Automorphisms of Manifolds and Algebraic K–theory

Part III

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Introduction

Let M be a compact topological manifold. Denote by $G(M)$ and $TOP(M)$ the spaces of self homotopy equivalences and self homeomorphisms of M which are the identity near ∂M. One of our goals is to compute

$$\mathcal{V}(M) := \text{hofiber } (TOP(M) \hookrightarrow G(M)),$$

where *hofiber* means homotopy fiber. Parallel to that, we shall study automorphisms of the tangent microbundle τ of M. Let $S\tau$ be the associated spherical fibration over M. Write $Aut(\tau)$ for the space of all bundle map germs from τ to itself which cover id_M and agree with the identity over ∂M. Write $Aut(S\tau^\infty)$ for the space of all stable fiber homotopy equivalences from $S\tau$ to itself which cover id_M and agree with the identity over ∂M. Another goal is to compute

$$\mathcal{V}(\tau) := \text{hofiber}(\text{ Aut } (\tau) \hookrightarrow \text{Aut } (S\tau^\infty)).$$

A third goal is to compute a certain derivative map

$$\nabla + \mathcal{V}(M) \longrightarrow \mathcal{V}(\tau).$$

The map ∇ expresses the fact that any $f \in TOP(M)$ determines a bundle map germ df from τ to τ covering f (its *topological derivative*), while any $g \in G(M)$ determines a stable fiber map Dg from $S\tau$ to $S\tau$ covering g (its *Spivak derivative*). See section 5 for details.

Program 1. Compute the map

$$\nabla : \mathcal{V}(M) \longrightarrow \mathcal{V}(\tau)$$

(and its source, and its target).

This is just the union of the three goals above. Suppose now for a moment that M is smooth. Replacing homeomorphisms by diffeomorphisms and microbundle maps by vector bundle maps in the definitions above, one obtains

$$\nabla^{\text{DIFF}} : \mathcal{V}^{\text{DIFF}}(M) \longrightarrow \mathcal{V}^{\text{DIFF}}(\tau),$$

the smooth version of ∇. This appears to be harder to compute than ∇. But it is worth noting that

$$\text{hofiber } \left(\nabla^{\text{DIFF}}\right) \simeq \text{hofiber } (\nabla) \qquad \text{if } \dim(M) \neq 4,$$

by a theorem of Morlet. (See 6.5, 6.4). The homotopy fiber of ∇^{DIFF} is closely related (by a forgetful map) to $\mathrm{DIFF}(M)$, the space of diffeomorphisms from M to M which are the identity near ∂M . This may help to motivate the program.

Program 1 above (for a topological manifold M) is at the same time a refinement of a standard program in L–theory, and a generalization of a standard program in K–theory (Waldhausen style). We recall these programs.

Let $\widetilde{TOP}(M)$ and $\widetilde{G}(M)$ be the simplicial sets whose k–simplices are the self homeomorphisms and self homotopy equivalence of $\Delta^k \times M$, respectively, which are the identity near $\Delta^k \times \partial M$ and preserve the faces $d_i\Delta^k \times M$ for $0 \le i \le k$. These are the block versions of $TOP(M)$ and $G(M)$; while $\widetilde{G}(M)$ is always homotopy equivalent to $G(M)$, there is usually a big difference between $\widetilde{TOP}(M)$ and $TOP(M)$. See [7] and [47]. Let $Aut(\tau^\infty)$ consists of the stable bundle map germs from τ to τ which cover id_M and are the identity over ∂M . Let

$$\begin{aligned}\widetilde{\mathcal{V}}(M) &: = \text{hofiber } (\widetilde{TOP}(M) \longrightarrow \widetilde{G}(M)),\\ \widetilde{\mathcal{V}}(\tau) &: = \text{hofiber } (Aut(\tau^\infty) \longrightarrow Aut(S\tau^\infty))\\ &\simeq map(M, \Omega(G/TOP)),\end{aligned}$$

where TOP and G are such that $BTOP$ and BG classify stable topological microbundles and stable spherical fibrations, respectively. There is a derivative map ∇ from $\widetilde{\mathcal{V}}(M)$ to $\widetilde{\mathcal{V}}(\tau)$, compatible with the ∇ in program 1. This, or a delooping of it, is better known under the name *normal invariant map.*

Program 2. Compute the map

$$\nabla : \widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau).$$

This was carried out by Sullivan (for simply connected M) and Wall (in general). Their result is described below, after a discussion of assembly maps.

In order to say how K–theory is related to program 1, we need relative notation. Suppose that M is contained as a compact codimension zero submanifold in ∂N , where N is another compact manifold. Denote by $TOP(N, M)$ the space of all homeomorphisms from N to itself which are the identity near the closure of $\partial N - M$. So there is a fibration

$$TOP(N) \hookrightarrow TOP(N, M) \xrightarrow{\text{restriction}} TOP(M).$$

Continuing in this manner, define $\mathcal{V}(N, M)$ and $\mathcal{V}(\tau^N, \tau^M)$ in such a way that there are fibrations

$$\begin{aligned}\mathcal{V}(N) \hookrightarrow \mathcal{V}(N, M) &\longrightarrow \mathcal{V}(M)\\ \mathcal{V}(\tau^N) \hookrightarrow \mathcal{V}(\tau^N, \tau^M) &\longrightarrow \mathcal{V}(\tau^M).\end{aligned}$$

Program 3. Compute the map

$$\nabla : \mathcal{V}(N, M) \longrightarrow \mathcal{V}(\tau^N, \tau^M)$$

in the case where (N, M) is homeomorphic to $(M \times [0,1], M \times \{1\})$.

This is the K–theory/concordance theory program which is central in the work of Waldhausen. His main result will also be described below, after a discussion of assembly maps.

Assembly. Let F be a functor from finite CW–spaces to spectra. Call F *homotopy invariant* if it respects homotopy equivalences, and call F *excisive* if it respects homotopy pushout squares (= homotopy cocartesian squares, [13]) and if $F(\emptyset)$ is contractible. (Excisive implies homotopy invariant.) Given a homotopy invariant F , there exists an excisive $F^{\%}$ and a natural transformation

$$\alpha : F^{\%}(Y) \longrightarrow F(Y)$$

(between functors from finite CW–spaces Y to spectra) which is a homotopy equivalence if Y is a point. Together, $F^{\%}$ and α are characterized by these properties up to natural homotopy equivalence. This existence and uniqueness statement presumably goes back to Quinn, [27], [28], [29], [31], who calls α the *assembly*. Example: The A–theory assembly map, where $F(Y)$ is the A–theory spectrum $\underline{\underline{A}}(Y)$ of Y , and $F^{\%}(Y)$ is homotopy equivalent to $Y_+ \wedge \underline{\underline{A}}(*)$. (It is always true that $F^{\%}(Y)$ is homotopy equivalent to $Y_+ \wedge F(*)$.)

We shall also need assembly in the more general situation where F is a functor on the category of finite CW–spaces Y equipped with spherical fibrations γ . (The morphisms in this category are maps between spaces covered by stable maps between spherical fibrations.) Such an F , with values in the category of spectra, may or may not be *homotopy invariant*, or *excisive*; the definitions are literally the same as before. If F is homotopy invariant, then it is the target of an essentially unique natural transformation

$$a : F^{\%}(Y, \gamma) \longrightarrow F(Y, \gamma),$$

with excisive $F^{\%}$, which is a homotopy equivalence whenever Y is a point. Example: The L–theory assembly map, where $F(Y, \gamma)$ is $\underline{\underline{L}}s^{\blacktriangledown}(\mathbb{Z}\pi_1(Y))$. The decoration s is for *simple*, and $\mathbb{Z}\pi_1(Y)$ has the w–twisted involution, where $w = w_1(\gamma)$ is the first Stiefel–Whitney class of γ . Here $F^{\%}(Y, \gamma)$ is homotopy equivalent to

$$\overline{Y}_+ \wedge_{Z_2} \underline{\underline{L}}s^{\blacktriangledown}(\mathbb{Z}),$$

where $\overline{Y}$ is the double cover of Y determined by $w_1(\gamma)$, and Z_2 acts on $\underset{=}{L}s_{\blacktriangledown}(\mathbb{Z})$ by a sign change involution (and on $\overline{Y}$ by permuting sheets). The example shows that $F^{\%}(Y,\gamma)$ is not always homotopy equivalent to $Y_+ \wedge F(*)$, but $F^{\%}$ is nevertheless very tractable if $F(*)$ is known. See section 2.

Notation: Write

$$\begin{aligned} F_{\%}(Y) &:= \text{hofiber of } (\alpha : F^{?5}(Y) \quad \longrightarrow F(Y)), \\ F_{\%}(Y,\gamma) &:= \text{hofiber of } (\alpha : F^{\%}(Y,\gamma) \longrightarrow F(Y,\gamma)), \end{aligned}$$

as appropriate. If F is a functor with a long name, then it can be more convenient to write $F(Y)^{\%}$ instead of $F^{\%}(Y)$, and so on. $\square$

In an up-to-date wording which is due to Quinn [27], [28], [31], and Ranicki [32], [33], [35], the Sullivan–Wall implementation of program 2 looks like this. From the definitions there is a forgetful map of spectra

$$\underset{=}{L}s_{\blacktriangledown}(\mathbb{Z}\pi_1(M))_{\%} \longrightarrow \underset{=}{L}s_{\blacktriangledown}(\mathbb{Z}\pi_1(M))^{\%}.$$

It induces a map of spaces

$$\Omega^{n+1}Q\left(\underset{=}{L}s_{\blacktriangledown}\left(\mathbb{Z}\pi_1(M)\right)_{\%}\right) \longrightarrow \Omega^{n+1}Q\left(\underset{=}{L}s_{\blacktriangledown}\left(\mathbb{Z}\pi_1(M)\right)^{\%}\right),$$

where $n = \dim(M)$ and Q denotes passage from spectra to their zeroth infinite loop spaces. *This map of spaces is homotopy equivalent to*

$$\nabla : \widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau^M)$$

provided $n \geq 5$. (In this statement, maps are treated as objects of a category whose morphisms are commutative squares.)

In program 3, the assumption on (N,M) leads to some simplifications and extra features. The space $G(N,M)$ is contractible, so that

$$\mathcal{V}(N,M) \simeq TOP(N,M) \cong Aut\left(\tau^{M\times I}, \tau^{M\times\{1\}}\right) =: \mathcal{C}(\tau^M).$$

So the task is to compute

$$\nabla : \mathcal{C}(M) \longrightarrow \mathcal{C}(\tau^M).$$

There are stabilization maps

$$\mathcal{C}(M) \longrightarrow \mathcal{C}(M\times I) \qquad \text{and} \qquad \mathcal{C}(\tau^M) \longrightarrow \mathcal{C}(\tau^{M\times I})$$

which commute with ∇ . The second of these stabilization maps is about $(\dim(M)/3)-$connected, and so is the first if M admits a smooth structure (but it is a general belief that the proviso is unnecessary). See Igusa [16]. Passage to homotopy direct limits under iterated stabilization gives

$$\nabla^{\infty} : \mathcal{C}^{\infty}(M) \longrightarrow \mathcal{C}^{\infty}(\tau^{M}),$$

a map from the stabilized concordance space of M to what one might call the stabilized concordance space of τ^{M} . Waldhausen implicitly computes ∇^{∞} : From the definitions, there is a forgetful map

$$\underline{\underline{A}}(M)_{\%} \longrightarrow \underline{\underline{A}}(M)^{\%}$$

of spectra. It induces a map of spaces

$$\Omega Q(\underline{\underline{A}}(M)_{\%}) \longrightarrow \Omega Q(\underline{\underline{A}}(M)^{\%}).$$

This map is homotopy equivalent to

$$\nabla^{\infty} : \mathcal{C}^{\infty}(M) \longrightarrow \mathcal{C}^{\infty}(\tau^{M}).$$

What we can contribute to program 1 uses, inevitably, a concoction of $L-$theory and $K-$theory or $A-$theory, and assembly maps. It gives precise information in the concordance stable range, less precise information above it, like Waldhausen's result on program 3. Rationally this was anticipated by Burghelea and Fiedorowicz [6], who computed the rational homotopy of $\mathcal{V}(M)$ in the concordance stable range. Their ideas have proved to be quite prophetic and generalizable; see especially remark 5.4 in [6]. (Earlier programmatic work on manifold automorphisms can be found in the book by Antonelli, Burghelea, and Kahn [3].) The concoction of $L-$theory and $K-$theory that Burghelea and Fiedorowicz use is the *hermitian* algebraic $K-$theory of Karoubi [17]. In this we do not follow them, because Part II (=[47]) gives us a more direct way to concoct L and K . (Incidentally, our treatment of *norm maps* in Part II has been vastly generalized and simplified by Adem, Cohen, and Dwyer [

Dimension shift. Let R be a ring with involution. Recall from the introduction to Part II the map

$$\boxminus : \underline{\underline{Lp}}_{\blacktriangledown}(R) \longrightarrow \widehat{H}^{\blacktriangledown}(Z_2; \underline{\underline{Kp}}(R)),$$

where $\underline{\underline{Kp}}(R)$ is the $K-$theory of f.g.projective left $R-$modules and their isomorphisms. Replacing such an isomorphism by its adjoint inverse defines the standard involution on $\underline{\underline{K}}(R)$. Write $\underline{\underline{Lp}}_{\blacktriangledown}(R, -n)$ for $\Omega^{n}\underline{\underline{Lp}}_{\blacktriangledown}(R)$, and write $\ddot{S}^{n}$ to mean S^{n} with the involution

$$(z_1, z_2, \ldots, z_{n+1}) \longmapsto (z_1, -z_2, \ldots, -z_{n+1})$$

in euclidean coordinates. Denote by $\boxminus_{-n}$ the composition

$$\underline{\underline{L}}p_{\blacktriangledown}(R,-n) \xrightarrow{\Omega^n\boxminus} \Omega^n\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}p(R) \xrightarrow{\iota^n} \Omega^n\widehat{H}^{\blacktriangledown}(Z_2;\ddot{S}^n\wedge\underline{\underline{K}}p(R)).$$

The diagonal involution on $\ddot{S}^n\wedge\underline{\underline{K}}p(R)$ is understood, and ι^n is induced by the Z_2–map

$$\underline{\underline{K}}p(R)\cong S^0\wedge\underline{\underline{K}}p(R)\hookrightarrow\ddot{S}^n\wedge\underline{\underline{K}}p(R).$$

It is shown in section 4 (after the proof of 4.4) that ι^n is a homotopy equivalence. With the identifications

$$\Omega^n\widehat{H}^{\blacktriangledown}(Z_2;\ddot{S}^n\wedge\underline{\underline{K}}p(R))\simeq\widehat{H}^{\blacktriangledown}(Z_2;\Omega^n(\ddot{S}^n\wedge\underline{\underline{K}}p(R))$$

(because $\widehat{H}^{\blacktriangledown}(Z_2;\dots)$ is excisive, see remark 2.8 in Part II) and

$$\Omega^n(\ddot{S}^n\wedge\underline{\underline{K}}p(R)) =: \underline{\underline{K}}p(R,-n))$$

(by definition), we can write

$$\boxminus_{-n}:\underline{\underline{L}}p_{\blacktriangledown}(R,-n)\longrightarrow\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}p(R,-n)).$$

Note that $\underline{\underline{K}}p(R,-n)$ is a spectrum with Z_2–action, homotopy equivalent to $\underline{\underline{K}}p(R)$ as a spectrum without Z_2–action.

LK–THEORY AND LA–THEORY. For $n\geq 0$, let

$$\underline{\underline{LK}}p(R,-n)$$

be the homotopy fiber of the composition

$$\underline{\underline{L}}p_{\blacktriangledown}(R,-n)\xrightarrow{\boxminus_{-n}}\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{K}}p(R,-n))\xrightarrow{d_{\mathcal{N}}}S^1\wedge H_{\blacktriangledown}(Z_2;\underline{\underline{K}}p(R,-n))$$

where $d_{\mathcal{N}}$ is the connecting map for the norm cofibration

$$H_{\blacktriangledown}(Z_2;\dots)\longrightarrow H^{\blacktriangledown}(Z_2;\dots)\longrightarrow\widehat{H}^{\blacktriangledown}(Z_2;\dots).$$

For a space Y with spherical fibration γ, let

$$\underline{\underline{LA}}p(Y,\gamma,-n)$$

be the homotopy fiber of the composition

$$\underline{\underline{L}}p_{\blacktriangledown}(\mathbb{Z}\pi_1(Y),-n)\xrightarrow{\boxminus_{-n}}\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n))\xrightarrow{d_{\mathcal{N}}}S^1\wedge H_{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n))$$

where $\underline{\underline{A}}p(Y,\gamma,-n)$ means $\Omega^n(\ddot{S}^n \wedge \underline{\underline{A}}p(Y,\gamma))$, and $\underline{\underline{A}}p(Y,\gamma)$ is $\underline{\underline{A}}p(Y)$ with the γ–twisted duality involution. ($\underline{\underline{A}}p(Y)$ is the K–theory of finitely dominated retractive spaces over Y .) The map $\boxminus_{-n}$ is obtained from the nonlinear version of $\boxminus$ (= construction E in the introduction to Part II) by dimension shift. If a sufficiently general notion of ring up to homotopy with involution existed, then $\underline{\underline{\underline{LA}}}\,p(Y,\gamma,-n)$ could be regarded as a special case of $\underline{\underline{\underline{LK}}}\,p(R,-n)$ for connected Y , with $R = Q(\Omega Y_+)$. (Exercise in speculation: What is the appropriate involution on $Q(\Omega Y_+)$, as a self map? For more details on rings up to homotopy with involution, see Fiedorowicz–Schw”anzl–Vogt [12].)

