $Th(M,\nu)/\Sigma\nu_y$ as a mapping cone. However, it is more illuminating to define it as a quotient space (in any case, the inclusion of $\Sigma\nu_y$ in $Th(M,\nu)$ is a cofibration). Then the image of f_y is contained in the subspace

$$Z = (\neg y/\neg y) \wedge (Th(M,\nu)/\Sigma\nu_y)$$

where $\neg y/\neg y$ is

$$\{y\} \times \text{ cofiber of } (M - \{y\} \xrightarrow{=} M - \{y\}),$$

a subspace of $M/\neg y$. Clearly Z is contractible.

Corollary 1. The compositive map

$$S^{n+i+1} \xrightarrow{\eta} TH(M,\nu) \xrightarrow{\text{diag.}} M_+ \wedge TH(M,\nu) \hookrightarrow (M/\neg y) \wedge Th(M,\nu)$$

has a stable deformation into $(M/\neg y) \wedge \Sigma\nu_y$, well defined up to contractible choice.

Corollary 2. The Whitney sum $\tau \oplus \nu$ has a canonical stable trivialization.

Proof: By the previous corollary, we have stable maps

$$e_y : S^{n+i+1} \longrightarrow (M/\neg y) \wedge \Sigma\nu_y = \tau_y \wedge \Sigma\nu_y$$

for every $y \in M$; they have degree one and depend continuously on y (from the construction).

6.2. Milnor duality. The Milnor duality map

$$\mu : \text{map}\ (Th(M,\nu), \Sigma^{n+i+1}\underline{\underline{E}}) \longrightarrow Q(M_+ \wedge \underline{\underline{E}})$$

is defined as follows: To a stable map

$$g : Th(M,\nu) \longrightarrow \Sigma^{n+i+1}\underline{\underline{E}}$$

it associates the composition

$$S^{n+i+1} \xrightarrow{\eta} Th(M,\nu) \xrightarrow{\text{diag.}} M_+ \wedge Th(M,\nu) \xrightarrow{\text{id}\wedge g} M_+ \wedge \Sigma^{n+i+1}\underline{\underline{E}}$$

which is a stable map from S^0 to $M_+ \wedge \underline{\underline{E}}$.

Clearly μ is defined in homotopy theoretic terms: it only uses the fact that M is a Poincaré space. Further, μ is a homotopy equivalence. Consequently, we can solve our problem about ρ_M by showing that

$$\rho_M\mu : \text{map}\ (Th(M,\nu), \Sigma^{n+i+1}\underline{\underline{E}}) \longrightarrow \Gamma(p_M)$$

has a description in homotopy theoretic terms, and is a homotopy equivalence. (By showing that $\tau \oplus \nu$ has a canonical stable trivialization, we have identified τ with the Spivak tangent bundle of M , which means that $\Gamma(p_M)$ has a description in homotopy theoretic terms.)

6.3. The Yoneda principle. Note that domain and codomain of $\rho_M\mu$ depend on $\underline{\underline{E}}$ (despite appearances). So far we have fixed $\underline{\underline{E}}$, but now it is a good idea to think of $\rho_M\mu$ as a *natural transformation* between functors in the variable $\underline{\underline{E}}$. Indeed, the domain functor is a representable (covariant) functor; therefore, by the Yoneda principle, the natural transformation $\rho_M\mu$ is completely determined by what it does in the particular case

$$\underline{\underline{E}} = \Sigma^{-n-i-1}\underline{Th}(M,\nu)$$

to the *universal element*

$$\text{id} \in \text{map}\ (Th(M,\nu), \underline{Th}(M,\nu)).$$

To put it more radically: We only have to show that the element

$$\rho_M \mu(\mathrm{id})$$

has a description in homotopy theoretic terms.

When $\underline{\underline{E}} = \Sigma^{-n-i-1}\underline{Th}(M,\nu)$, the fibration p_M has fibers

$$p_M^{-1}(y) = \Omega^{n+i+1} Q(\tau_y \wedge Th(M,\nu))$$

(where $\tau_y = M/\neg y$), and $\rho_M\mu(\mathrm{id})$ is the section of p_M whose value at y is the composite (stable) map

$$S^{n+i+1} \longrightarrow \tau_y \wedge Th(M,\nu)$$

from 6.1, Cor. 1. By the same corollary, we can deform the section so that the value at y becomes

$$S^{n+i+1} \xrightarrow{e_y} \tau_y \wedge \Sigma\nu_y \hookrightarrow \tau_y \wedge Th(M,\nu)$$

(with e_y as in the proof of 6.1, cor. 2), for all y .