The definition of $\underline{\underline{\underline{LA}}}\,p(Y,\gamma,-n)$ admits the usual variations: the decoration p can be replaced by h or s throughout, giving

$$\underline{\underline{\underline{LA}}}\,h(Y,\gamma,-n) \text{ and } \underline{\underline{\underline{LA}}}\,s(Y,\gamma,-n)$$

(see the remarks following construction E in the introduction to Part II). The s–version is closest to geometry.

It will be shown in the course of this introduction that precorollary G , just below, is a consequence of pretheorem F , a little further down. We intend to prove pretheorem F in Part IV or V. The purpose of the present instalment is merely to formulate it. It is altogether *the* main theorem of the whole enterprise.

Precorollary G. There exist highly connected maps e_1^M, e_2^M which fit into a commutative square

$$\begin{tikzcd}
\mathcal{V}(M) \arrow[r, "\nabla"] \arrow[d, "e_1^M"] & \mathcal{V}(\tau^M) \arrow[d, "e_2^M"] \\
\Omega Q(\underline{\underline{\underline{LA}}}\,s(M,\nu,-n)_{\%}) \arrow[r, "f"] & \Omega Q(\underline{\underline{\underline{LA}}}\,s(M,\nu,-n)^{\%})
\end{tikzcd}$$

where ν is the normal bundle, f is forgetful, and $n = \dim(M)$.

A geometric square. Recall from Part I (= [46]) the map

$$\Phi^s : \widetilde{TOP}(M)/TOP(M) \longrightarrow QH_{\blacktriangledown}(Z_2; \Omega \underline{\underline{\underline{WH}}}s^{TOP}(M)),$$

where $\underline{\underline{\underline{Wh}}}s^{TOP}(M)$ is a spectrum designed geometrically in such a way that $Q\Omega^2\underline{\underline{\underline{Wh}}}s^{TOP}(M)$ is homotopy equivalent to the stabilized concordance space $\mathcal{C}^\infty(M)$. Note also that

$$\widetilde{TOP}(M)/TOP(M) \simeq \widetilde{\mathcal{V}}(M)/\mathcal{V}(M)$$

because $\widetilde{G}(M) \simeq G(M)$. In section 4, we shall construct by similar methods a similar map

$$\Phi : \widetilde{\mathcal{V}}(\tau^M)/\mathcal{V}(\tau^M) \longrightarrow QH_\blacktriangledown(Z_2;\Omega\underline{\underline{\underline{Wh}}}^{TOP}(\tau^M)),$$

where $\underline{\underline{\underline{Wh}}}^{TOP}(\tau^M)$ is a spectrum designed geometrically in such a way that $Q\Omega^2\underline{\underline{\underline{Wh}}}^{TOP}(\tau^M)$ is the stabilized concordance space $\mathcal{C}^\infty(\tau^M)$ of τ^M. (This map Φ is also highly connected, like its mate; more details are given below, after pretheorem F.) The square

$$\begin{tikzcd}
\widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \arrow[r, "\nabla"] \arrow[d, "\Phi^s"] & \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \arrow[d, "\Phi"] \\
QH_\blacktriangledown(Z_2;\Omega\underline{\underline{\underline{Wh}}}s^{TOP}(M)) \arrow[r, "\nabla"] & QH_\blacktriangledown(Z_2;\Omega\underline{\underline{\underline{Wh}}}^{TOP}(\tau))
\end{tikzcd}$$

commutes (with $\tau = \tau^M$). The lower horizontal arrow is induced by a Z_2–map

$$\nabla : \Omega\underline{\underline{\underline{Wh}}}s^{TOP}(M) \longrightarrow \Omega\underline{\underline{\underline{Wh}}}^{TOP}(\tau)$$

which on applying $Q\Omega$ becomes the usual map

$$\nabla^\infty : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(\tau).$$

Now compose with the projection maps from $\widetilde{\mathcal{V}}(M)$ to $\widetilde{\mathcal{V}}(M)/\mathcal{V}(M)$ and from $\widetilde{\mathcal{V}}(\tau)$ to $\widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)$ to obtain another commutative square

$$(\&)\qquad\begin{tikzcd}
\widetilde{\mathcal{V}}(M) \arrow[r, "\nabla"] \arrow[d, "\Phi^s\cdot\mathrm{pr}"] & \widetilde{\mathcal{V}}(\tau) \arrow[d, "\Phi\cdot\mathrm{pr}"] \\
QH_\blacktriangledown(Z_2;\Omega\underline{\underline{\underline{Wh}}}s^{TOP}(M)) \arrow[r, "\nabla"] & QH_\blacktriangledown(Z_2;\Omega\underline{\underline{\underline{Wh}}}^{TOP}(\tau)).
\end{tikzcd}$$

This square deserves to be computed because its vertical homotopy fibers receive highly connected maps from $\mathcal{V}(M)$ and $\mathcal{V}(\tau)$, respectively (by construction). What is more, the four corners of the square admit algebraic expressions (not quite if $\dim(M) < 5$). There are the Sullivan–Wall–Quinn–Ranicki maps

$$c_1 : \widetilde{\mathcal{V}}(M) \longrightarrow \Omega Q(\underline{\underline{L}}s_\blacktriangledown(\mathbb{Z}\pi_1(M), -n)_{\%}),$$
$$c_2 : \widetilde{\mathcal{V}}(\tau) \longrightarrow \Omega Q(\underline{\underline{L}}s_\blacktriangledown(\mathbb{Z}\pi_1(M), -n)^{\%}),$$

with $n = \dim(M)$, which are homotopy equivalences if $n \geq 5$. (Actually, c_2 is always a homotopy equivalence.) By Waldhausen's theory (implementation of the stabilized program 3) the spectra $\Omega\underline{\underline{Whs}}^{TOP}(M)$ and $\Omega\underline{\underline{Wh}}^{TOP}(\tau)$ are homotopy equivalent to $\underline{\underline{A}}s(M)_{\%}$ and $\underline{\underline{A}}s(M)^{\%}$, respectively. (Earlier we stated an infinite loop space version of this result.) It will be shown in Part IV or Part V that these homotopy equivalences can be realized by Z_2–maps from $\Omega\underline{\underline{Whs}}^{TOP}(M)$ and $\Omega\underline{\underline{Wh}}^{TOP}(\tau)$ to $\underline{\underline{A}}s(M,\nu,-n)_{\%}$ and $\underline{\underline{A}}s(M,\nu,-n)^{\%}$, respectively. (Vogell [40] proves a weaker result in this direction.) Granting this, we have homotopy equivalences

$$
\begin{aligned}
c_3 : QH_{\blacktriangledown}(Z_2;\Omega\underline{\underline{Whs}}^{TOP}(M)) &\longrightarrow QH_{\blacktriangledown}(Z_2;\underline{\underline{A}}s(M,\nu,-n),\\
c_4 : QH_{\blacktriangledown}(Z_2;\Omega\underline{\underline{Wh}}^{TOP}(\tau)) &\longrightarrow QH_{\blacktriangledown}(Z_2;\underline{\underline{A}}s(M,\nu,-n)^{\%}).
\end{aligned}
$$

An algebraic square. For a qualified guess concerning the maps in the geometric square (&) , it is necessary to know more about the operations $^{\%}$ and $_{\%}$ (from the assembly briefing). To begin with, they are or can be made natural. In particular, if F is a homotopy invariant functor from finite CW –spaces with spherical fibrations to spectra, and if the finite group T acts naturally on $F(Y,\gamma)$ for all Y and γ , then T also acts naturally on $F(Y,\gamma)^{\%}$ and on $F(Y,\gamma)_{\%}$. In this situation there are natural homotopy equivalences

$$
\begin{aligned}
H_{\blacktriangledown}(T;F(Y,\gamma))^{\%} &\simeq H_{\blacktriangledown}(T;F(Y,\gamma)^{\%}),\\
H_{\blacktriangledown}(T;F(Y,\gamma))_{\%} &\simeq H_{\blacktriangledown}(T;F(Y,\gamma)_{\%}).
\end{aligned}
$$

(Proof of the first line: The functor on the right has the properties characterizing the functor on the left. It is excisive, transforms naturally to $H_{\blacktriangledown}(T;F(Y,\gamma))$, and the transformation is a homotopy equivalence whenever Y is a point. The second line follows from the first.)

All this being understood, we can write down a commutative square of spectra

(**)

$$
\begin{tikzcd}
\Omega\underline{\underline{L}}s_{\blacktriangledown}(\mathbb{Z}\pi_1(M),-n)_{\%} \arrow[r] \arrow[d, "(d_{\mathcal{N}}\cdot\boxminus_{-n})_{\%}"] & \Omega\underline{\underline{L}}s_{\blacktriangledown}(\mathbb{Z}\pi_1(M),-n)^{\%} \arrow[d, "(d_{\mathcal{N}}\cdot\boxminus^{-n})^{\%}"] \\
H_{\blacktriangledown}(Z_2;\underline{\underline{A}}s(M,\nu,-n)_{\%}) \arrow[r] & H_{\blacktriangledown}(Z_2;\underline{\underline{A}}s(M,\nu,-n)^{\%})
\end{tikzcd}
$$

with forgetful horizontal arrows. The map $d_{\mathcal{N}}\cdot\boxminus_{-n}$ was used earlier, in a delooped form, to define $\underline{\underline{LA}}\,s(Y,\gamma,-n)$ for all Y and γ .

Pretheorem F. The cubical diagram with back face (&) , front face Q(&&) , and edges c_1,c_2,c_3,c_4 connecting back face and front face, is homotopy commutative. In particular, the squares (&) and Q(&&) are homotopy equivalent if $n\geq 5$.

(Explanation of terms: $Q(\&\&)$ is the square of spaces and maps obtained from $(\&\&)$ by applying the functor Q . A diagram of spaces and maps is *homotopy commutative* if there exists a strictly commutative diagram of spaces and maps to which it is isomorphic in the homotopy category of spaces and maps.)

Assuming pretheorem F, we have maps from the vertical homotopy fibers in $(\&)$ to the vertical homotopy fibers in $Q(\&\&)$, which are

$$\Omega Q(\underline{\underline{\underline{LA}}}\,s(M,\nu,-n)_{\%}) \qquad \text{and} \qquad \Omega Q(\underline{\underline{\underline{LA}}}\,s(M,\nu,-n)^{\%})$$

because the operations $_{\%}$ and $^{\%}$ preserve natural fibrations up to homotopy. These maps between vertical homotopy fibers are homotopy equivalences for $n \geq 5$. Since the vertical homotopy fibers in $(\&)$ also receive highly connected maps from $\mathcal{V}(M)$ and $\mathcal{V}(\tau^M)$, respectively, this proves precorollary G modulo pretheorem F . One finds that e_1^M in precorollary G is k–connected if $n \geq 5$ and if

$$\Phi^s : \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \longrightarrow QH_{\blacktriangledown}(Z_2; \Omega\underline{\underline{\underline{Wh}}}\,s^{TOP}(M))$$

is $(k+1)$–connected. Further, Φ^s is $(k+1)$–connected if k belongs to the topological concordance stable range for M , by theorem A in Part I. Similarly, e_2^M in precorollary G is k–connected if

$$\Phi : \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \longrightarrow QH_{\blacktriangledown}(Z_2; \Omega\underline{\underline{\underline{Wh}}}^{TOP}(\tau))$$

is componentwise $(k+1)$–connected (which means that all homotopy fibers are k–connected or empty). This will be the case if $n \geq 5$ and $k < n-1$, and if $k-1$ is in the smooth concordance stable range for the disk D^n . See 5.13.

There are relative versions of pretheorem F and precorollary G, for a pair (N^{n+1}, M^n) of compact manifolds with $M \subset \partial N$. When (N, M) is homeomorphic to $(M \times I, M \times \{1\})$, the relative pretheorem F degenerates in a rather remarkable way into Waldhausen's computation of

$$\nabla^\infty : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(\tau^M)$$

and nothing else. The LA–theory of the pair $(M \times I, M \times \{1\})$ is just the A–theory of M . See 4.6.

*

The cube in pretheorem F is pictured in Fig. 1. We have omitted the letter Q in six places. As a result, six vertices of the cube are spectra, and only two are spaces, and some edges are maps from spaces to spectra. But our conventions allow that. Homotopy commutativity of the surgery theory face is well known, but we recall it at the end of section 4. Homotopy commutativity of the concordance theory face, *before* passage to homotopy Z_2–orbit spectra, is implicit in Waldhausen's work; we make it explicit in section 6. Commutativity of the algebraic face is obvious. Homotopycommutativity of the geometric face follows from the definitions (in section 5). The manifold theory face and the bundle theory face are the new faces, to be investigated in future instalments. And then there is the interior of the cube: A cubical diagram all of whose faces are homotopy commutative need not be homotopy commutative!

Leitfaden. Section 1 is about the naturality properties of the constructions from Part II. These constructions relate the L–theory of $\mathbb{Z}\pi_1(Y)$ or of $Q(\Omega Y_+)$ to the K–theory of $\mathbb{Z}\pi_1(Y)$ or of $Q(\Omega Y_+)$, for a pointed space Y. The base point in Y makes it difficult to state naturality properties, so we have to make efforts to abolish it. This is tedious but necessary: no assembly without naturality. Section 2 is about assembly, and should be easy to read. Section 3 is a digest of constructions leading from geometric bordism theories to various types of L–theory. These constructions go back to Ranicki [33], [34] and in part to Mishchenko [23]. It is not refreshing to see them here again in full detail and with ornaments. Our excuse is that we make heavy demands on these constructions. In particular, L–theory to us means nonlinear L–theory as in section 5 and 6 of Part II, because we must keep a door open in nonlinear K–theory alias A–theory. Section 4 again shoul

top face = SURGERY THEORY
back face = GEOMETRIC SQUARE
right face = BUNDLE THEORY
left face = MANIFOLD THEORY
front face = ALGEBRAIC SQUARE
bottom face = CONCORDANCE THEORY

Fig. 1

1. Naturality

In Part II we associated many K–theoretic and L–theoretic spectra to a fixed space Y , and constructed maps between them. The task is now to make these spectra and maps behave naturally in Y , if necessary by modifying some definitions. In particular, definitions involving a base point cannot be permitted any more.

We begin with the *linear* (i.e. pre–Waldhausen) K– and L–theories. The point of view is due to Quinn [30], but the language we use is that of L"uck [19], chapter II, and tom Dieck [39]. See also B. Mitchell [24].

Following a suggestion of MacLane [20], we use the word *ringoid* to mean a small category in which all morphism sets come equipped with an abelian group structure, and composition of morphisms is bilinear. A *ringoid with involution* is a ringoid $\mathcal{R}$ together with a ringoid isomorphism

$$j : \mathcal{R} \longrightarrow \mathcal{R}^{\mathrm{op}}$$

such that the composite functor

$$\mathcal{R} \xrightarrow{j} \mathcal{R}^{\mathrm{op}} \xrightarrow{j^{\mathrm{op}}} \mathcal{R}$$

is the identity.

1.1 Examples. A ringoid with one object is just a ring; a ringoid with involution, with one object, is a ring with involution.

Any small category $\mathcal{C}$ gives rise to a ringoid $\mathbb{Z}\mathcal{C}$ having the same objects as $\mathcal{C}$. The set of morphisms from x_0 to x_1 in $\mathbb{Z}\mathcal{C}$ is the free abelian group generated by the set of morphisms from x_0 to x_1 in $\mathcal{C}$.

In particular, taking $\mathcal{C}$ to be the fundamental *groupoid* $\pi_1(Y)$ of a space Y , cf. Spanier [37], we obtain a ringoid $\mathbb{Z}\pi_1(Y)$. Objects in $\mathbb{Z}\pi_1(Y)$ are points of Y , and a morphism from y_0 to y_1 is a finite formal linear combination $\Sigma n_g \cdot g$, where the g are path classes beginning in y_0 and ending in y_1 , and the n_g are integers.

Let $w : \overline{Y} \to Y$ be a double covering. Unfortunately w does not, as one might expect, determine an involution on $\mathbb{Z}\pi_1(Y)$. But is does determine an involution on an equivalent category $\mathbb{Z}\pi_1(Y, w)$. The objects of $\mathbb{Z}\pi_1(Y, w)$ are the points of $\overline{Y}$, not Y ; but a morphism from x_0 to x_1 in $\mathbb{Z}\pi_1(Y, w)$ is the same as a morphism from $w(x_0)$ to $w(x_1)$ in $\mathbb{Z}\pi_1(Y)$. The involution is trivial on objects, and maps

$$\Sigma n_g \cdot g : x_0 \longrightarrow x_1$$

(a typical morphism) to

$$\Sigma \text{ sign } (g) \cdot n_g \cdot g^{-1} : x_1 \longrightarrow x_0,$$

where the *sign* of a path class g from $w(x_0)$ to $w(x_1)$ is $+1$ if g lifts to a path class from x_0 to x_1 in $\overline{Y}$, and -1 otherwise.

Let $\mathcal{R}$ be a ringoid. A *left* $\mathcal{R}$*–module* is a covariant functor from $\mathcal{R}$ to abelian groups which is homomorphic on morphism sets; a *right* $\mathcal{R}$*–module* is a left $\mathcal{R}^{\mathrm{op}}$–module. A left $\mathcal{R}$–module is *free on one generator* if it is representable (that is, isomorphic to a morphism functor $\hom(x,-)$ for some object x in $\mathcal{R}$). It is *finitely generated free* if it is isomorphic to a finite direct sum of representable ones, and just *free* if it is isomorphic to an arbitrary direct sum of representable ones. It is *projective* if it is a direct summand of a free one, and *finitely generated projectiv* if it is a direct summand of a f.g. free one.

Left $\mathcal{R}$–modules form a category in which the morphisms are natural transformations. Exercise for the reader: prove that a left $\mathcal{R}$–module P is projective if and only if any $\mathcal{R}$–module epimorphism with target P splits.

The *tensor product* of a right $\mathcal{R}$–module P with a left $\mathcal{R}$–module Q is the abelian group

$$P \otimes_{\mathcal{R}} Q := \left(\bigoplus_x P(x) \otimes Q(x) \right) / \sim .$$

Here x runs over the objects of $\mathcal{R}$, and $\sim$ denotes the "coend" relations:

$$f^*(p) \otimes q \sim p \otimes f_*(q)$$

where $f : x_0 \to x_1$ is a morphism in $\mathcal{R}$ and $p \in P(x_1), q \in Q(x_0)$. (See [20] for coends in general.) The elements of $P \otimes_{\mathcal{R}} Q$ are called *pairings* between P and Q. Such a pairing z between P and Q is *nondegenerate* if the following holds: for any left $\mathcal{R}$–module T, the homomorphism of abelian groups

$$\hom_{\mathcal{R}}(Q,T) \longrightarrow P \otimes_{\mathcal{R}} T \ ; \ f \longmapsto (\mathrm{id}_P \otimes f)_*(z)$$

is an isomorphism, and for any right $\mathcal{R}$–module U, the homomorphism

$$\hom_{\mathcal{R}}(P,U) \longrightarrow U \otimes_{\mathcal{R}} Q \ ; \ f \longmapsto (f \otimes \mathrm{id}_Q)_*(z)$$

is an isomorphism.

1.2. Examples. Choose objects x_0 and x_1 in $\mathcal{R}$, and let

$$P = \hom(-, x_1) \quad \text{and} \quad Q = \hom(x_0, -).$$

There is an isomorphism

$$\hom(x_0, x_1) = P(x_0) \longrightarrow P \otimes_{\mathcal{R}} Q$$

given by

$$f \longmapsto f \otimes \mathrm{id} \ \in \ P(x_0) \otimes Q(x_0) \longrightarrow P \otimes_{\mathcal{R}} Q.$$

Therefore pairings between P and Q correspond to morphisms from x_0 to x_1 , and one finds that nondegenerate pairings correspond to isomorphisms. Consequently, any f.g. free $\mathcal{R}$–module, left or right, occurs in a nondegenerate pairing. (The companion is unique up to unique isomorphism: this is implicit in the notion of nondegeneracy.) Exercise: extend this statement to f.g. projective $\mathcal{R}$–modules.

If $\mathcal{R}$ comes equipped with an involution j , then a left $\mathcal{R}$–module P can also be regarded as a right $\mathcal{R}$–module P^t (compose with $j^{-1} = j^{\mathrm{op}}$). It is then possible to talk about pairings between two left $\mathcal{R}$–modules P and Q : these are elements of $P^t \otimes Q$.

A *based free* left $\mathcal{R}$–module is a left $\mathcal{R}$–module of the form

$$\bigoplus_{s \in S} \hom(x_s, -)$$

where S is any set equipped with a map $s \longmapsto x_s$ to $\mathcal{O}b(\mathcal{R})$. Strictly speaking, a based free left $\mathcal{R}$–module is just a set S and a map from S to $\mathcal{O}b(\mathcal{R})$. A *based projective* left $\mathcal{R}$–module is an object in the idempotent completion of the category of all based free left $\mathcal{R}$–modules. These notions behave covariantly in $\mathcal{R}$. Note that the category of based free left $\mathcal{R}$–modules is equivalent to the category of all free left $\mathcal{R}$–modules, and the category of based projective left $\mathcal{R}$–modules is equivalent to the category of all projective left $\mathcal{R}$–modules.

1.3. Notations. Let $\underline{\underline{K}}h(\mathcal{R})$ and $\underline{\underline{K}}p(\mathcal{R})$ be the K–theories of f.g. based free and f.g. based projective left $\mathcal{R}$–modules (and their isomorphisms), respectively. Alternatively, use Waldhausen's construction, starting with chain complexes of f.g. based free or projective left $\mathcal{R}$–modules. If $\mathcal{R} = \mathbb{Z}\pi_1(Y)$ for a space Y , then it is possible to define $\underline{\underline{K}}s(\mathcal{R})$, the K–theory of f.g. based free left $\mathcal{R}$–modules and their *simple* isomorphisms. We leave it to the reader to make sense of the word *simple.*

It is also clear from 1.2 how $\underline{\underline{L}}p_{\blacktriangledown}(\mathcal{R})$, $\underline{\underline{L}}p^{\blacktriangledown}(\mathcal{R})$, $\underline{\underline{L}}h_{\blacktriangledown}(\mathcal{R})$, and $\underline{\underline{L}}h^{\blacktriangledown}(\mathcal{R})$ have to be defined if $\mathcal{R}$ is a ringoid with involution. (Use chain complexes of based free or projective left $\mathcal{R}$–modules satisfying suitable finiteness conditions, and use symmetric or quadratic structures as appropriate.) If $\mathcal{R}$ is $\mathbb{Z}\pi_1(Y, w)$ as in 1.1, then $\underline{\underline{L}}s_{\blacktriangledown}(\mathcal{R})$ and $\underline{\underline{L}}s^{\blacktriangledown}(\mathcal{R})$ are also defined.

We turn to *nonlinear* studies. The input we used in Part II, sections 4 and 6, was a topological group G , but the real interest was in the classifying space BG. Waldhausen shows in section 2.1 of [42] that, in most K–theoretic investigations, categories of spaces or spectra with a certain type of G–action can be replaced by categories of retractive spaces or spectra over BG . The retractive space approach gives much better naturality properties. Our problem is to make Waldhausen's comparison theorem work in L–theory, too.

Fix a space Y . We study retractive spaces over Y , say

$$X \underset{s}{\overset{r}{\leftrightarrows}} Y \qquad (\text{with } rs = \mathrm{id}_Y)$$

where X is assumed to have a CW–structure relative to Y . We often write X instead of (X, r, s) . A morphism

$$f : X_1 \longrightarrow X_2$$

between retractive spaces over Y is an h*–equivalence* if it is a homotopy equivalence relative to Y (upon forgetting the retractions to Y). If X_1 and X_2 have finitely many cells only (relative to Y), then any h–equivalence from X_1 to X_2 determines a *torsion* in the Whitehead group of Y . If this is zero, the h–equivalence is an s*–equivalence.* A morphism between retractive spaces over Y has a mapping cylinder and a mapping cone. These are also retractive spaces over Y ; see [42], p. 348–349. The suspension of a retractive space X over Y , written $\Sigma_Y X$, is the mapping cone of the morphism to the final object Y (which is also a retractive space over Y).

The *internal smash product* $X \wedge_Y X'$ of two retractive spaces X and X over Y is the quotient space P/P_0 , where P is the homotopy theoretic pullback of

$$X \times X' \xrightarrow{r \times r'} Y \times Y \xleftarrow{\text{diagonal}} Y$$

and P_0 is the homotopy theorectic pullback of

$$X \times Y \cup Y \times X' \xrightarrow{r \times r'} Y \times Y \xleftarrow{\text{diagonal}} Y.$$

To avoid technicalities, think of P/P_0 as a virtual space (see Part I, section 0). Twisted versions exist: Assume that the base space Y comes equipped with a spherical fibration γ having a distinguished section. (The fiber of γ need only be weakly homotopy equivalent to a sphere.) We then have the option to define, for any space V and map $f : V \longrightarrow Y$, another (virtual) space

$$\Sigma^{\gamma} V := (\text{total space of } f^{*}\gamma)/V$$

where we use the canonical section of $f^*\gamma$ to include V into the total space. In particular, with P and P_0 as above, we have the canonical projections

$$P \longrightarrow Y \quad \text{and} \quad P_0 \longrightarrow Y,$$

and we can define the *twisted internal smash product*

$$X \wedge_{Y,\gamma} X' := \Sigma^\gamma P / \Sigma^\gamma P_0.$$

This agrees with the untwisted internal smash product if γ is the trivial fibration with fiber S^0.

Most of these notions have a stable variant. A *retractive CW–spectrum* over Y is a sequence of retractive spaces X_k over Y, with $k \in \mathbb{Z}$, together with injections

$$\varepsilon_k : \Sigma_Y X_k \hookrightarrow X_{k+1} \qquad (k \in \mathbb{Z})$$

of retractive spaces over Y. (We assume that the X_k have relative CW–structures, and that the ε_k respect those.) If Y is a point, this is just Boardman's definition of a CW–spectrum, as described in Adams [1]. Boardman's definition of maps between CW–spectras as equivalence classes of so–called functions has an easy generalization to the case of retractive CW–spectra over Y, which we leave to the reader. A map between retractive CW–spectra over Y, say

$$f : X \longrightarrow X',$$

is an *h–equivalence* if

$$f_* : \lim_{k\to\infty} \pi_{n+k}(X_k, Y) \longrightarrow \lim_{k\to\infty} \pi_{n+k}(X'_k, Y)$$

is an isomorphism for all $n \in \mathbb{Z}$ and all choices of base point in Y. If X and X' are *finite* (generated by finitely many cells only, relative to Y), then there is a more direct definition: f is an h–equivalence if

$$f_k : X_k \longrightarrow X'_k$$

is an h–equivalence between retractive spaces over Y, for sufficiently large k. Such an h–equivalence between finite retractive CW–spectra over Y determines a *torsion* in the Whitehead group of Y (careful with signs). If this is zero, the h–equivalence is an *s–equivalence.*

A retractive CW–spectrum X over Y is *finitely dominated* if there exist retractive CW–spectra X' and X'' over Y, with X'' finite, and an h–equivalence

$$X'' \longrightarrow X \amalg X'.$$

The *internal smash product* of two retractive CW–spectra X and X' over Y is the bispectrum

$$X \wedge_Y X'$$

with terms $X_k \wedge_Y X'_j$, where k and j are integers. If Y is equipped with a spherical fibration γ (with section), then there is a *twisted internal smash product*

$$X \wedge_{Y,\gamma} X'$$

which is also a bispectrum. Fixing Y and γ (with fibers weakly homotopy equivalent to S^q), define a *pairing* between X and X' to be a map

$$S^q \longrightarrow X \wedge_{Y,\gamma} X'.$$

(Remember that this means: a map from the suspension bispectrum of S^q to $X \wedge_{Y,\gamma} X'$.)

We return to retractive spaces

$$X \underset{s}{\overset{r}{\rightleftarrows}} Y$$

over Y , in a homological spirit. To simplify, assume that s is an inclusion. Further technical assumptions are in order. As before, X should have a CW–structure relative to Y . But here we need a specific CW–atlas giving, for each cell e , a homeomorphism to $\mathbb{R}^{|e|}$ whose inverse extends to a map from the radial compactification of $\mathbb{R}^{|e|}$ (with boundary $\partial\overline{\mathbb{R}}^{|e|} \cong S^{|e|-1}$) to X . Then each cell e in $X - Y$ has a preferred midpoint z_e . Letting $\mathcal{R} = \mathbb{Z}\pi_1(Y)$ as in 1.1, we now have a graded based free left $\mathcal{R}$–module

$$cl^Y(X)$$

which in degree n equals

$$cl^Y_n(X) := \bigoplus_e \hom(r(z_e), -)$$

where e runs over the n–cells of $X - Y$. To make it into a chain complex, we need differentials

$$d_n : cl^Y_n(X) \longrightarrow cl^Y_{n-1}(X)$$

for all $n > 0$; these should be left $\mathcal{R}$–module morphisms. To specify d_n is to specify a matrix (a_{ef}) with one entry

$$a_{ef} \in \hom(r(z_e), r(z_f))$$

for each $(n-1)$–cell e and n–cell f in $X - Y$. This *incidence morphism*

$$a_{ef} : r(z_e) \longrightarrow r(z_f)$$

in $\mathcal{R}$ is obtained as follows. Form

$$Y \cup e \cup f,$$

a retractive space over Y (and a subquotient of X in the world of retractive spaces over Y). The attaching map for the n–cell f is a map from $\partial\overline{\mathbb{R}}^n$ to $Y \cup e$. Deforming slightly if necessary, we can assume that it is transverse to the midpoint z_e of e . The transverse preimage of z_e in $\partial\overline{\mathbb{R}}^n$ is a finite set of oriented points x_i . Each x_i determines a path g_i in Y from $r(z_e)$ to $r(z_f)$: the image under r of the ray connecting x_i with the midpoint z_f of the cell $f \cong \mathbb{R}^n$. Let

$$a_{ef} = \Sigma \ \mathrm{sign}(x_i) \cdot [g_i],$$

where $\mathrm{sign}(x_i)$ is $+1$ if x_i is positively oriented, and -1 otherwise.

Let X' be another retractive space over Y (with relative CW–structure). Remember that

$$X \wedge_Y X' = P/P_0$$

where P and P_0 are the homotopy pullbacks of

$$X \times X' \longrightarrow Y \times Y \xleftarrow{\text{diagonal}} Y,$$
$$(X \times Y) \cup (Y \times X') \longrightarrow Y \times Y \xleftarrow{\text{diagonal}} Y,$$

respectively. Looking at the vertical homotopy fibers in the homotopy pullback square

$$\begin{tikzcd}
P_0 \arrow[r] \arrow[d, "\text{projection}"] & P \arrow[d, "\text{projection}"] \\
(X \times Y) \cup (Y \times X') \arrow[r] & X \times X'
\end{tikzcd}$$

we find that the cellular filtration on

$$(X \times X')/((X \times Y) \cup (Y \times X')) \cong (X/Y) \wedge (X'/Y)$$

determines an inverse image filtration on P/P_0 which is still hemicellular (Part II, beginning of section 5). Here we assume that Y has the homotopy type of a CW–space, to be on the safe side. The hemicellular chain complex

$$hcl(P/P_0) = hcl(X \wedge_Y X')$$

(part II, 5.1) is free over $\mathbb{Z}$, with one generator in dimension $m+n$ for each triple $(e, f, [g])$ where e is an m–cell in $X - Y$, where f is an n–cell in $X' - Y$, and where $[g]$ is a path class in Y from $r'(z_f)$ to $r(z_e)$. It is not hard to deduce a canonical isomorphism

$$hcl(X \wedge_Y X') \cong cl^Y(X)^t \otimes_{\mathcal{R}} cl^Y(X').$$

The twisted version of this result states that

$$\Sigma^{-q} hcl(X \wedge_{Y,\gamma} X') \cong cl^Y(X)^t \otimes_{\mathcal{R}} cl^Y(X')$$

where γ has fibers weakly homotopy equivalent to S^q . Some explanations are in order: Firstly, the filtration on

$$X \wedge_{X,\gamma} X'$$

can be obtained much as before, but it is hemicellular only after some reindexing. Secondly, read

$$\mathcal{R} = \mathbb{Z}\pi_1(Y, w)$$

here, where $w : \overline{Y} \to Y$ is the orientation cover associated to γ . Use the forgetful equivalence of ringoids

$$\mathbb{Z}\pi_1(Y, w) \longrightarrow \mathbb{Z}\pi_1(Y)$$

to make $cl^Y(X)$ and $cl^Y(X')$ into chain complexes of modules over $\mathbb{Z}\pi_1(Y, w)$. Finally, the formula

$$\Sigma^{-q} hcl(X \wedge_{Y,\gamma} X') \cong cl^Y(X)^t \otimes_{\mathcal{R}} cl^Y(X')$$

remains meaningful and correct if X and X' are retractive CW–spectra over Y . Define $cl^Y(X)$ to be

$$\lim_{k \to \infty} \Sigma^{-k} cl^Y(X_k),$$

where X_k is the k–th term of X . Working over $\mathbb{Z}\pi_1(Y)$, we can arrange this to be a chain complex of free based left modules; over $\mathbb{Z}\pi_1(Y, w)$, it is still a chain complex of free left modules.

Keeping the notation, we now return to pairings. Let

$$\eta : S^q \longrightarrow X \wedge_{Y,\gamma} X'$$

be such a pairing. Suppose also that X and X' are finitely dominated retractive CW–spectra over Y . Suppose finally that the map η is hemicellular (see Part II, beginning of section 5); this is not a serious restriction. Then η determines a 0–cycle

$$cl(\eta) \in \Sigma^{-q} hcl(X \wedge_{Y,\gamma} X') \cong cl^Y(X)^t \otimes_{\mathcal{R}} cl^Y(X')$$

which we regard as pairing between $cl^Y(X)$ and $cl^Y(X')$. Call η *nondegenerate* if $cl(\eta)$ is nondegenerate. This means that the chain map

$$\begin{aligned} \hom_{\mathcal{R}}(cl^Y(X'), A) &\longrightarrow cl^Y(X)^t \otimes_{\mathcal{R}} A \\ f &\longmapsto (\mathrm{id} \otimes f)_* cl(\eta) \end{aligned}$$

is an isomorphism in homology, for all left $\mathcal{R}$–modules A . (It is sufficient to check this for free $\mathcal{R}$–modules A on one generator; compare Part II, 3.13.)

1.4. Proposition. Every finitely dominated retractive CW–spectrum X over Y occurs in an essentially unique nondegenerate pairing, with finitely dominated companion.