After this adjustment, which extends to a natural homotopy from $\rho_M\mu$ to another natural transformation, we can say that $\rho_M\mu(\mathrm{id})$ has a description in homotopy theoretic terms: in terms of the stable trivialization $e = \{e_y\}$ of $\tau \oplus \nu$.

6.4. Spelling out. Suppose that $h : M \longrightarrow N$ is a homotopy equivalence between closed manifolds. This is automatically covered by a map of Spivak normal fibrations, hence by a map of Spivak tangent fibrations, and so it induces

$$h_* : \Gamma(p_M) \longrightarrow \Gamma(p_N).$$

Strictly speaking, one should use 6.1, cor. 2 here. What we have proved in 6.3 implies that the square

$$\begin{array}{ccc} Q(M_+ \wedge \underline{\underline{E}}) & \xrightarrow{\rho_M} & \Gamma(p_M) \\ \big\downarrow h_* & & \big\downarrow h_* \\ Q(N_+ \wedge \underline{\underline{E}}) & \xrightarrow{\rho_N} & \Gamma(p_N) \end{array}$$

commutes up to a preferred homotopy.

6.5. The noncompact case. For a noncompact manifold M (without boundary) we have, with exactly the same definitions as before, a scanning map

$$\rho_M : Q(M_+ \wedge \underline{\underline{E}}) \longrightarrow \Gamma_c(p_M)$$

where Γ_c denotes a space of sections with compact support. As before, the map is a homotopy equivalence and can be described in *proper* homotopy theoretic terms. More precisely, it can be described in terms of a canonical stable trivialization of $\tau \oplus \nu$, where τ has fibers

$$\tau_y = M/\neg y$$

(as before) and ν is the Spivak normal fibration of the *proper* Poincaré duality space M (see 3.7). The result can be spelled out as in 6.4: If $h : M \longrightarrow N$ is a *proper* homotopy equivalence between noncompact manifolds, then

$$h_*\rho_M \simeq \rho_N h_* : Q(M_+ \wedge \underline{\underline{E}}) \longrightarrow \Gamma_c(p_N)$$

by a canonical homotopy.

7. Automatic continuity

Let $\mathcal{D}$ be a topological category (with discrete class of objects). A natural transformation

$$e : F \longrightarrow G$$

between continuous functors

$$F, G : \mathcal{D} \longrightarrow \text{ pointed spaces}$$

is a *weak equivalence* if

$$e_X : F(X) \longrightarrow G(X)$$

is a homotopy equivalence for all X in $\mathcal{D}$. More generally, two continuous functors F and G from $\mathcal{D}$ to pointed spaces are *weakly equivalent* if there exists a third continuous functor H and weak equivalences

$$F \longrightarrow H \longleftarrow G.$$

("Weakly equivalent" is indeed an equivalence relation.)

Let $\delta\mathcal{D}$ be the underlying discrete category. We will encounter the following problem (in the case where $\mathcal{D}$ is the category $\mathcal{J}$ of finite dimensional real vector spaces with inner product, from Part I, def. 1.11). Suppose that

$$t : F_1 \longrightarrow F_2$$

is a natural transformation between functors

$$F_1, F_2 : \delta\mathcal{D} \longrightarrow \text{ pointed spaces,}$$

and suppose that both F_1 and F_2 are known to be weakly equivalent to *continuous* functors (from $\mathcal{D}$ to pointed spaces). Does it follow that t is in some sense weakly equivalent to a natural transformation between continuous functors? For $\mathcal{D} = \mathcal{J}$ the question may be hard to answer, but an enlargement of $\mathcal{J}$ works wonders.

7.1. Cofibrant functors. Let F and G be continuous functors from $\mathcal{D}$ to pointed spaces, and let

$$e : F \longrightarrow G$$

be a weak equivalence. Then it may be impossible to find a continuous natural transformation

$$e' : G \longrightarrow F$$

and a natural homotopy

$$h_X : G(X) \times [0, 1] \longrightarrow G(X)$$

from $e_X e'_X$ to the identity transformation. If it is always possible, for fixed G and arbitrary F and e as above, then we call G a *cofibrant* functor. (Example: Suppose that $\mathcal{D} = \delta\mathcal{D}$ is the category with two objects made from the ordered set $\{0, 1\}$. A functor G from $\mathcal{D}$ to pointed spaces is cofibrant if and only if the map $G(0) \to G(1)$ induced by $0 < 1$ is a cofibration.)