This calls for a comment. Let $\mathcal{P}_X$ be the simplicial category whose objects in degree j have the form (η, X') , where X' is another finitely dominated retractive CW–spectrum over Y and

$$\eta : \Delta^j_+ \wedge S^q \longrightarrow X \wedge_{Y,\gamma} X'$$

is a j–parameter family of nondegenerate pairings. A morphism in degree j , say from (η, X') to (μ, X'') , is an h–equivalence $f : X' \to X''$ such that

$$(\mathrm{id}_X \wedge f) \cdot \eta = \mu.$$

One way to give a precise formulation of 1.4 is to say that the nerve of $\mathcal{P}_X$ (a bisimplicial set) is *contractible.* (Note that contractible implies nonempty.) Compare Part II, 3.10. By the way, we still assume that Y is homotopy equivalent to a CW–space.

The proof of 1.4 is by reduction to a similar statement in Part II (3.10 and its nonlinear version, proved in section 4 of Part II). Recall therefore how retractive spaces and spectra are related to spaces and spectra with a tame (alias semifree) group action; cf. section 2.1 of [42]. Suppose for this purpose that G is a topological group and that Y comes with a principal G–bundle

$$p : E \longrightarrow Y$$

where E is weakly homotopy equivalent to a point. A functor F_p from the category of retractive spaces over Y (with relative CW–structures) to the category of tame G–CW–spectra (section 5 of Part II) is defined by

$$F_p(X \underset{s}{\overset{r}{\rightleftarrows}} Y) = r^*E/E,$$

where r^*E is the total space of the principal G–bundle r^*p over X , and E is contained in r^*E by means of s . The functor F_p respects suspension and can therefore also be regarded as a functor from retractive CW–spectra over Y to tame G–CW–spectra.

We shall next verify that F_p respects internal smash products. Here the commutative diagram

is useful, where d is the inclusion of the diagonal, where f is given by $f(p(e)) = (e,e)$ for e in E, where $Y^{\&}$ is the space of continuous maps w from $[0,1]$ to $Y \times Y$ with $w(0)$ in $d(Y)$, where $d^{\&}$ is given by $d^{\&}(w) = w(1)$, and where $f^{\&}$ exists because the inclusion of Y in $Y^{\&}$ is a homotopy equivalence and $p \times p$ is a fiber bundle. Note that f is a weak homotopy equivalence; consequently so is $f^{\&}$. Given two retractive spaces X and X' over Y, we write P and P_0 for the homotopy pullbacks of

$$X \times X' \xrightarrow{r \times r'} Y \times Y \xleftarrow{d} Y,$$

$$X \times Y \cup Y \times X' \xrightarrow{r \times r'} Y \times Y \xleftarrow{d} Y,$$

respectively, as always. Equivalently, P and P_0 are the strict pullbacks of

$$X \times X' \longrightarrow Y \times Y \xleftarrow{d^{\&}} Y^{\&},$$

$$X \times Y \cup Y \times X' \longrightarrow Y \times Y \xleftarrow{d^{\&}} Y^{\&},$$

respectively. Using $f^{\&}$ we now have obvious weak homotopy equivalences

$$P \xrightarrow{\simeq} T, P_0 \xrightarrow{\simeq} T_0 \text{ and } P/P_0 \xrightarrow{\simeq} T/T_0$$

where T and T_0 are the strict pullbacks of

$$X \times X' \longrightarrow Y \times Y \xleftarrow{p \times p} E \times_G E,$$

$$X \times Y \cup Y \times X' \longrightarrow Y \times Y \xleftarrow{p \times p} E \times_G E,$$

respectively. But

$$P/P_0 = X \wedge_Y X' \quad \text{and} \quad T/T_0 = F_P(X) \wedge_G F_P(X').$$

This takes care of untwisted (internal) smash products. In the twisted case, a spherical fibration γ on Y will be involved. It is not a severe restriction to assume that γ has the form

$$E \times {}_G J \xrightarrow{\text{projection}} E \times_G (\text{point}) \cong Y$$

for some well–pointed G–space J having the weak homotopy type of a sphere S^q . (See Part II, 6.8.) Assuming this, let μ be the spherical fibration over $E \times_G E$ given by

$$(E \times E) \times_G J \xrightarrow{\text{projection}} (E \times E) \times_G (\text{point}) = E \times_G E.$$

The map f from Y to $E \times_G E$ is covered by a map from γ to μ , and therefore

$$\Sigma^\gamma P/\Sigma^\gamma P_0 = X \wedge {}_{Y,\gamma} X'$$

maps by a homotopy equivalence to

$$\Sigma^\mu T/\Sigma^\mu T_0 = F_P(X) \wedge {}_{G,J} F_P(X').$$

This takes care of twisted internal smash products.

Proof of 1.4. We can assume that Y *is* CW–space. It is not hard to find a retractive CW–subspectrum $Z \subset X$ such that the inclusion of Z in X is an h–equivalence, and Z is generated by countably many cells (relative to Y). Replacing X by Z if necessary, we may assume that X itself is generated by countably many cells relative to Y . But then only countably many cells of Y are involved, so we may suppose that Y has only countably many cells. Moreover, we can concentrate on the case where Y is connected.

In this situation there exist a topological group G and a principal G–bundle

$$p : E \longrightarrow Y$$

such that E is weakly homotopy equivalent to a point (by [21]). The associated functor F_p respects (internal) smash products and therefore pairings, by the preceding discussion. It also maps a nondegenerate pairing between finitely dominated objects to a nondegenerate pairing between finitely dominated objects. All this implies a functor between simplicial categories

$$t : \mathcal{P}_X \longrightarrow \mathcal{P}_{R_P}(X)$$

whose target is the simplicial category which in degree j has objects (η, V) , where V is a finitely dominated tame G–CW–spectrum and

$$\eta : \Delta^j_+ \wedge S^q \longrightarrow F_P(X) \wedge {}_{G,J} V$$

is a j–parameter family of nondegenerate pairings. ($\mathcal{P}_X$ is defined just after 1.4, and we still assume that the spherical fibration γ is given by projecting $E\times_G J$ to Y .) From sections 3 and 5 of Part II, we know that the target of t is contractible (after geometric realization).

Write $w\mathcal{A}$ for the category of finitely dominated retractive CW–spectra over Y , with h–equivalences as morphisms, and write $w\mathcal{B}$ for the category of finitely dominated tame G–CW–spectra, with G–homotopy equivalences as morphisms. The forgetful functors

$$\mathcal{S}_1 : \mathcal{P}_X \longrightarrow w\mathcal{A} \quad \text{and} \quad \mathcal{S}_2 : \mathcal{P}_{F_P(X)} \longrightarrow w\mathcal{B}$$

which forget the pairings but record the companions of X and $F_P(X)$, respectively, fit into a commutative square

$$\begin{tikzcd}
\mathcal{P}_X \arrow[r, "t"] \arrow[d, "\mathcal{S}_1"] & \mathcal{P}_{F_P(X)} \arrow[d, "\mathcal{S}_2"] \\
w\mathcal{A} \arrow[r, "F_P"] & w\mathcal{B} \quad .
\end{tikzcd}$$

The lower horizontal arrow is a homotopy equivalence. (This ia proved in section 2.1 of [42]; see especially 2.1.4.) The vertical arrows are *quasi–fibrations* by inspection; see Dold–Thom [11]. It is clear that t maps each fiber of $\mathcal{S}_1$ to the corresponding fiber of $\mathcal{S}_2$ by a homotopy equivalence. Therefore t is a homotopy equivalence, and $\mathcal{P}_X$ is contractible. □

1.5. Notation. The category of retractive CW–spectra over Y will be denoted by $\mathrm{ret}(Y)$. It is a category with cofibrations and weak equivalences: the weak equivalences are, as a rule, the h–equivalences, and a morphism is a cofibration if it is isomorphic to the inclusion of a subobject. The finitely dominated objects in $\mathrm{ret}(Y)$ form a full subcategory (with cofibrations and weak equivalences) with K–theory spectrum

$$\underset{=}{A}p(Y).$$

Replacing finitely dominated by finite objects gives another K–theory spectrum

$$\underset{=}{A}h(Y),$$

and if furthermore only s–equivalences are allowed as weak equivalences, then one obtains a third K–theory spectrum

$$\underset{=}{A}s(Y).$$

If Y is equipped with a spherical fibration γ, with section, then we have notions of pairing and duality (call this γ–duality) and we define

$$\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma),\underline{\underline{L}}h_{\blacktriangledown}(Y,\gamma),\underline{\underline{L}}s_{\blacktriangledown}(Y,\gamma),\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma),\underline{\underline{L}}h^{\blacktriangledown}(Y,\gamma),\underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma)$$

to be the bordism spectra of finitely dominated γ–quadratic, finite γ–quadratic, finite simple simple γ–quadratic, finitely dominated γ–symmetric, finite γ–symmetric, finite simple γ–symmetric Poincaré objects in $\mathrm{ret}(Y)$, respectively. Using 1.4 and variations, we can (as in Part II) construct γ–duality involutions on (spectra homotopy equivalent to)

$$\underline{\underline{A}}p(Y),\underline{\underline{A}}h(Y),\underline{\underline{A}}s(Y),$$

and we write

$$\underline{\underline{A}}p(Y,\gamma),\underline{\underline{A}}h(Y,\gamma),\underline{\underline{A}}s(Y,\gamma),$$

whenever the involutions matter. We often omit the prefix γ– in γ–duality, γ–symmetric, and so on.

1.6. Remark. These definitions are strictly natural in Y, and not just base–point free, provided one is sufficiently careful in defining retractive spaces over Y, and in defining the suspension Σ_Y. Suggestion: A retractrive space over Y consists of a set S, a map $f : S \to Y$, and a topology $\mathcal{T}$ on the coproduct

$$S \amalg Y \quad (:= S\times\{0\}\cup Y\times\{1\})$$

such that both the inclusion of Y in $S \amalg Y$ and the map

$$r = f \amalg \mathrm{id}_Y : S \amalg Y \longrightarrow Y$$

are continuous.

2. Assembly

We adopt a very category theoretic point of view in describing assembly maps. It has been formulated explicitly by Quinn in the appendix to [29], also in [31], and more implicitly in articles of Waldhausen, e.g. [42], [44]. From this point of view, the goal of assembly is: Given a homotopy invariant functor F from spaces to spectra, to approximat F from the left by an excisive homotopy invariant functor $F^{\%}$.

For a more detailed formulation, we take spaces to mean simplicial sets, and spectra to mean CW–spectra. A functor F from simplicial sets to CW–spectra is *homotopy invarant* if it takes homotopy equivalences to homotopy equivalences. A homotopy invariant F is *excisive* if $F(\emptyset)$ is contractible and if F takes homotopy pushout squares

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "g"] & X_2 \arrow[d] \\ X_3 \arrow[r] & X_4 \end{tikzcd}$$

of simplicial sets to homotopy pushout squares of spectra. (We call the square just above a *homotopy pushout square* if it induces a homotopy equivalence from the double mapping cylinder of f and g to X_4 .) The excision condition implies that F preserves finite coproducts, up to homotopy equivalence. Call F *strongly excisive* if it also preserves infite coproducts, up to homotopy equivalence.

So given a homotopy invariant F , we seek an excisive (maybe strongly excisive) homotopy invariant $F^{\%}$, and a natural transformation

$$\alpha : F^{\%} \longrightarrow F$$

which looks universal among such transformations. It is not hard to find: let

$$F^{\%}(Y) = \operatorname*{hocolim}_{y \in Y} F(\Delta^{|y|})$$

where $|y|$ is the dimension of a simplex $y \in Y$. Here we regard the simplices of Y as the objects of a category: a morphism from y_1 to y_2 is an injective monotone map

$$g : \{0, 1, \ldots, |y_1|\} \longrightarrow \{0, 1, \ldots, |y_2|\}$$

with $g^*(y_2) = y_1$. (The injectivity condition could be omitted, but it simplifies the proof of 2.1 below.) The rule

$$y \longmapsto F(\Delta^{|y|})$$

is a functor from this category to that of CW–spectra, so the homotopy direct limit hocolim is defined (see remark 3. below, however). The standard reference for homotopy limits is Bousfield–Kan [4], in the case of homotopy direct limits especially p. 327. A natural map called *assembly*,

$$\alpha : F^{\%}(Y) \longrightarrow F(Y),$$

can be concocted from the compatible family of maps

$$\Big\{ F(\chi_y) : F(\Delta^{|y|}) \longrightarrow F(Y) \mid y \in Y \Big\}$$

where χ_y is the characteristic map for y (from $\Delta^{|y|}$ to Y).

2.1. Proposition. The functor $F^{\%}$ is homotopy invariant and strongly excisive. (Compare Goodwillie [13].)

First proof. Let $sk_n(Y)$ be the n–skeleton of Y . We have natural inclusions

$$F^{\%}(sk_n(Y)) \hookrightarrow F^{\%}(sk_{n+1}(Y))$$

such that

$$F^{\%}(Y) = \bigcup_n F^{\%}(sk_n(Y)). \tag{!}$$

Applying the homotopy invariance of F to the constant map from a simplex to a point, one finds that

$$F^{\%}(sk_n(Y))/F^{\%}(sk_{n-1}(Y)) \simeq \bigvee_z \Sigma^n F(\text{point}) \tag{!!}$$

where z ranges over the n–simplices of Y . So the natural filtration of $F^{\%}(Y)$ given by (!) leads to a spectral sequence converging to the homotopy groups of $F^{\%}(Y)$, with

$$E^2_{p,q} = H_p(Y; \pi_q(F(\text{point}))$$

as E^2–term. But if the E^2–term is already homotopy invariant, then so is the E^∞–term, which implies the homotopy invariance of $F^{\%}$. Further, the formula (!!) shows that the functor

$$Y \longmapsto F^{\%}(sk_n(Y))/F^{\%}(sk_{n-1}(Y))$$

takes squares of the form

$$\begin{tikzcd} Y_1 \cap Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_1 \cup Y_2 \end{tikzcd}$$

to homotopy pushout squares, and preserves arbitrary coproducts (up to homotopy equivalence). Using induction on n, and then (!), we conclude that the functors

$$Y \longmapsto F^{\%}(sk_n(Y)) \quad \text{and} \quad Y \longmapsto F^{\%}(Y)$$

have these properties, too. Together with homotopy invariance this implies that $F^{\%}$ is strongly excisive. $\square$

Second proof. For each simplex y in Y we have a homotopy equivalence

$$F(\Delta^{|y|}) \longrightarrow F(\text{point})$$

induced by a map from $\Delta^{|y|})$ to a point. By the homotopy invariance property of homotopy colimits we may write

$$\underset{y\in Y}{\text{hocolim}}\ F(\Delta^{|y|}) \simeq \underset{y\in Y}{\text{hocolim}}\ F(\text{point}) \cong |\mathcal{N}(Y)|_+ \wedge F(\text{point})$$

where $\mathcal{N}(Y)$ is the nerve of the category whose objects are the simplices of Y. It is an exercise to show that

$$|\mathcal{N}(Y)| \simeq |Y|$$

by a natural chain of homotopy equivalences. Therefore

$$F^{\%}(Y) \simeq |Y|_+ \wedge F(\text{point})$$

by a natural chain of homotopy equivalences, which proves that $F^{\%}$ is homotopy invariant and strongly excisive. $\square$

2.2. Observation. If F is already excisive, then

$$\alpha : F^{\%}(Y) \longrightarrow F(Y)$$

is a homotopy equivalence for all finite Y, and if F is strongly excisive, then α is a homotopy equivalence for all Y.

Proof: By arguments going back to Eilenberg and Steenrod it is sufficient to verify that α is a homotopy equivalence for $Y =$ point. $\square$

To show that $\alpha = \alpha_F$ is the best possible approximation (from the left) of F by a strongly excisive homotopy invariant functor, suppose that

$$\beta : E \longrightarrow F$$

is another natural transformation with strongly excisive and homotopy invariant E . The commutative square

$$\begin{tikzcd}
E^{\%} \arrow[r, "\alpha_E"] \arrow[d, "\beta^{\%}"] & E \arrow[d, "\beta"] \\
F^{\%} \arrow[r, "\alpha_F"] & F
\end{tikzcd}$$

in which α_E is a homotopy equivalence by 2.2, shows that β essentially factors over α_F .

2.3. Variants. (i) Suppose that the functor F is defined, not on the category of simplicial sets, but on the category $\mathcal{Y}$ of spaces having the homotopy type of a CW–space. If F is homotopy invariant, define $F^{\%}$ by

$$F^{\%}(Y) = (F \cdot R)^{\%}(\operatorname{sing}(Y))$$

where $\operatorname{sing}(Y)$ is the singular simplicial set of Y , and R denotes the geometric realization functor on the category of simplicial sets. Define the assembly

$$F^{\%}(Y) \longrightarrow F(Y)$$

to be the composition

$$(F \cdot R)^{\%}(\operatorname{sing}(Y)) \xrightarrow{\alpha} F(R(\operatorname{sing}(Y)) \xrightarrow{F(t)} F(Y)$$

where t is the tautological map from $R(\operatorname{sing}(Y))$ to Y .

(ii) Suppose that the functor F is defined, not on the category $\mathcal{Y}$ as in (i), but on the category $\mathcal{Y}^{\&}$ of spaces with spherical fibrations. (Details follow.) Then it is still possible to do assembly provided F is homotopy invariant. In detail, $\mathcal{Y}^{\&}$ has objects of the form (Y, γ) , where Y is an object in $\mathcal{Y}$ and γ is a fiber bundle over Y , with a locally trivial section, and with fibers homotopy equivalent as pointed spaces to S^q , for some q depending on γ . (The section is *locally trivial* if it is chartwise constant in suitable bundle charts.) These conditions are preserved under pullback. (Remark 6.8 in Part II shows that they are not too restrictive.) A morphism in $\mathcal{Y}^{\&}$, from (Y, γ) to (Y', γ') , is a map

$$f : Y \longrightarrow Y'$$

together with a fiber homotopy equivalence

$$\Sigma^k_Y \gamma \longrightarrow f^*(\gamma'),$$

with suitable k, respecting the distinguished sections. Here Σ_Y^k is the k–fold fiberwise suspension.

A morphism in $\mathcal{Y}^{\&}$ is a *homotopy equivalence* if the underlying morphism in $\mathcal{Y}$ is a homotopy equivalence. A functor F from $\mathcal{Y}^{\&}$ to the category of CW–spectra is *homotopy invariant* if it preserves homotopy equivalences. Given such a homotopy invariant F, define another homotopy invariant functor $F^{\%}$ by

$$F^{\%}(Y,\gamma) := \operatorname*{hocolim}_{x\in \operatorname{sing}(Y)} F(\Delta^{|x|}, x^*(\gamma)).$$

(Explanation: A simplex x in $\operatorname{sing}(Y)$ is a continuous map from $\Delta^{|x|}$ to Y, and $x^*(\gamma)$ is the pullback of γ under this map.) The morphisms

$$F(x) : F(\Delta^{|x|}, x^*(\gamma)) \longrightarrow F(Y,\gamma)$$

give rise, by dint of their compatibility, to a map

$$\alpha : F^{\%}(Y,\gamma) \longrightarrow F(Y,\gamma)$$

called the *assembly*. The first proof of 2.1 (but not the second) can be modified to show that $F^{\%}$ is homotopy invariant and strongly excisive, which means that it preserves homotopy equivalences, homotopy pushout squares, and arbitrary coproducts up to homotopy equivalence. (A square in $\mathcal{Y}^{\&}$ is a *homotopy pushout square* if the underlying square in $\mathcal{Y}$ is a homotopy pushout square.) If F is already strongly excisive, then α is a homotopy equivalence for all objects in $\mathcal{Y}^{\&}$, and if F is excisive, then α is a homotopy equivalence for all objects (Y,γ) such that Y has the homotopy type of a finite CW–space.

2.4. Examples. The functors F from $\mathcal{Y}$ or from $\mathcal{Y}^{\&}$ to spectra which we have in mind are those from section 1, e.g. the functor sending an object Y of $\mathcal{Y}$ to $\underline{\underline{K}}p(\mathbb{Z}\pi_1(Y))$, or to $\underline{\underline{A}}s(Y)$; or the functor sending (Y,γ) in $\mathcal{Y}^{\&}$ to $\widehat{H}^{\blacktriangledown}(Z_2; \underline{\underline{A}}s(Y,\gamma))$, or to $\underline{\underline{L}}h_{\blacktriangledown}(\mathbb{Z}\pi_1(Y,w))$, where w is the orientation cover associated to γ. See 1.3, 1.5, 1.6.

2.5. Remark. Write $F_1(Y,\gamma) = \underline{\underline{A}}s(Y,\gamma)$ and $F_2(E) = H_{\blacktriangledown}(Z_2;E)$ (or $\widehat{H}^{\blacktriangledown}(Z_2;E)$, or $H^{\blacktriangledown}(Z_2;E)$) for a Z_2–spectrum E. By a mild variation of 2.2 and sequel, we need not distinguish pedantically between $F_2(F_1^{\%}(Y,\gamma))$ and $(F_2F_1)^{\%}(Y,\gamma)$, if Y has the homotopy type of a finite CW–space. Indeed, both $F_2(F_1^{\%})$ and $(F_2F_1)^{\%}$ are excisive approximations to F_2F_1, and they agree when Y is a point.

3. Poincaré Duality and S–Duality

Throughout this section we fix a space Y and a spherical fibration γ over Y with a distinguished section. We impose a local triviality condition on γ , as in 2.3 (ii), and assume for convenience that Y is a compact ENR.

What we have to say about Poincaré duality and S–duality is well known to many L–theorists, but not particularly accessible in the literature. It goes back in part to §9 of Wall's book [45], and in part to Prop. II.2.3 (together with II.3.4) of Ranicki [33]. Briefly, Poincaré spaces give rise to nondegenerate symmetric structures, and degree one normal maps between Poincaré spaces give rise to nondegenerate quadratic structures. In the formulation which suits us best, the "give rise to" clauses become maps between spectra, which we are about to describe.

We refer to [45] for the definition of a *finite Poincaré space* (alias Poincaré complex) of formal dimension n . A mild generalization is convenient: we allow our Poincaré spaces to be arbitrary compact ENR's. (It is important to remember Chapman's result [9] that compact ENR's have preferred simple homotopy types, because finite Poincaré spaces are supposed to satisfy *simple* Poincaré duality.) With this convention, a closed topological manifold is a finite Poincaré space. Spivak's theorem asserts that such a finite Poincaré space X has a *Spivak normal fibration* ν : a spherical fibration over X equipped with a map

$$\eta : S^n \longrightarrow \underset{===}{\underline{Th}}\,(X,\nu) = \text{Thom spectrum of } \nu$$

such that the cap product of the Thom class of ν with $[\eta]$ is a fundamental class for X . See Spivak [38], but also Browder [5] or Ranicki [33]. (We have kept a convention from part II, to the effect that a map from a space U to a spectrum V is the same as a map from the suspension spectrum $\underline{U}$ to V .) The pair (ν,η) is determined by the finite Poincaré space X up to *contractibel choice*, in the sense that a suitably defined space of all such pairs (ν,η) is contractible. (The space in question fibres over the space of maps from X to BG, and the typical fibre is a union of components of the space of maps from S^n to $\underset{===}{\underline{Th}}\,(X,\nu)$. Since $\underset{===}{\underline{Th}}\,(X,\nu)$ is related to X by S–duality, the fibers are loop spaces of the base.) It is therefore admissible to *define* a Poincaré space of formal dimenison n as a triple (X,ν,η) having the above properties.

A *finite Poincaré space over* (Y,γ) , *of formal dimension* n , is a triple (X,f,η) where X is a finite Poincaré space, $f : X \to Y$ is a map and

$$\eta : S^n \longrightarrow \underset{===}{\underline{Th}}\,(X, f^*\gamma)$$

is a map such that the cap product of the Thom class of $f^*\gamma$ with $[\eta]$ is a fundamental class for X . The bordism group of such Poincaré spaces is the n–th homotopy group of a simplicial set (without degeneracy operators)

$$\mathfrak{N}^{PD}(Y,\gamma).$$

The k–simplices of this simplicial set are triples (X,f,η) where X is a well behaved functor from the partially ordered set of faces of Δ^k to the category of compact ENR's, where f is a natural transformation from X to the constant functor with value Y , and where η is a natural map

$$s_+ \longrightarrow \underline{\underline{Th}}\,(X(s),f_s^*\gamma)$$

with s ranging over the faces of Δ^k . (*Well–behaved* means, as in Part II, that the canonical maps from

$$\partial X(s) := \operatorname*{colim}_{t<s} X(t)$$

to $X(s)$ are injections for all faces s ; note that an injective map between compact ENR's has the homotopy extension property.) It is understood that $(X(s),\partial X(s))$ is a finite Poincaré pair for every s , with fundamental class equal to the cap product of the Thom class of $f_s^*\gamma$ with $[\eta_s]$.

Furthermore, the space $|\mathfrak{N}^{PD}(Y,\gamma)|$ is an infinite loop space. Namely, the rule

$$n \longmapsto \left|\mathfrak{N}^{PD}\left(\amalg_{i=1}^n Y, \amalg_{i=1}^n \gamma\right)\right|$$

defines a Γ–space in the sense of Segal [36]. Its underlying space $\mathfrak{N}^{PD}(Y,\gamma)$ is group–like, so according to [36] it is the zeroth term of a connective Ω–spectrum

$$\mathbf{\mathfrak{N}}^{PD}(Y,\gamma).$$

Replacing Poincaré spaces and Poincaré pairs by closed topological manifolds and compact topological manifolds with boundary whenever that makes sense, we obtain a subspectrum

$$\mathfrak{N}^{TOP}(Y,\gamma) \subset \mathfrak{N}^{Pd}(Y,\gamma).$$

Its homotopy groups are the bordism groups of closed topological manifolds equipped with normal maps to (Y,γ) . Warning: Do not confuse these groups with the homotopy groups of the Thom spectrum of Y and γ . See 3.3 below.

We also need the bordism theory of surgery problems over (Y,γ) . An n–dimensional *Poincaré surgery problem over* (Y,γ) is a diagram

$$X_1 \xrightarrow{g} X_2 \xrightarrow{f} Y$$

together with a map

$$\eta : S^n \longrightarrow \underline{\underline{Th}}\,(X_1,(fg)^*\gamma)$$

such that X_1 is a finite Poincaré space of formal dimension n, with fundamental class equal to the cap product of the Thom class of $(fg)^*\gamma$ with $[\eta]$, and X_2 is a finite Poincaré space of formal dimension n, with fundamental class equal to the cap product of the Thom class of $f^*\gamma$ with $[g_*\eta]$. Then g is a degree one normal map between finite Poincaré spaces. If X_1 and X_2 are closed topological manifolds of dimension n, then we speak of a *manifold surgery problem over* (Y,γ). The bordism group of n–dimensional Poincaré surgery problems over (Y,γ) is the n–th homotopy group of a spectrum

$$\mathfrak{S}^{PD}(Y,\gamma),$$

and the bordism group of n–dimensional manifold surgery problems over (Y,γ) is the n–th homotopy group of a spectrum

$$\mathfrak{S}^{TOP}(Y,\gamma).$$

The construction of these spectra follows the usual pattern and is therefore left to the reader. Note in any case that

$$\mathfrak{S}^{TOP}(Y,\ ) \subset \mathfrak{S}^{PD}(Y,\gamma).$$

There is an important map

$$\kappa : \mathfrak{S}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{\mathfrak{N}}}^{PD}(Y,\gamma)$$

whose effect on n–th homotopy groups is to send the bordism class of an n–dimensional Poincaré surgery problem

$$(X_1 \xrightarrow{g} X_2 \xrightarrow{f} Y, \eta)$$

over (Y,γ) to the difference of the bordism classes of the two Poincaré spaces over (Y,γ) involved, that is,

$$[(X_1, fg, \eta)] - [(X_2, f, g_*\eta)] \qquad \in \pi_n(\underline{\underline{\mathfrak{N}}}^{PD}(Y,\gamma)).$$

To describe κ itself, we choose a specific map of spectra

$$z : \underline{S}^0 \longrightarrow \underline{S}^0 \vee \underline{S}^0$$

together with homotopies

$$s_1 : (1 \vee 0)\cdot z \simeq 1 \quad \text{and} \quad s_2 : (1 \vee 1)\cdot z \simeq 0,$$

where 1 and 0 denote the identity and the zero map from $\underline{S}^0$ to itself, respectively. (The space of such triples (z, s_1, s_2) is contractible.) The rule

$$(X_1 \xrightarrow{g} X_2 \xrightarrow{f} Y, \eta) \longmapsto (X_1 \amalg X_2, fg \amalg g, (\eta \vee g_*\eta)z)$$

defines, properly interpreted, a simplicial map

$$\mathfrak{S}^{PD}(Y,\gamma) \longrightarrow \mathfrak{N}^{PD}(Y,\gamma).$$

This extends to a map of Γ–spaces and can therefore be written as a map κ from $\mathfrak{S}^{PD}(Y,\gamma)$ to $\mathfrak{N}^{PD}(Y,\gamma)$. We note that κ sends manifold surgery problems over (Y,γ) to manifolds over (Y,γ) .

3.1. Description. (of constructions which follow). The symmetric construction of Mishchenko [23] becomes, in the nonlinear setting, a map of spectra

$$\zeta_{mi} : \mathfrak{N}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma).$$

The quadratic construction of Ranicki [33] becomes, in the nonlinear setting, a map of spectra

$$\zeta_{ra} : \mathfrak{S}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{L}}s_{\blacktriangledown}(Y,\gamma).$$

They satisfy the equation

$$(1+T)\cdot\zeta_{ra} = \zeta_{mi}\cdot\kappa : \underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma),$$

where $(1+T)$ is the symmetrization map from quadratic to symmetric L–theory. Both ζ_{mi} and ζ_{ra} are natural transformations between functors in the variable (Y,γ) .

3.2. Remark. The assembly map

$$\underline{\underline{\mathfrak{N}}}^{TOP}(Y,\gamma)^{\%} \longrightarrow \underline{\underline{\mathfrak{N}}}^{TOP}(Y,\gamma)$$

is a homotopy equivalence (by dint of 2.2 and topological transversality, see Quinn [26]). Using it as an identification, we may pretend that $\underline{\underline{\mathfrak{N}}}^{TOP}(Y,\gamma)$ is contained in $\underline{\underline{\mathfrak{N}}}^{PD}(Y,\gamma)^{\%}$. Restriction and abuse of notation give a map

$$\zeta_{mi}^{\%} : \underline{\underline{\mathfrak{N}}}^{TOP}(Y,\gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma)^{\%}.$$

The same transversality argument gives

$$\zeta_{ra}^{\%} : \underline{\underline{\mathfrak{S}}}^{TOP}(Y,\gamma) \longrightarrow \underline{\underline{L}}s_{\blacktriangledown}(Y,\gamma)^{\%}.$$

3.3. Remark. The spectrum $\underset{=}{\mathfrak{N}}^{TOP}(Y,\gamma)$ is homotopy equivalent to the Thom spectrum with base space

$$P = \text{homotopy pullback of } (Y \xrightarrow{\gamma} BG \hookleftarrow BTOP)$$

and bundle classified by the projection $P \longrightarrow BTOP$.

3.4. Adjustment. In constructing the maps in 3.1, it will be necessary to adjust $\underset{=}{Ls}_{\blacktriangledown}(Y,\gamma)$ and $\underset{=}{Ls}^{\blacktriangledown}(Y,\gamma)$ a little, without changing homotopy types. For example, we defined $\underset{=}{Ls}_{\blacktriangledown}(Y,\gamma)$ and $\underset{=}{Ls}^{\blacktriangledown}(Y,\gamma)$ using the notion of finite retractive CW–space over Y ; see section 1. Now the CW–structures are likely to cause trouble, because our definitions of Poincaré space and manifold do not require specific CW–structures. The lemma below, which follows from ch. VI, Thm. 1.2 in Hu[15], implies that there is no harm in replacing finite retractive CW–spaces over Y by compact ENR's retracting to Y . (Remember that Y is a compact ENR.)

Lemma. The pushout of a diagram

$$X_1 \xleftarrow{f} X_2 \xrightarrow{g} X_3$$

of ENR's is an ENR, provided f and g are proper and f is injective.

Note also that an injective proper map between ENR's is closed and has the homotopy extension property (see Dold [10], 8.13.3). The lemma implies that the category of retractive compact ENR's over Y is a category with cofibrations and weak equivalences. The cofibrations are the injective maps over Y , and the weak equivalences are the maps over Y which become homotopy equivalences upon forgetting the retractions to Y , with torsion equal to zero in the Whitehead group of Y . The lemma also implies that mapping cylinders, mapping cones and suspensions can be formed as usual in the category of retractive compact ENR's over Y . Last not least, the lemma implies that a finite retractive CW–space over Y is a retractive compact ENR over Y .

A *retractive ENR–spectrum over* Y is a sequence of retractive compact ENR's over Y , say

$$X = \{(X_n, r_n) \mid n \in \mathbb{Z}\}$$

together with injections

$$\varepsilon_n : (\Sigma_Y X_n, \Sigma_Y r_n) \longrightarrow (X_{n+1}, r_{n+1})$$

over Y. A retractive ENR–subspectrum $X' \subset X$ (over Y) is *cofinal* if for any n there exists a k such that the image of $\Sigma_Y^k X_n$ in X_{n+k} is contained in X'_{n+k}. A *function* between retractive ENR–spectra over Y, say from

$$X(1) = \{(X(1)_n, r(1)_n)\} \quad \text{to} \quad X(2) = \{(X(2)_n, r(2)_n)\},$$

is a sequence of maps

$$f_n : (X(1)_n, r(1)_n) \longrightarrow (X(2)_n, r(2)_n)$$

over Y, satisfying $f_{n+1} \cdot \varepsilon_n = \varepsilon_n \cdot \Sigma_Y f_n$ for all n. A *map* (alias morphism) from $X(1)$ to $X(2)$ is an equivalence class of functions $f : X(1)' \longrightarrow X(2)$ where $X(1)'$ is cofinal in $X(1)$; two such functions are considered *equivalent* if they agree on a cofinal retractive ENR–subspectrum contained in the intersection of their respective domains.

A retractive ENR–spectrum X over Y is *finite* if the injections

$$\varepsilon_n : (\Sigma_Y X_n, \Sigma_Y r_n) \longrightarrow (X_{n+1}, r_{n+1})$$

are homeomorphisms for all sufficiently large n. We leave it to the reader to define pairings between finite retractive ENR–spectra over Y, and to obtain ENR–versions of $\underset{=}{L}s_{\blacktriangledown}(Y,\gamma)$ and $\underset{=}{L}s^{\blacktriangledown}(Y,\gamma)$. These ENR–versions contain the older CW–versions, but the enlargement does not affect the homotopy type.