There is a standard procedure for replacing an arbitrary

$$G : \mathcal{D} \longrightarrow \text{ pointed spaces}$$

(still continuous) by a weakly equivalent $G\hat{}$ which is cofibrant. The formula is

$$G\hat{}(Y) = \operatorname*{hocolim}_{X \to Y} G(X)$$

where the (topological) homotopy colimit is taken over the (topological) category $\mathcal{D} \downarrow Y$ whose objects are the morphisms $X \longrightarrow Y$ in $\mathcal{D}$ with arbitrary X . Explicitly, $G\hat{}(Y)$ is the geometric realization of the simplicial space

$$n \longmapsto \bigvee_X st_{n+1}(X, Y)_+ \wedge G(X)$$

where the wedge is taken over all objects X in $\mathcal{D}$, and $st_{n+1}(X, Y)$ is the space of all diagrams in $\mathcal{D}$ of the form

$$X \to X_1 \to X_2 \to \cdots \to X_n \to Y.$$

Since $\mathcal{D} \downarrow Y$ has a final object $\mathrm{id} : Y \to Y$, there is a canonical projection

$$G\hat{}(Y) \longrightarrow G(Y).$$

This is a homotopy equivalence, and it is natural in Y .

(Example: Again let $\mathcal{D}$ be the category made from the ordered set $\{0, 1\}$. For arbitrary G we have $G\hat{}(0) \cong G(0)$, and $G\hat{}(1)$ is homeomorphic to the reduced mapping cylinder of the map $G(0) \to G(1)$ induced by $0 < 1$.)

Of course we ought to assume that $\mathcal{D}$ is a small category, otherwise $G\hat{}$ will be too large. Also, pointed spaces are understood to be well–pointed.

Note that if G is a continuous functor from $\mathcal{D}$ to pointed spaces, and F is given by

$$F(X) = G(X) \wedge P$$

for some pointed space P , then

$$F\hat{}(X) \cong G\hat{}(X) \wedge P.$$

7.2. Spaces of natural transformations. Let

$$F, G : \mathcal{D} \longrightarrow \text{ pointed spaces}$$

be continuous functors. Define the *space* of natural transformations

$$\text{nat}\hat{}(F, G)$$

to be the geometric realization of the simplicial set whose n–simplices are the natural transformations

$$t_X : F\hat{}(X) \wedge \Delta^n_+ \longrightarrow G(X) \qquad\qquad (X \text{ in } \mathcal{D}).$$

This has the following useful properties:

(i) $\text{nat}\hat{}(F, G)$ is a bifunctor, covariant in G and contravariant in F.

(ii) If $e : F_1 \longrightarrow F_2$ is a weak equivalence, then

$$e^* : \text{nat}\hat{}(F_2, G) \longrightarrow \text{nat}\hat{}(F_1, G),$$
$$e_* : \text{nat}\hat{}(G, F_1) \longrightarrow \text{nat}\hat{}(G, F_2)$$

are homotopy equivalences for arbitrary G.

(iii) The construction is natural in $\mathcal{D}$. (Let v be a continuous functor from some $\mathcal{D}'$ to $\mathcal{D}$. This induces an evident map

$$v^* : \text{nat}\hat{}(F, G) \longrightarrow \text{nat}\hat{}(Fv, Gv)$$

whenever F and G are continuous functors from $\mathcal{D}$ to pointed spaces.)

Unfortunately, there is no easy composition map

$$\text{nat}\hat{}(F_2, F_3) \times \text{nat}\hat{}(F_1, F_2) \longrightarrow \text{nat}\hat{}(F_1, F_3).$$

But (ii) implies

$$\text{nat}\hat{}(F_1, F_2) \simeq \text{nat }(F_1, F\hat{_2}),$$

and there is an easy composition map

$$\text{nat}\hat{}(F_2, F_3) \times \text{nat}\hat{}(F_1, F\hat{_2}) \longrightarrow \text{nat}\hat{}(F_1, F_3).$$

7.3. The category $\mathcal{J}L$. We want to specialize to $\mathcal{D} = \mathcal{J}L$, an enlargement of $\mathcal{J}$. The objects of $\mathcal{J}L$ are those of $\mathcal{J}$: finite dimensional real vector spaces with inner product. A morphism from V to W in $\mathcal{J}L$ is an equivalence class of pairs (U,g) , where U is another object in $\mathcal{J}$ and

$$g : U \oplus V \longrightarrow W$$

is a norm–preserving radial homeomorphism (which means $\|g(z)\| = \|z\|$ and $g(\lambda z) = \lambda g(z)$ for all $z \in U \oplus V$ and $\lambda \geq 0$). Two such pairs (U,g) and (U',g') are *equivalent* if there exists a norm–preserving radial homeomorphism $h : U \to U'$ such that

$$g'(h \oplus \mathrm{id}_V) = g.$$

The set of morphisms $\hom_{\mathcal{J}L}(V,W)$ has an evident topology making it homeomorphic to

$$TOP(S^{m-1})/TOP(S^{m-k-1})$$

where $k = \dim(V)$ and $m = \dim(W)$. (To see this, choose orthonormal bases for V and W . Think of $TOP(S^{m-1})$ as the group of norm–preserving radial homeomorphisms of $\mathbb{R}^m$, etc. . Then the inclusion

$$TOP(S^{m-k-1}) \longrightarrow TOP(S^{m-1})$$

is obvious.)

A morphism $f : V \to W$ in $\mathcal{J}$ (which is a norm–preserving *linear* map) gives rise to a morphism from V to W in $\mathcal{J}L$: let

$$\begin{aligned} &U = \text{orthogonal complement of } im(f) \text{ in } W, \\ &g : U \oplus V \xrightarrow{\cong} U + im(f) = W. \end{aligned}$$

The pair (U,g) represents a morphism in $\mathcal{J}L$ from V to W . We still call it f , and moreover we shall often use single letters (such as f) to denote morphisms in $\mathcal{J}L$. We see that $\mathcal{J}L$ contains $\mathcal{J}$ as a subcategory. The inclusion

$$\hom_{\mathcal{J}}(V,W) \longrightarrow \hom_{\mathcal{J}L}(V,W)$$

looks like

$$O(m)/O(m-k) \longrightarrow TOP(S^m)/TOP(S^{m-k-1})$$

where $k = \dim(V)$ and $m = \dim(W)$.

Note that $\mathcal{J}L$ is equivalent to a small category.

7.4. Automatic continuity theorems. Write

$$\iota : \delta\mathcal{J}L \longrightarrow \mathcal{J}L$$

for the identity functor (where $\delta\mathfrak{J}L$ is $\mathfrak{J}L$ made discrete). Let

$$F, G : \mathfrak{J}L \longrightarrow \text{pointed spaces}$$

be continuous functors. *Technical hypotheses*: Suppose that G can be delooped, i.e. there exists a continuous functor H from $\mathfrak{J}L$ to pointed spaces such that G is weakly equivalent to

$$V \longmapsto \Omega H(V).$$

THEOREM. The map

$$\iota^* : \mathrm{nat}\hat{\ }(F, G) \longrightarrow \mathrm{nat}\hat{\ }(F\iota, G\iota)$$

is a homotopy equivalence.

COROLLARY 1. If $F\iota$ and $G\iota$ are weakly equivalent, then so are F and G.

Two natural transformations

$$t : F_1 \longrightarrow F_2 \ , \ t' : F_1' \longrightarrow F_2'$$

between functors (say from $\delta\mathfrak{J}L$ to pointed spaces) are *weakly equivalent* if there exists a diagram of functors and natural transformations

$$\begin{array}{ccccc} F_1 & \longrightarrow & F_1'' & \longleftarrow & F_1' \\ \downarrow t & & \downarrow t'' & & \downarrow t' \\ F_2 & \longrightarrow & F_2'' & \longleftarrow & F_2' \end{array}$$

where all horizontal arrows are weak equivalences.

COROLLARY 2. Let $t : F_1 \longrightarrow F_2$ be a natural transformation between functors from $\delta\mathfrak{J}L$ to pointed spaces. If both F_1 and F_2 are weakly equivalent to continuous functors (=functors of the form $F\iota$), then t is weakly equivalent to a natural transformation between continuous functors.

One could be more precise with regard to uniqueness in corollary 2.

Suppose again that G is a continuous functor from $\mathfrak{J}L$ to pointed spaces which can be delooped (as in the technical hypothesis).