3.5. Construction. Let X_1 and X_2 be pointed spaces and assume for simplicity that X_1 is a finite CW–space. To a map

$$f : X_1 \wedge \underline{S}^0 \longrightarrow X_2 \wedge \underline{S}^0$$

of spectra we want to associate a map of bispectra

$$\overline{f} : X_1 \wedge \underline{S}^0 \wedge \underline{S}^0 \longrightarrow X_2 \wedge \underline{S}^0 \wedge \underline{S}^0$$

which is invariant under the *flip*; that is,

$$\overline{f}_{m,n} : X_1 \wedge S^m \wedge S^n \longrightarrow X_2 \wedge S^m \wedge S^n$$

agrees with

$$\overline{f}_{n,m} : X_1 \wedge S^n \wedge S^m \longrightarrow X_2 \wedge S^n \wedge S^m$$

under the canonical identifications, for all $m, n \gg 0$. It will be sufficient to describe $\overline{f}_{n,n}$ for all $n \gg 0$. This is given by means of a commutative diagram

$$\begin{tikzcd}
(X_1 \wedge S^n) \wedge S^n \arrow[r, "f_n \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge e"] & (X_2 \wedge S^n) \wedge S^n \arrow[d, "\mathrm{id} \wedge e"] \\
X_1 \wedge S^n \wedge S^n \arrow[r, "\overline{f}_{n,n}"] & X_2 \wedge S^n \wedge S^n
\end{tikzcd}$$

where $e : S^n \wedge S^n \longrightarrow S^n \wedge S^n$ sends (a,b) to $(b+a, b-a)$, the identification $S^n \approx \mathbb{R}^n \cup \{\infty\}$ being understood.

3.6. Construction. Suppose first that Y is a point and γ has fiber S^0 . A retractive compact ENR over Y , say X , is then just a pointed compact ENR. The diagonal map

$$\Delta : X \longrightarrow X \wedge X = X \wedge_{Y,\gamma} X$$

has the usual symmetry properties. If Y and γ are arbitrary, and (X, r) is a retractive compact ENR over Y , then we have a diagonal map of the form

$$\Delta : \Sigma^\gamma X / \Sigma^\gamma Y \longrightarrow X \wedge_{Y,\gamma} X$$

because

$$X \wedge_{Y,\gamma} X = \Sigma^\gamma P / \Sigma^\gamma P_0;$$

see section 1, definition of twisted internal smash product. (There are diagonal inclusions $X \subset P$ and $Y \subset P_0$, so that $\Sigma^\gamma X \subset \Sigma^\gamma P$ and $\Sigma^\gamma Y \subset \Sigma^\gamma P_0$.) The map Δ has good symmetry properties, too.

We now produce the map ζ_{mi} promised in 3.1. Suppose first that (X, f, η) is an n–dimensional finite Poincaré space over Y . Then $X_f := X \amalg Y$ is a retractive compact ENR over Y , the retraction being equal to f on X . Denote by $\underline{X}_f$ the retractive suspension spectrum over Y made from X_f . The composite map

$$S^{n+q} \wedge (EZ_2)_+ \xrightarrow{\text{proj.}} S^{n+q} \xrightarrow{\eta} \Sigma^\gamma X = \Sigma^\gamma X_f / \Sigma^\gamma Y \xrightarrow{\Delta} X_f \wedge_{Y,\gamma} X_f$$

is a stable Z_2–map between pointed spaces (only stable because η is stable). Using 3.5 we can interpret it as a map between the asociated suspension bispectra which respects both the Z_2–action and the flip; that is, we have a Z_2–map

$$\overline{\Delta\eta} : S^{n+q} \wedge (EZ_2)_+ \longrightarrow \underline{X}_f \wedge_{Y,\gamma} \underline{X}_f .$$

This is a nondegenerate n–dimensional γ–symmetric structure on $\underline{X}_f$. Compare section 6 of Part II. The rule

$$[(X, f, \eta)] \longmapsto [(\underline{X}_f, \overline{\Delta\eta})]$$

describes the effect of ζ_{mi} on homotopy groups (=bordism groups). In the same way, an n–simplex (X, f, η) in the incomplete simplicial set $\mathfrak{N}^{PD}(Y,\gamma)$ gives rise to a nondegenerate γ–symmetric structure on $\underline{X}_f$, where $\underline{X}_f$ is a functor from the category of faces of Δ^n to the category of retractive spaces over Y because X itself is a functor on the category of faces of Δ^n . So we have a simplicial map

$$\mathfrak{N}^{PD}(Y,\gamma) \longrightarrow Ls^\blacktriangledown(Y,\gamma) = Q\underline{\underline{L}}s^\blacktriangledown(Y,\gamma)$$

which extends to a map of Γ–spaces in the sense of 36, and which can therefore be written as a map of spectra,

$$\zeta_{mi} : \underline{\underline{\mathfrak{N}}}^{PD}(Y, \gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y, \gamma).$$

According to 3.1, it must be possible to refine the map

$$\zeta_{mi} \cdot \kappa : \underline{\underline{\mathfrak{S}}}^{PD}(Y, \gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y, \gamma)$$

to a map whose target is $\underline{\underline{L}}s_{\blacktriangledown}(Y, \gamma)$. The search for such a refinement generates 3.7–3.10.

3.7. Definition. Let (A, ψ) and $(B, \mathcal{S})$ be n–dimensional γ–symmetric Poincaré objects in the category of finite retractive ENR–spectra over Y . Let

$$g : A \longrightarrow B$$

be a map sending ψ to $\mathcal{S}$. A *symmetric kernel* for g consists of

(1) a finite retractive ENR–spectrum C over Y , and a map $p : A \to C$ such that the double mapping cylinder of

$$B \xleftarrow{g} A \xrightarrow{p} C$$

is weakly contractible (with Whitehead torsion zero);

(2) a nullhomotopy h of the composition

$$S^{n+q} \xrightarrow{\psi_0} A \wedge_{Y,\gamma} A \xrightarrow{p \wedge g} C \wedge_{Y,\gamma} B.$$

3.8. Fact. Such a symmetric kernel for g exists and is essentially unique. Moreover, C is an n–dimensional symmetric Poincaré object with the symmetric structure $p_*(\psi)$.

Comment: The uniqueness part is nothing more than a relative version of the existence part. The relative version states that a map between $(n+1)$–dimensional symmetric Poincaré *pairs* (in the usual category) also has a symmetric kernel, which is again an $(n+1)$–dimensional symmetric Poincaré pair; what is more, the boundary symmetric kernel, which is an n–dimensional symmetric Poincaré object, can be prescribed.

Morally, condition 3.7. (i) means that $A \simeq B \amalg C$. The nullhomotopy h ensures that, under this moral identification, ψ corresponds to the symmetric structure $\mathcal{S} \amalg p_*(\psi)$ on $B \amalg C$.

We omit the proof of 3.8. It is essentially identical with the proof of the chain complex analogue, given in [33].

We have defined an n–dimensional symmetric structure on a finite retractive ENR–spectrum C over Y to be a map

$$\Theta : (S^{n+q} \wedge (EZ_2)_+) \wedge \underline{S}^0 \wedge \underline{S}^0 \longrightarrow C \wedge_{Y,\gamma} C$$

of bispectra respecting the canonical skew involutions. Let

$$V(n) \subset \underline{S}^0 \wedge \underline{S}^0$$

be the smallest sub–bispectrum containing the diagonal n–sphere

$$S^n \subset S^n \wedge S^n = (\underline{S}^0 \wedge \underline{S}^0)_{n,n}.$$

Then $V(0) = \underline{S}^0 \wedge \underline{S}^0$, and

$$V(0) \supset V(1) \supset V(2) \supset \dots\ .$$

3.9. Adjustment. We redefine a quadratic structure on C to be a symmetric structure Θ as above, plus a Z_2–nullhomotopy h of the restriction $r_m\Theta$ of Θ to

$$(S^{n+q} \wedge (EZ_2)_+) \wedge V(m),$$

for some $m \geq 0$. (More precisely, we are only interested in what h does on $V(m)$ for very large m ; so we would consider two such nullhomotopies as equal if they agree on

$$(S^{n+q} \wedge (EZ_2)_+) \wedge V(k)$$

for some integer $k \gg 0$.)

To show that this definition of quadratic structure agrees (morally) with older ones, we use the resolution theory of Part II, section 2. Let

$$F_m = \underset{===}{map}(V(m), C \wedge_{Y,\gamma} C).$$

Let Z_2 act on F_m via its skew–actions on $V(m)$ and $C \wedge_{Y,\gamma} C$. Restriction from $V(m)$ to $V(m+1)$ defines the maps in a direct system of Z_2–spectra

$$F_0 \to F_1 \to F_2 \to F_3 \to \dots\ .$$

We can interpret Θ as a map

$$S^{n+q} \longrightarrow H^{\blacktriangledown}(Z_2; F_0)$$

and we can interpret h as a nullhomotopy of the composition

$$S^{n+q} \longrightarrow H^{\blacktriangledown}(Z_2; F_0) \longrightarrow H^{\blacktriangledown}(Z_2; F_m)$$

for some m . So what we have to verify is that

$$F_0 \to F_1 \to F_2 \to \dots$$

is an augmented resolution of F_0 (see Part II, 2.9 and the remarks following 1.9 concerning filtered spectra and simplicial spectra). For in that case 2.10 in Part II tells us that the inclusion

$$H^{\blacktriangledown}(Z_2; F_0) \longrightarrow \underset{m}{\mathrm{hocolim}} H^{\blacktriangledown}(Z_2; F_m)$$

(where the hocolim is a telescope) can be identified with the inclusion

$$H^{\blacktriangledown}(Z_2; F_0) \longrightarrow \widehat{H}^{\blacktriangledown}(Z_2; F_0)$$

which has homotopy fibre

$$H_{\blacktriangledown}(Z_2; F_0) = H_{\blacktriangledown}(Z_2; C \wedge_{Y,\gamma} C).$$

Consequently (Θ, h) constitutes a map

$$S^{n+q} \longrightarrow H_{\blacktriangledown}(Z_2; C \wedge_{Y,\gamma} C),$$

which is a quadratic structure in the usual sense.

We still have to verify that

(i) the maps $F_m \to F_{m+1}$ are nullhomotopic for all $m \geq 0$ (upon forgetting Z_2–actions);

(ii) the mapping cone of $F_m \to F_{m+1}$ is an induced Z_2–spectrum, for all $m \geq 0$.

The verification consists in going back to the inclusion map from $V(m+1)$ to $V(m)$, which is nullhomotopic and whose cofiber is an induced Z_2–skew–bispectrum.

3.10. Example. Let

$$X \xrightarrow{g} U \xrightarrow{f} Y, \qquad \eta : S^{n+q} \longrightarrow \Sigma^{\gamma} X$$

be an n–dimensional Poincaré surgery problem over (Y, γ) . So X and U are Poincaré spaces with fundamental classes

$$t \cap [\eta] \qquad \text{and} \qquad t \cap [g_* \eta]$$

respectively, where t is the Thom class of γ . Write $\mu := f^*(\gamma)$. The symmetric construction gives two objects

$$\underline{X}_g \text{ and } \underline{Y}_{\mathrm{id}}$$

(the retractive suspension spectra of $X \amalg U$ and $U \amalg U$) in the category of retractive ENR–spectra over U , and two nondegenerate μ–symmetric structures

$$\psi = \overline{\Delta\eta} \text{ on } \underline{X}_g, \text{ and } \mathcal{S} = \overline{\Delta \cdot g_*\eta} \text{ on } \underline{U}_{\mathrm{id}}.$$

(All this is happening over (U, μ) , not over (Y, γ) .) The map

$$g^{\&} : \underline{X}_g \to \underline{U}_{\mathrm{id}}$$

induced by g sends ψ to $\mathcal{S}$. Choose a symmetric kernel for $g^{\&}$, say

$$(p : \underline{X}_g \to C, h)$$

where h is a nullhomotopy of

$$S^{n+q} \xrightarrow{\psi_0} \underline{X}_g \wedge_{U,\mu} \underline{X}_g \xrightarrow{p \wedge g^{\&}} C \wedge_{U,\mu} \underline{U}_{\mathrm{id}}.$$

Let $\Sigma^\mu(C/U)$ be the spectrum whose j–th term is

$$\Sigma^\mu C_j / \Sigma^\mu U \quad \text{for} \quad j \in \mathbb{Z}.$$

The following observation is crucial. Since $\underline{U}_{\mathrm{id}}$ is the retractive suspension spectrum of $U \amalg U = S^0 \times U$ (as a retractive space over U), and since the retraction from $S^0 \times U$ to U is a fibration, there is an inclusion of bispectra

$$\Sigma^\mu(C/U) \wedge \underline{S}^0 \longrightarrow C \wedge_{U,\mu} \underline{U}_{\mathrm{id}}$$

which is a homotopy equivalence. The map

$$(p \wedge g^{\&}) \cdot \psi_0 : S^{n+q} \longrightarrow C \wedge_{U,\mu} \underline{U}_{\mathrm{id}}$$

actually lands in the subspectrum

$$\Sigma^\mu(C/U) \wedge \underline{S}^0,$$

and we now impose a tiny extra condition on the symmetric kernel $(p : \underline{X}_g \to C, h)$ by requiring h to stay inside this subspectrum.

Choose an integer m large enough so that

$$\Sigma^m p : \Sigma^m \underline{X}_g \longrightarrow \Sigma^m C$$

is a map between retractive *spaces* over U . (Here $\Sigma^m \underline{X}_q$ is the retractive spectrum over U whose n–th term is the $(n{+}m)$–th term of $\underline{X}_g$, and so on.) By inspection, the composite map

$$\begin{tikzcd}
(EZ_2)_+ \wedge S^{n+q+m} \arrow[r, "\text{proj.}"] & S^{n+q+m} \arrow[r, "(p\wedge g^{\&})\cdot\psi_0"] & \Sigma^{\mu}(\Sigma^m(C/U)) \wedge \underline{S}^0 \arrow[d, "\overline{\Delta}"'] \\
& & \Sigma^m C \wedge_{U,\mu} \Sigma^m C
\end{tikzcd}$$

(with $\overline{\Delta}$ as in 3.5, 3.6) agrees with the (m,m)–fold suspension of $r_m(p_*(\psi))$, in the notation of 3.9. (Note that the (m,m)–fold suspension of $V(m)$ is the suspension bispectrum of S^m , up to canonical isomorphism.) We conclude that the nullhomotopy h of $(p\wedge g^{\&})\cdot\psi_0$ implies a Z_2–nullhomotopy of $r_m(p_*(\psi))$. We still call this h , and we are then entitled to say that $(p_*(\psi),h)$ is a quadratic structure on C , in the sense of 3.9. Summarizing, we see that our symmetric kernel cannot help being a quadratic kernel. It can be pushed forward to Y , using the map $f : U \to Y$ and 1.6, and is then a γ–quadratic n–dimensional Poincaré object in the category of retractive ENR–spectra over Y .

Call the symmetric kernel in 3.10 a symmetric kernel for the surgery problem

$$X \xrightarrow{g} U \xrightarrow{f} Y \ , \quad \eta : S^{n+q} \longrightarrow X$$

over (Y,γ) . (It lives over U .) It is convenient to modify the definition of $\underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma)$ in such a way that all Poincaré surgery problems in sight (and pairs of such) are equipped with symmetric kernels (and pairs of such). By 3.8, this has no effect on the homotopy type of $\underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma)$. Now

$$\zeta_{ra} : \underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{L}}s_{\blacktriangledown}(Y,\gamma)$$

is the map which forgets the surgery problems but records the symmetric kernels which cannot help being quadratic, pushed forward to Y .

Unfortunately, with the present definitions of ζ_{mi} and ζ_{ra} , the square

$$\begin{tikzcd}
\underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma) \arrow[r, "\kappa"] \arrow[d, "\zeta_{ra}"] & \underline{\underline{\mathfrak{N}}}^{PD}(Y,\gamma) \arrow[d, "\zeta_{mi}"] \\
\underline{\underline{L}}s_{\blacktriangledown}(Y,\gamma) \arrow[r, "1+T"] & \underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma)
\end{tikzcd}$$

(see 3.1) is not strictly commutative. Some adjustments have to be made. It is sufficient to construct a natural homotopy

$$w : (1+T)\cdot\zeta_{ra} \longrightarrow \zeta_{mi}\cdot\kappa$$

in the square above (natural in Y and γ). Reason: Convert the left vertical arrow into a cofibration by the usual method; in particular, replace the target of ζ_{ra} by the mapping cylinder of ζ_{ra} . Use the homotopy w to extend the map $(1+T)$, defined on the target of ζ_{ra} , to the mapping cylinder of ζ_{ra} . The resulting square is strictly commutative and still natural in Y and γ .

We now generalize in order to simplify. Let $(A, \mathcal{S})$ be an n–dimensional γ–symmetric Poincaré object in the category of finite retractive ENR–spectra over Y . A splitting of $(A, \mathcal{S})$ shall consist of two morphisms

$$C \xleftarrow{p} A \xrightarrow{g} B$$

such that the double mapping cylinder of p and g is weakly contractible with torsion zero, plus a nullhomotopy h of

$$(p \wedge g) \cdot \mathcal{S}_0 : S^{n+q} \longrightarrow C \wedge_{Y,\gamma} B.$$

Compare 3.7. We denote by

$$\underline{\underline{L}}s^{\blacktriangledown\blacktriangledown}(Y,\gamma)$$

the bordism theory of such symmetric Poincaré objects with splitting. We shall compare two maps

$$\lambda_1, \lambda_2 : \underline{\underline{L}}s^{\blacktriangledown\blacktriangledown}(Y,\gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown}(Y,\gamma)$$

given in shorthand notation by

$$\lambda_1(C \xleftarrow{p} A \xrightarrow{g} B, \mathcal{S}, h) = (A, \mathcal{S}) - (B, g_*(\mathcal{S})),$$
$$\lambda_2(C \xleftarrow{p} A \xrightarrow{g} B, \mathcal{S}, h) = (C, p_*(\mathcal{S})).$$

A detailed definition of λ_1 would resemble that of κ (just before 3.1). It is more than sufficient to produce a natural homotopy

$$w' : \lambda_1 \simeq \lambda_2,$$

because

$$\zeta_{mi} \cdot \kappa = \lambda_1 u \qquad \text{and} \quad (1+T) \cdot \zeta_{ra} = \lambda_2 u$$

where

$$u : \underline{\underline{\mathfrak{S}}}^{PD}(Y,\gamma) \longrightarrow \underline{\underline{L}}s^{\blacktriangledown\blacktriangledown}(Y,\gamma)$$

is a map implicit in 3.10.

Now the map

$$i_1 \vee i_2 : \underset{=}{L}s^{\blacktriangledown}(Y,\gamma) \vee \underset{=}{L}s^{\blacktriangledown}(Y,\gamma) \longrightarrow \underset{=}{L}s^{\blacktriangledown\blacktriangledown}(Y,\gamma)$$

given by

$$\begin{aligned} i_1(A,\mathcal{S}) &= (A \xleftarrow{\text{id}} A \to 0, \mathcal{S}) \\ i_2(A,\mathcal{S}) &= (0 \leftarrow A \xrightarrow{\text{id}} A, \mathcal{S}) \end{aligned}$$

is a homotopy equivalence. Therefore it is sufficient to find a natural homotopy w'' from $\lambda_1(i_1 \vee i_2)$ to $\lambda_2(i_1 \vee i_2)$. (If w'' is available, then w' can also be found using general nonsense in the shape of a suitable enlargement of the receiving spectrum $\underset{=}{L}s^{\blacktriangledown}(Y,\gamma)$.) Constructing w'' amounts to constructing natural homotopies

$$w_1'' : \lambda_1 i_1 \simeq \lambda_2 i_1 \quad \text{and} \quad w_2'' : \lambda_1 i_2 \simeq \lambda_2 i_2 .$$

The maps $\lambda_1 i_1$, $\lambda_2 i_1$, $\lambda_1 i_2$ and $\lambda_2 i_2$ are given by

$$(A,\mathcal{S}) \longmapsto \begin{cases} (A,\mathcal{S}) & -\ (0,0) \\ (A,\mathcal{S}) & \\ (A,\mathcal{S}) & -\ (A,\mathcal{S}) \\ (0,0) & \end{cases}$$

respectively (in shorthand). We stop here, leaving the explicit construction of w_1'' and w_2'' to the reader.

4. Dimension shift

One of our favourite integers in Part II and in the preceding sections has been *zero*. We used general versions of Spanier–Whitehead *zero*–duality, i.e. zero dimensional pairings. Higher dimensional pairings occurred mostly in the form of higher homotopies in spaces of zero dimensional pairings. In Part I, however, another integer is prominent: the dimension of a specific manifold M^n . Since we are trying to unify Parts I and II, we literally face a dimension gap.

As always, let γ be a spherical fibration with section over Y , with fibers homotopy equivalent to S^q . In addition let n be an integer $\geq -q$. We have previously defined an n–dimensional pairing between objects C and D in $\mathrm{ret}(Y)$ (see 1.5) to be a map

$$S^{n+q} \longrightarrow C \wedge_{Y,\gamma} D.$$

If we now use such n–dimensional pairings instead of 0–dimensional pairings to define duality, then we obtain by the usual method an n–duality involution on $\underline{\underline{A}}p(Y)$. More precisely, we obtain a Z_2–spectrum (=spectrum with Z_2–action)

$$\underline{\underline{A}}p(Y, \gamma, -n)$$

equipped with a forgetful map to $\underline{\underline{A}}p(Y)$ which is a homotopy equivalence. Taking w to be the orientation of γ , we can similarly construct a Z_2–spectrum

$$\underline{\underline{K}}p(\mathbb{Z}\pi_1(Y, w), -n)$$

homotopy equivalent to $\underline{\underline{K}}p(\mathbb{Z}\pi_1(Y))$, and an equivariant linearization map

$$\underline{\underline{A}}p(Y, \gamma, -n) \longrightarrow \underline{\underline{K}}p(\mathbb{Z}\pi_1(Y, w), -n).$$

This is an isomorphism on π_0 , and Z_2 acts by sending the Euler characteristic of a finitely dominated object in $\mathrm{ret}(Y)$ or in $\mathrm{chain}(Y)$ to the Euler characteristic of its n–dual. (Here $\mathrm{chain}(Y)$ is the category of chain complexes of based free left $\mathbb{Z}\pi_1(Y)$–modules.)

In the same vein, we let

$$\begin{aligned} &Q\underline{\underline{L}}p_{\blacktriangledown}(Y, \gamma, -n), Q\underline{\underline{L}}p^{\blacktriangledown}(Y, \gamma, -n),\\ &Q\underline{\underline{L}}p_{\blacktriangledown}(\mathbb{Z}\pi_1(Y, w), -n), Q\underline{\underline{L}}p^{\blacktriangledown}(\mathbb{Z}\pi_1(Y, w), -n) \end{aligned}$$

be the simplicial sets whose k–simplices are the n–dimensional symmetric or quadratic Poincaré objects in $\rho_k\mathrm{ret}(Y)$ or $\rho_k\mathrm{chain}(Y)$, the ρ_k–notation being as in section 3 of Part II. What can be done for spaces can also be done for maps. For example, there is a map of infinite loop spaces

$$\boxminus^p_{-n} : Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n) \longrightarrow Q\widehat{H}^{\blacktriangledown}(Z_2; \underline{\underline{A}}p(Y,\gamma,-n))$$

constructed just like $\boxminus^p$ in Part II, with which it agrees if $n = 0$. (The notation is reformed though, in accordance with 1.5.) This was or is defined as the composition of symmetrization $(1+T)$ with another map, which we also informally denote by

$$\boxminus^p_{-n} : Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n) \longrightarrow Q\widehat{H}^{\blacktriangledown}(Z_2; \underline{\underline{A}}p(Y,\gamma,-n)).$$

We will usually omit the decoration p in $\boxminus^p_{-n}$. For purposes of calculation, a description of $\boxminus_{-n}$ in terms of $\boxminus_0 = \boxminus$ is desirable. In practice, n is the dimension of a manifold, so $n \geq 0$; and we can bridge the gap between n and $n-1$ by looking at relative situations in which n–duality and $(n-1)$–duality occur simultaneously.

Let $Y' \subset U$ be a subspace, and let γ' be the pullback of γ to Y' . For simplicity, assume that both Y and Y' are compact ENR's. Recall that $\rho_1\mathrm{ret}(Y)$ is the category whose objects are diagrams

$$X' \longrightarrow X \longleftarrow X''$$

in $\mathrm{ret}(Y)$. Those diagrams with $X'' = 0$, and X' in $\mathrm{ret}(Y')$, form a full subcategory

$$\mathrm{ret}(Y,Y') \subset \rho_1\mathrm{ret}(Y)$$

closed under n–duality (we leave it to the reader to make this precise). The functor

$$\partial : \mathrm{ret}\,(Y,Y') \longrightarrow \mathrm{ret}\,(Y'); (X' \longrightarrow X) \longmapsto X'$$

is also compatible with n–duality, and under the functor

$$e : \mathrm{ret}\,(Y) \longrightarrow \mathrm{ret}\,(Y,Y'); X \longmapsto (0 \longrightarrow X),$$

$(n+1)$–duality in $\mathrm{ret}\,(Y)$ corresponds to n–duality in $\mathrm{ret}\,(Y,Y')$. Applying L–theory and K–theory to the diagram of categories

$$\mathrm{ret}\,(Y) \xrightarrow{e} \mathrm{ret}\,(Y,Y') \xrightarrow{\partial} \mathrm{ret}\,(Y'),$$

we therefore obtain diagrams of spaces and spectra (in self–explanatory notation, and with $n \geq 0$):

(a) $Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n-1) \xrightarrow{e} Q\underline{\underline{L}}p_{\blacktriangledown}(Y,Y',\gamma,-n) \xrightarrow{\partial} Q\underline{\underline{L}}p_{\blacktriangledown}(Y',\gamma',-n)$

(b) $Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n-1) \xrightarrow{e} Q\underline{\underline{L}}p^{\blacktriangledown}(Y,Y',\gamma,-n) \xrightarrow{\partial} Q\underline{\underline{L}}p^{\blacktriangledown}(Y',\gamma',-n)$

(c) $\underline{\underline{A}}p(Y,\gamma,-n-1) \xrightarrow{e} \underline{\underline{A}}p(Y,Y',\gamma,-n) \xrightarrow{\partial} \underline{\underline{A}}p(Y',\gamma',-n)$.

Diagram (c) is a diagram of Z_2–spectra.

4.1. Propositions. The diagrams (a) and (b) are fibrations up to homotopy (of spaces). The map ∂ in (c) has a natural (but non–equivariant) section u (so that $\partial u = \mathrm{id}$). The natural map of spectra

$$e \vee u : \underline{\underline{A}}P(Y,\gamma,-n-1) \vee \underline{\underline{A}}p(Y',\gamma',-n) \longrightarrow \underline{\underline{A}}p(Y,Y',\gamma,-n)$$

is a homotopy equivalence.

Proof: Note that ∂e is zero as a functor, and therefore as a map. The L–theoretic part of 4.1 can be proved by inspecting homotopy groups. To define the map u , think of $\mathrm{ret}\,(Y')$ as a full subcategory of $\mathrm{ret}\,(Y)$. To a finitely dominated object X' in $\mathrm{ret}\,(Y')$, associate the object

$$(X' \longrightarrow 0) \quad \text{in} \quad \mathrm{ret}\,(Y,Y').$$

This is functorial, and induces, in self–explanatory notation, a map

$$u : \underline{\underline{A}}p(Y) \longrightarrow \underline{\underline{A}}p(Y,Y').$$

If we want to use models for A–theory with involution, then we also have to say what u does to nondegenerate pairings: It sends a nondegenerate pairing

$$\eta : S^{n+q} \longrightarrow X \wedge_{Y',\gamma'} V$$

between finitely dominated objects in $\mathrm{ret}\,(Y')$ to the unique pairing $\overline{\eta}$ between $(X \to 0)$ and $(\mathrm{id} : V \to V)$ with $\partial\overline{\eta} = \eta$. (Then $\overline{\eta}$ is nondegenerate.) The map $e \vee u$ is a homotopy equivalence by Waldhausen's additivity theorem (section 1.4 of [42]). Here it is easier to use the standard models for A–theory, without involution. □

We now assume that $Y' = Y$.

4.2. Proposition. The spaces

$$Q\underline{\underline{L}}p_{\blacktriangledown}(Y,Y,\gamma,-n) \quad \text{and} \quad Q\underline{\underline{L}}p^{\blacktriangledown}(Y,Y,\gamma,-n)$$

are contractible. The map of spectra

$$u \vee \tau u : \underline{\underline{A}}p(Y,\gamma,-n) \vee \underline{\underline{A}}p(Y,\gamma,-n) \longrightarrow \underline{\underline{A}}p(Y,Y,\gamma,-n)$$

is a homotopy equivalence (where τ denotes the generator of Z_2, acting on the target of u).

Proof: Inspection (see also 4.6 in Part II).

4.3. Corollary. There are homotopy equivalences of infinite loop spaces

$$\begin{aligned} \beta : Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n-1) &\longrightarrow \Omega Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n) \\ \beta : Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n-1) &\longrightarrow \Omega Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n). \end{aligned}$$

There is a homotopy equivalence of spectra

$$\beta : \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n-1)) \longrightarrow \Omega\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n)).$$

The diagram

$$\begin{tikzcd} Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n-1) \arrow[r, "1+T"] \arrow[d, "\beta"] & Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n-1) \arrow[r, "\boxminus_{-n-1}"] \arrow[d, "\beta"] & Q\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n-1)) \arrow[d, "\beta"] \\ \Omega Q\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n) \arrow[r, "1+T"] & \Omega Q\underline{\underline{L}}p^{\blacktriangledown}(Y,\gamma,-n) \arrow[r, "\boxminus_{-n}"] & \Omega Q\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n)) \end{tikzcd}$$

commutes.

Proof. Look at the diagrams (a) and (b) just before 4.1, with $Y' = Y$. The middle terms in (a) and (b) are then contractible by 4.2. Since $\partial e = 0$ in both diagrams, we obtain maps β from the source of e to the homotopy fiber of ∂, which is Ω on the target of ∂. These maps induce isomorphisms in π_k for $k \geq 0$ by 4.1. Next, look at diagram (c) just before 4.1, again with $Y' = Y$. This is a fibration up to homotopy by 4.1. Applying $\widehat{H}^{\blacktriangledown}(Z_2;\dots)$ to (c) gives another fibration up to homotopy $\widehat{H}^{\blacktriangledown}(Z_2;(c))$, by remark 2.8 Part II. Its middle term

$$\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,Y,\gamma,-n))$$

is contractible by oberservation 2.6 in Part II, since $\underline{\underline{A}}p(Y,Y,\gamma,-n)$ is an induced Z_2–spectrum by 4.2 (see 2.5 in Part II). This gives

$$\beta : \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n-1) \xrightarrow{\simeq} \Omega\widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n)).$$

The compatibility between the three β 's is a consequence of the fact that symmetrization $1+T$ can be viewed as a map from (a) to (b), while $\boxminus$ (suitably decorated with dimension indices) can be viewed as a map from (b) to $\widehat{H}^{\blacktriangledown}(Z_2;(c))$.
$\square$

In the next corollary, and elsewhere, we shall say that two Z_2–spectra are weakly Z_2–equivalent if they can be connected by a chain of Z_2–maps (between Z_2–spectra) which are ordinary homotopy equivalences. Recall also from the introduction that $\ddot{S}^n$ is S^n with a specific involution, with fixed point set $S^0 \subset S^n$.

4.4. Corollary. The spectra

$$\underline{\underline{A}}p(Y,\gamma,-n-1) \quad \text{and} \quad \Omega(\overset{..1}{S} \wedge \underline{\underline{A}}p(Y,\gamma,-n))$$

are weakly Z_2–equivalent.

Proof. Since diagram (c) just before 4.1 is a fibration up to homotopy, we can say that $\underline{\underline{A}}p(Y,\gamma,-n-1)$ is weakly Z_2–equivalent to Ω on the cofiber of

$$\partial : \underline{\underline{A}}p(Y,Y,\gamma,-n) \longrightarrow \underline{\underline{A}}p(Y,\gamma,-n).$$

(We assume that $Y' = Y$.) By 4.2 we may replace ∂ by

$$\partial(u \vee \tau u) : \underline{\underline{A}}p(Y,\gamma,-n) \vee \underline{\underline{A}}p(Y,\gamma,-n) \longrightarrow \underline{\underline{A}}p(Y,\gamma,-n).$$

But $\partial(u \vee \tau u) = \partial u \vee \partial \tau u = \partial u \vee \tau \partial u = \mathrm{id} \vee \tau$, because ∂ is a Z_2–map and $\partial u = \mathrm{id}$. So the cofiber of $\partial(u \vee \tau u)$ is isomorphic to

$$\overset{..1}{S} \wedge \underline{\underline{A}}p(Y,\gamma,-n).$$

$\square$

The upshot of 4.4 is that the ad hoc definition of $\underline{\underline{A}}p(Y,\gamma,-n)$ given in the introduction for $n \geq 0$, namely,

$$\underline{\underline{A}}p(Y,\gamma,-n) := \Omega^n(\overset{..n}{S} \wedge \underline{\underline{A}}p(Y,\gamma))$$

is an agreement with the official definition (using n–duality) given earlier in this section. (Use induction on n , and note that the n–fold smash power of $\overset{..1}{S}$ is $\overset{..n}{S}$.)

It is possible to recover the homotopy equivalence

$$\beta : \widehat{H}^{\blacktriangledown}(Z_2; \underset{=}{A}p(Y,\gamma,-n-1)) \longrightarrow \Omega\widehat{H}^{\blacktriangledown}(Z_2; \underset{=}{A}p(Y,\gamma,-n))$$

in 4.3 from 4.4, as follows. The inclusion

$$\underset{=}{A}p(Y,\gamma,-n) \cong \overset{..0}{S} \wedge \underset{=}{A}p(Y,\gamma,-n) \longrightarrow \overset{..1}{S} \wedge \underset{=}{A}p(Y,\gamma,-n)$$

has a quotient

$$(\overset{..1}{S}/\overset{..0}{S}) \wedge \underset{=}{A}p(Y,\gamma,-n)$$

which is an induced Z_2–spectrum by inspection. Therefore it induces a homotopy equivalence

$$\iota : \widehat{H}^{\blacktriangledown}(Z_2; \underset{=}{A}p(Y,\gamma,-n)) \longrightarrow \widehat{H}^{\blacktriangledown}(Z_2; \overset{..1}{S} \wedge \underset{=}{A}p(Y,\gamma,-n)).$$

This gives

$$\begin{aligned}\widehat{H}^{\blacktriangledown}(Z_2; \underset{=}{A}p(Y,\gamma,-n-1)) &\simeq \widehat{H}^{\blacktriangledown}(Z_2; \Omega(\overset{..1}{S} \wedge \underset{=}{A}p(Y,\gamma,-n)) \\ &\simeq \Omega\widehat{H}^{\blacktriangledown}(Z_2; \overset{..1}{S} \wedge \underset{=}{A}p(Y,\gamma,-n)) \simeq \Omega\widehat{H}^{\blacktriangledown}(Z_2; \underset{=}{A}p(Y,\gamma,-n)).\end{aligned}$$

(The first homotopy equivalence in this chain comes from 4.4, the second comes from remark 2.8 in Part II, and the third is $\Omega\iota^{-1}$.) The composite homotopy equivalence agrees with in 4.3. The verification is left to the reader.