COROLLARY 3. If $G\iota$ is weakly equivalent to a functor E from $\delta\mathfrak{J}L$ to pointed T–spaces (where T is some discrete group), then G is weakly equivalent to a continuous functor from $\mathfrak{J}L$ to pointed T–spaces.

Proof: Think of the T–action as an A_∞–map from T to the A_∞–space

$$\mathrm{nat}\hat{}(E,E) \simeq \mathrm{nat}\hat{}(G\iota, G\iota) \simeq \mathrm{nat}\hat{}(G,G).$$

(The A_∞–structures are given by composition, as in 7.2.)

The proof of the theorem itself relies mainly on the McDuff theorem in 5.1, with $M = S^i$ for $i = 0,1,2,\ldots$. This is equally true for the next theorem, which gives means for making a discrete functor continuous. Let

$$F : \delta\mathcal{J}L \longrightarrow \text{pointed spaces}$$

be a functor.

THEOREM. If F sends isotopic morphisms to homotopic maps, then F is weakly equivalent to a continuous functor.

(Two morphisms $f_1, f_2 : V \longrightarrow W$ in $\delta\mathcal{J}L$ are *isotopic* if they belong to the same connected component of the *space* of morphisms $\hom_{\mathcal{J}L}(V,W)$. Note that $\hom_{\mathcal{J}L}(V,W)$ is connected if $\dim(V) < \dim(W)$, empty if $\dim(V) > \dim(W)$, and has two components if $\dim(V) = \dim(W) \neq 0$.)

7.5. Natural homology equivalences. Let

$$F_1, F_2, F_3 : \delta\mathcal{J}L \longrightarrow \text{pointed spaces}$$

be functors, and let

$$t : F_1 \longrightarrow F_2$$

be a natural transformation such that

$$t_V : F_1(V) \longrightarrow F_2(V)$$

is a homology equivalence for every V . (In other words, the homotopy fiber of t_V over any point in $F_2(V)$ has the integral homology of a point.) Suppose also that F_3 can be "delooped" (technical hypothesis in 7.4).

LEMMA. The map

$$t^* : \mathrm{nat}\hat{}(F_2, F_3) \longrightarrow \mathrm{nat}\hat{}(F_1, F_3)$$

is a homotopy equivalence.

This can often be used together with the automatic continuity theorem in 7.4. In contrast to these theorems, the lemma has nothing to do with the special features of the category $\delta\mathcal{J}L$; any other category instead of $\delta\mathcal{J}L$ would do.

8. The manifold face

8.1. The task. Fix a closed manifold M. The manifold face is the left face of the cube pictured in the introduction to Part III. In more relaxed notation, it looks like

$$\begin{array}{ccc} \widetilde{\mathbb{V}}(M) & \xrightarrow{\quad c_1 \quad} & \Omega^{n+1}\underline{\underline{L}}s(M) \\ \Big\downarrow \text{mostly } \Phi & & \Big\downarrow \text{mostly } \boxminus \\ H_{\blacktriangledown}(Z_2; \Omega \underline{\underline{\underline{Wh}}}s(M)) & \xrightarrow{\quad c_3 \quad} & H_{\blacktriangledown}(Z_2; \underline{\underline{A}}s(M)_{\%}) \end{array}$$

where $\underline{\underline{\underline{Wh}}}s(M) = \underline{\underline{\underline{Wh}}}s^{TOP}(M)$, throughout this section. (Convention: An arrow $X \longrightarrow \underline{\underline{E}}$, where X is a space and $\underline{\underline{E}}$ is a spectrum, means a map from X to $Q\underline{\underline{E}}$.) The task is to prove that the manifold face is homotopy commutative.

Recall therefore that $\widetilde{\mathbb{V}}(M)$ is the homotopy fiber of

$$\widetilde{TOP}(M) \longrightarrow G(M),$$

that c_1 is the Sullivan–Wall map, that c_3 is induced by the Waldhausen map

$$\omega : \Omega \underline{\underline{\underline{Wh}}}s(M) \longrightarrow \underline{\underline{A}}s(M)_{\%},$$

that "mostly Φ" is the Φ–map from Part I (introduction) composed on the right with

$$\widetilde{\mathbb{V}}(M) \longrightarrow \widetilde{TOP}(M) \longrightarrow \widetilde{TOP}(M)/TOP(M),$$

and that "mostly $\boxminus$" is the $\boxminus$–map from Part II (also Part III section 4) composed on the left with

$$\widehat{H}^{\blacktriangledown}(Z_2; \underline{\underline{A}}s(M)) \longrightarrow H_{\blacktriangledown}(Z_2; \underline{\underline{A}}s(M))$$

from the norm fibration (Part II section 2) and embellished with a subscript $\%$. The *sub*script $\%$ means passage to homotopy fibers of assembly maps. See also Part III, remark 2.5.

We have not yet explained how and why the Waldhausen map ω is a Z_2–map, although Vogell proves a weaker statement in this direction.

8.2. Homotopy limits and quasifibrations. In this subsection, a *quasifibration* on a simplicial set X is a covariant functor

$$q : \text{ simplices of } X \longrightarrow CW\text{–spaces}$$

taking all morphisms between simplices to homotopy equivalences. (A morphism from an m–simplex x to an n–simplex y is a monotone map $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that $f^*(y) = x$.) A *quasisection* of q is a natural transformation from the functor

$$x \longmapsto \Delta^{|x|}$$

to q . The space of quasisections is denoted by $\Gamma(q)$.

EXAMPLE 1: Any fibration $p : E \to |X|$ over the geometric realization X gives rise to a quasifibration q over X , by

$$q(x) = \text{pullback of } E \to |X| \xleftarrow{cx} \Delta^{|x|}$$

where cx is the characteristic map for x .

EXAMPLE 2: Let $\mathcal{C}$ be a small category, and let

$$u : \mathcal{C} \longrightarrow CW\text{–spaces}$$

be a covariant functor taking all morphisms to homotopy equivalences. This determines a quasifibration q_u on the nerve of $\mathcal{C}$:

$$x = (C_0 \to C_1 \to \cdots \to c_n) \longmapsto u(c_n),$$

where $C_0 \to C_1 \to \cdots \to C_n$ is a typical n–simplex of the nerve. In this case we have

$$\Gamma(q_u) \cong \text{holim } u$$

by inspection. Note in passing that q_u is *regular*, which means that it takes degeneracy morphisms to homeomorphisms.

Quasifibrations and quasisections can be pulled back under simplicial maps, just like fibrations and sections. As an illustration, suppose that we are in the following complicated situation. We have

(i) a simplicial map $f : X \longrightarrow Y$ between based simplicial sets;

(ii) a natural transformation

$$t : q_1 \longrightarrow q_2$$

between quasifibrations q_1 and q_2 on Y ;

(iii) quasisections s_1 of f^*q_1 , and s_2 of q_2 , and a path h in $\Gamma(f^*q_2)$ from t_*s_1 to f^*s_2 .

OBSERVATION: This gives rise to a map

$$\text{hofiber } (f) \longrightarrow \text{hofiber } (q_1(*) \xrightarrow{t} q_2(*))$$

where $s_2(*)$ serves as base point for $q_2(*)$.

Sketch: Convert f into a Kan fibration $\widehat{f}$,

$$\begin{array}{ccc} \widehat{X} & & \\ \uparrow \simeq & \searrow \widehat{f} & \\ X & \xrightarrow{f} & Y. \end{array}$$

Extend s_1 to a quasisection $\widehat{s}_1$ of $\widehat{f}^* q_1$, and extend h to a path $\widehat{h}$ in $\Gamma(\widehat{f}^* q_2)$ from $t_* \widehat{s}_1$ to $\widehat{f}^* s_2$. Restrict $\widehat{s}_1$ and $\widehat{h}$ to

$$\widehat{f}^{-1}(*) := \text{hofiber}(f).$$

Note finally that the quasifibrations $\widehat{f}^* q_1$ and $\widehat{f}^* q_2$ are trivial over $\widehat{f}^{-1}(*)$ (here it may be wise to assume that q_1 and q_2 are *regular*), and that $\widehat{f}^* s_2$ is constant over $\widehat{f}^{-1}(*)$. So s_1 restricts to a map from $\text{hofiber}(f)$ to $q_1(*)$, and h restricts to a nullhomotopy in $q_2(*)$.

The choices in this construction (extensions X, f, s_1, h of X, f, s_1, h) can be made in a functorial way if they are made at the same time; we leave this to the reader.

8.3. The Euler transformation. Fix a vector space V in $\mathcal{J}$, in addition to the manifold M . Let $\mathcal{A}$ be the category of control spaces $(E \subset \overline{E})$ with boundary

$$\overline{E} - E = S(V) = \text{unit sphere in } V,$$

allowing only morphisms between control spaces restricting to the identity $S(V) \to S(V)$ on the boundaries. In $\mathcal{A}$, there is a good notion of homotopy.

Let $\mathcal{D} \subset \mathcal{A}$ consist of the objects

$$a_k = \big(M \times \Delta^k \times V \subset \big(M \times \Delta^k\big) * S(V)\big)$$

(for $k \geq 0$); allow all homotopy equivalences as morphisms. Note that a morphism $a_j \longrightarrow a_k$ in $\mathcal{A}$ is a homotopy equivalence if and only if it restricts to an ordinary homotopy equivalence from $M \times \Delta^j \times V$ to $M \times \Delta^k \times V$. This implies easily a homotopy equivalence

$$|\text{nerve of } \mathcal{D}| \simeq BG(M),$$

where $G(M)$ is the space of homotopy automorphisms.

Let $\mathcal{C} \subset \mathcal{D}$ consist of the single object a_0 and its automorphisms. Obviously, the nerve of $\mathcal{C}$ is isomorphic to $B\delta TOP^c(M \times V)$ as in 5.2. Letting

$$f : \text{nerve } \mathcal{C} \longrightarrow \text{nerve } \mathcal{D}$$

be the inclusion, we get

$$\text{hofiber}(f) \simeq G(M)/\delta TOP^c(M \times V).$$

Now write objects in $\mathcal{D}$ in the form $(E \subset \overline{E})$, but note that all objects in $\mathcal{D}$ are Poincaré control spaces of formal dimension $\dim(M) + \dim(V)$. Using example 2 in 8.2, we see that the functor

$$(E \subset \overline{E}) \longmapsto A(E \subset \overline{E})^{SW}$$

on $\mathcal{D}$ determines a quasifibration q_2 on the nerve of $\mathcal{D}$, and that the Euler characteristics

$$(E \subset \overline{E}) \longmapsto \langle E \subset \overline{E} \rangle^{SW}$$

determine a quasisection s_2 of q_2 . See 3.8. The functor

$$(E \subset \overline{E}) \longmapsto A^{\%}(E \subset \overline{E})^{SW}$$

(notation of 4.1) determines another quasifibration q_1 on the nerve of $\mathcal{D}$, and the Euler microcharacteristics

$$(E \subset \overline{E}) \longmapsto \ll E \subset \overline{E} \gg^{SW}$$

(defined and lax natural for $(E \subset \overline{E})$ in $\mathcal{C}$) determine a quasisection s_1 of f^*q_1 . See 2.9. By 4.3, we have

$$t_* s_1 = f^* s_2$$

where $t : q_1 \longrightarrow q_2$ is the assembly transformation.

We are therefore exactly in the "complicated situation" described at the end of 8.2 (with X = nerve of $\mathcal{C}$, and Y = nerve of $\mathcal{D}$, etc.). It follows that we have constructed a map

$$\text{hofiber}(f) \longrightarrow \text{hofiber}(q_1(a_0) \xrightarrow{t} q_2(a_0))$$

which we can write in the form

$$\mu_V : G(M)/\delta TOP^c(M \times V) \longrightarrow A_{\%}(M \times V \subset M * S(V))^{SW}.$$

See 4.4 for notation.

Of course, we now want to think of μ_V as a natural transformation between functors in the variable V , and preferrably V should be allowed to vary in the category $\delta\mathcal{J}L$. The rule

$$V \longmapsto G(M)/\delta TOP^c(M \times V)$$

is indeed a functor from $\delta\mathfrak{I}L$ to spaces, also with the specific models we have chosen. The rule

$$V \longmapsto A_{\%}(M \times V \subset M * S(V))^{SW}$$

can also be made into a functor from $\delta\mathfrak{I}L$ to spaces, but this is harder. For a morphism

$$V \longrightarrow W$$

in $\delta\mathfrak{I}L$, represented by a (norm–preserving radial) homeomorphism

$$g : U \oplus V \longrightarrow W$$

for some U , the induced map

$$A_{\%}(M \times V \subset M * S(V))^{SW} \longrightarrow A_{\%}(M \times W \subset M * S(W))^{SW}$$

should be given by external product (as in 4.4) with the microcharacteristic $\ll U \gg^{SW}$, followed by g_* . It is not too difficult to make this well defined, but it is more difficult to make it compatible with composition of morphisms and to ensure that μ_V becomes a natural transformation. We have to omit this point now and refer to for *some* of the details.