The upshot of this is that the definition of $\boxminus_{-n}$ given in the introduction (at the spectrum level) is in agreement with the definition given earlier in this section (at the infinite loop space level). Again, the proof is by induction on n (for $n \geq 0$).

The story has a relative version. In the situation of 4.1, there is a fibration up to homotopy of Z_2–spectra

$$\underset{=}{A}p(Y,\gamma,-n-1) \xrightarrow{e} \underset{=}{A}p(Y,Y',\gamma,-n-1) \longrightarrow \underset{=}{A}p(Y',\gamma',-n).$$

With the identifications

$$\underset{=}{A}p(Y,\gamma,-n-1) \simeq \Omega^{n+1}(\overset{..n+1}{S} \wedge \underset{=}{A}p(Y,\gamma)), \underset{=}{A}p(Y',\gamma',-n) \simeq \Omega^{n}(\overset{..n}{S} \wedge \underset{=}{A}p(Y',\gamma'))$$

we can guess what the connecting map for the fibration is: it is the Z_2–map given by inclusion,

$$\Omega^{n+1}(\overset{..n}{S} \wedge \underset{=}{A}p(Y',\gamma')) \longrightarrow \Omega^{n+1}(\overset{..n+1}{S} \wedge \underset{=}{A}p(Y,\gamma)).$$

The guess is correct by definition if $Y = Y'$, and therefore in general (by naturality). With this in mind, form the homotopy commutative square

$$\begin{tikzcd}
\Omega \underline{\underline{L}}p_{\blacktriangledown}(Y',\gamma',-n) \arrow[r, "\boxminus_{-n}"] \arrow[d] & \begin{array}{c} \Omega \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y',\gamma',-n)) \\ \simeq \widehat{H}^{\blacktriangledown}(Z_2;\Omega \underline{\underline{A}}p(Y',\gamma',-n)) \end{array} \arrow[d] \\
\underline{\underline{L}}p_{\blacktriangledown}(Y,\gamma,-n-1) \arrow[r, "\boxminus_{-n-1}"] & \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,\gamma,-n-1))
\end{tikzcd}$$

using the notation established in the introduction (under the heading "dimension shift"). Then (and only then) the vertical inclusion maps make sense, and their cofibers are

$$\underline{\underline{L}}p_{\blacktriangledown}(Y,Y',\gamma,-n-1) \quad \text{and} \quad \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,Y',\gamma,-n-1)),$$

respectively (the first by definition, the second by preparation).

4.5. Definitions. Write

$$\boxminus_{-n-1} : \underline{\underline{L}}p_{\blacktriangledown}(Y,Y',\gamma,-n-1) \longrightarrow \widehat{H}^{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,Y',\gamma,-n-1))$$

for the induced map between cofibers, and write

$$\underline{\underline{LA}}\,p(Y,Y',\gamma,-n-1)$$

for the homotopy fiber of the composition

$$\delta_{\mathcal{N}} \boxminus_{-n-1} : \underline{\underline{L}}p_{\blacktriangledown}(Y,Y',\gamma,-n-1) \longrightarrow S^1 \wedge H_{\blacktriangledown}(Z_2;\underline{\underline{A}}p(Y,Y',\gamma,-n-1)).$$

There is a fibration up to homotopy

$$\underline{\underline{LA}}\,p(Y,\gamma,-n-1) \hookrightarrow \underline{\underline{LA}}\,p(Y,Y',\gamma,-n-1) \longrightarrow \underline{\underline{LA}}\,p(Y',\gamma',-n).$$

As in the absolute case, these definitions can be justified at the infinite loop space level, but we omit the justification.

4.6. Observation. If the inclusion $Y' \hookrightarrow Y$ is a homotopy equivalence, then

$$\underline{\underline{LA}}\,p(Y,Y',\gamma,-n-1) \simeq \underline{\underline{A}}p(Y).$$

Proof: Taking $Y' = Y$ and using 4.2, wse get from the definitions

$$\underline{\underline{LA}}\,p(Y, Y, \gamma, -n-1) \simeq H_{\blacktriangledown}(Z_2; \underline{\underline{A}}p(Y, Y, \gamma, -n-1)) \simeq \underline{\underline{A}}p(Y).$$

$\square$

4.7. Remark. The notation $\underline{\underline{A}}p(Y, \gamma, d)$ used in this section is meant to suggest that the spherical fibration γ (with fibers still homotopy equivalent to S^q) has a virtual dimension d, possibly different from q. Taking $d = q$ gives an involution on $\underline{\underline{A}}p(Y)$ which presumably agrees with that constructed earlier by Vogell [40]. In Part II, we chose $d = 0$ for uniformity; but if Y is an n–manifold with normal bundle γ, then it is most reasonable to take $d = -n$.

The remainder of this section is about program 2 (from the introduction) and the Sullivan–Wall–Quinn–Ranicki result. We return to the manifold M, assuming it to be closed for simplicity. To facilitate the description of certain structure spaces associated to M, we begin with a categorical notion. Let $\mathcal{C}$ be a Δ–category (a simplicial category without degeneracy operators), and let X be a Δ–set. By a *functor*

$$F : X \longrightarrow \mathcal{C}$$

we shall mean a rule which to each simplex x in X associates an object $F(x)$ in $\mathcal{C}_{|x|}$, and to each pair (x, f), where x is a simplex and f is a face operator such that f^*x is defined, a morphism

$$F(f) : F(f^*x) \longrightarrow f^*F(x) \qquad \text{in} \quad \mathcal{C}_k,$$

where $k = |f^*x|$. The various $F(f)$ are to satisfy the expected rules for composition, and $F(f)$ must be an identity morphism if f is. (These rules for composition are certainly satisfied if all the morphisms $F(f)$ are identity morphisms, in which case F is nothing but a Δ–map from X to the Δ–set of objects of $\mathcal{C}$.)

4.8. Definition. The *bordism Δ–set* of $\mathcal{C}$ is the Δ–set whose k–simplices are the functors from Δ^k to $\mathcal{C}$.

The bordism Δ–set of $\mathcal{C}$ seems to be more useful than the nerve of $\mathcal{C}$ in certain geometric situations; but it might well have the same homotopy type. In the examples which follow, $\mathcal{C}$ is always a Δ–groupoid (all morphisms are invertible).

4.9. Prototype. Let $\mathcal{C}$ be the Δ–category whose objects in degree k are compact smooth manifolds modelled on Δ^k (Part II, 3.11), and whose morphisms in degree k are diffeomorphisms between manifolds modelled on Δ^k. The bordism

Δ–set of $\mathcal{C}$ is a Kan Δ–set whose n–th homotopy group is the bordism group of n–dimensional smooth closed manifolds.

4.10. Examples. (i) We define the Δ–category of *block* h*–structures* on M. An object in degree k is a simple homotopy equivalence

$$f : M' \times \Delta^k \longrightarrow M \times \Delta^k$$

such that $f(M' \times s) \subset M \times s$ for all faces $s \subset \Delta^k$. (The n–manifold M' is also closed, but otherwise a variable.) A morphism in degree k, from

$$f : M' \times \Delta^k \longrightarrow M \times \Delta^k \text{ to } g : M'' \times \Delta^k \longrightarrow M \times \Delta^k,$$

is a homeomorphism

$$h : M' \times \Delta^k \longrightarrow M'' \times \Delta^k$$

over $M \times \Delta^k$ such that $h(M' \times s) = M'' \times s$ for all s. The bordism Δ–set is denoted by $\widetilde{\mathscr{S}}(M)$, and is called the *space of block* h*–structures* on M.

(ii) We define the Δ–category of *normal maps* to M. An object in degree k consists of

(a) a topological $\mathbb{R}^p$–bundle γ on $M \times \Delta^k$, for some p;

(b) a degree one map

$$f : N \longrightarrow M \times \Delta^k$$

between compact manifolds modelled on $\mathbb{R}^n \times \Delta^k$, and a face respecting embedding

$$e : E(f^*\gamma) \longrightarrow \mathbb{R}^{n+p} \times \Delta^k.$$

Explanation: $E(f^*\gamma)$ is the total space of $f^*\gamma$. Manifolds modelled on $\mathbb{R}^n \times \Delta^k$ are defined in Part II, 3.11. In (b), it is understood that f sends $N(s)$ to $M \times s$ for all faces $s \subset \Delta^k$, and that e embeds the portion over $N(s)$ into $\mathbb{R}^{n+p} \times s$. Note that e is a codimension zero embedding; the effect of (b) is that $f^*\gamma$ becomes the normal bundle of N. The degree one condition means that

$$f_* : H_*(N, \partial N; \mathbb{Z}(f^*\gamma)) \longrightarrow H_*(M \times \Delta^k, \partial(M \times \Delta^k); \mathbb{Z}(\gamma))$$

sends *the* fundamental class ω_N to *a* fundamental class ω_M; we have written $\mathbb{Z}(f^*\gamma)$ and $\mathbb{Z}(\gamma)$ for the orientation $\mathbb{Z}$–bundles of $f^*\gamma$ and γ.

A morphism in degree k, from (γ, f, e) to (γ', f', e'), is simply a stable isomorphism over $M \times \Delta^k$, where the word *stable* means that it is permitted to stabilize γ, e and/or γ', e'. The bordism Δ–set is denoted by $\perp(M)$, and is called the *space of normal maps* to M.

(iii) Fix Y, γ as in section 3 and fix $n \geq 0$. We define the Δ–category of n–dimensional manifold surgery problems over (Y, γ) . An object in degree k consists of

(a) a diagram

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} Y$$

where f is a face–respecting map between compact manifolds modelled on $\mathbb{R}^n \times \Delta^k$, and g is any map;

(b) a map of spectra

$$\eta : \underline{S}^n \wedge \Delta^k_+ \longrightarrow \underset{===}{Th}\,(X_1, f^*g^*)$$

which respects faces and carries fundamental classes for both X_1 and X_2 (cf. beginning of section 3).

A morphism in degree k is an isomorphism over Y . The bordism Δ–set is denoted by

$$\mathfrak{S}^{TOP}(Y, \gamma, -n).$$

It agrees for $n = 0$ with

$$\mathfrak{S}^{TOP}(Y, \gamma)$$

from section 3. With a suitable definition of Ω (as the adjoint of the combinatorial suspension functor) there is an isomorphism of pointed Δ–sets

$$\mathfrak{S}^{TOP}(Y, \gamma, -n) \cong \Omega^n \mathfrak{S}^{TOP}(Y, \gamma).$$

(The functor Ω is well behaved on pointed Kan Δ–sets.) Replacing manifolds by finite Poincaré spaces throughout leads to a bordism Δ–set

$$\mathfrak{S}^{PD}(Y, \gamma, -n) \cong \Omega^n \mathfrak{S}^{PD}(Y, \gamma).$$

Now in order to construct maps between the spaces in 4.10, we have to add some more ballast to the definitions. To simplify, we concentrate on degree zero objects.

An h–structure on M is a simple homotopy equivalence

$$f : M' \longrightarrow M$$

where M' is another closed manifold. Given f , choose an $\mathbb{R}^p$–bundle γ on M and an identification of $f^*\gamma$ with the normal bundle of M' in $\mathbb{R}^{p+n}$, for some p . This is a contractible choice which can be added to the definition of an h–structure. An h–structure on M is then also a normal map to M . This leads to a map

$$u : \widetilde{\mathcal{S}}(M) \longrightarrow \perp (M).$$

Next, let (γ, f, e) be a normal map to M, where f is a degree one map from some N to M and e is an embedding of $E(f^*\gamma)$ into $\mathbb{R}^{p+n}$ (and γ is an $\mathbb{R}^p$–bundle on M). Choose a stable fiber homotopy equivalence

$$j : S\gamma \longrightarrow S\nu$$

(covering id_M) between the spherical fibrations made from γ and $\nu = \nu^M$, and a homotopy h from

$$\underline{S}^n \xrightarrow{\text{collapse}} \underline{\underline{Th}}\,(N, f^*\gamma) \xrightarrow{f_*} \underline{\underline{Th}}\,(M, \gamma) \xrightarrow{j_8} \underline{\underline{Th}}\,(M, \nu)$$

to the usual collapsing map

$$\underline{S}^n \longrightarrow \underline{\underline{Th}}\,(M, \nu).$$

By the uniqueness of the Spivak normal fibration of M, this is again a contractible (and therefore a possible) choice, which can be added to the definition of a normal map. Then (γ, f, e, j, h) forgetfully gives rise to a stable TOP–reduction, namely: j, of the spherical fibration $S\nu$ on M. Identifying the space of such TOP–reductions with

$$\mathrm{map}(M, G/TOP),$$

we have a forgetful map

$$\perp(M) \longrightarrow \mathrm{map}(M, G/TOP).$$

By transversality, the map is a homotopy equivalence. Furthermore, a normal map (γ, f, e, j, h) as above gives rise to a surgery problem over (M, ν) :

$$N \xrightarrow{f} M \xrightarrow{\mathrm{id}} M \ ,\ \eta : \underline{S}^n \longrightarrow \underline{\underline{Th}}\,(N, f^*\nu),$$

where η is the composition

$$\underline{S}^n \xrightarrow{\text{collapse}} \underline{\underline{Th}}\,(N, f^*\gamma) \xrightarrow{j_*} \underline{\underline{Th}}\,(N, f^*\nu).$$

This leads to a map

$$v : \perp(M) \longrightarrow \mathfrak{S}^{TOP}(M, \nu, -n).$$

4.11. Record. We have indicated the construction of maps

$$\begin{tikzcd}
\widetilde{\not S}(M) \arrow[r, "u"] & \perp(M) \arrow[r, "v"] \arrow[d, "\simeq"] & \mathfrak{S}^{TOP}(M, \nu, -n) \\
 & \mathrm{map}(M, G/TOP). &
\end{tikzcd}$$

The map u in 4.11 is particularly important here because

$$\begin{aligned} &\Omega\widetilde{\mathscr{S}}(M) \simeq \widetilde{\mathcal{V}}(M) \text{ from the introduction,} \\ &\Omega \perp (M) \simeq \Omega\mathrm{map}(M, G/TOP) \simeq \widetilde{\mathcal{V}}(\tau^M), \end{aligned}$$

and

$$\Delta = -\Omega u : \widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau^M)$$

under these identifications. (The minus sign stands for inversion of loops; more about Δ and $-\Omega u$ in section 5.) So, in order to do the Sullivan–Wall–Quinn–Ranicki analysis of

$$\Delta : \widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau^M),$$

we only have to produce the vertical arrows in a commutative diagram of pointed spaces

$$\begin{tikzcd}
\widetilde{\mathscr{S}}(M) \arrow[r, "u"] \arrow[d, "\beta_1"] & \perp (M) \arrow[d, "\beta_2"] \\
Q(\underline{\underline{L}}s_{\blacktriangledown}(M, \nu, -n)_{\%}) \arrow[r, "\text{forget}"] & Q(\underline{\underline{L}}s_{\blacktriangledown}(M, \nu, -n)^{\%})
\end{tikzcd}$$

and to claim that $\Omega\beta_2$ and $\Omega^k\beta_1$ are homotopy equivalences, with $k = \max(1, 5-n)$.

Take β_2 to be the composition

$$\perp (M) \xrightarrow{v} \mathfrak{S}^{TOP}(M, \nu, -n) \longrightarrow Q(\underline{\underline{L}}a_{\blacktriangledown}(M, \nu, -n)^{\%})$$

where the second arrow is $\zeta_{\mathrm{ra}}^{\%}$ from 3.2, subjected to dimension shift. Constructing β_1 then amounts to showing that

$$\text{assembly} \cdot \beta_2 u : \widetilde{\mathscr{S}}(M) \longrightarrow Q\underline{\underline{L}}s_{\blacktriangledown}(M, \nu, -n)$$

is nullhomotopic. Write this map differently, as a composition

$$\widetilde{\mathscr{S}}(M) \xrightarrow{vu} \mathfrak{S}^{TOP}(M, \nu, -n) \xrightarrow{i} \mathfrak{S}^{PD}(M, \nu, -n) \longrightarrow Q\underline{\underline{L}}s_{\blacktriangledown}(M, \nu, -n)$$

where the last arrow is ζ_{ra} from 3.1. Now the map ivu already factors through the Poincaré duality counterpart $\widetilde{\mathscr{S}}^{PD}(M)$ of $\widetilde{\mathscr{S}}(M)$, and $\widetilde{\mathscr{S}}^{PD}(M)$ is contractible.
□

5. Derivatives

We begin by describing derivatives of homeomorphisms and homotopy equivalences in a homotopy theoretic way.

Let M^n be the usual topological manifold, closed for simplicity. Assume that M is embedded in some euclidean space $\mathbb{R}^{n+s}$ with specified normal $\mathbb{R}^s$–bundle ν . Choose an $\mathbb{R}^q$–bundle τ on M , for some q , and a trivialization

$$w : \tau \oplus \nu \xrightarrow{\cong} \varepsilon^{q+s}.$$

Write $S\nu$ and $S\tau$ for the spherical fibrations made from ν and τ by fiberwise one–point compactification.

In addition to the simplicial sets

$$\begin{array}{ccc} TOP(M) & \subset & \widetilde{TOP}(M) \\ \cap & & \cap \\ G(M) & \subset & \widetilde{G}(M) \end{array}$$

from the introduction, we need four others:

$$\begin{array}{ccc} TOP(M,\nu^\infty,\tau^\infty) & \subset & \widetilde{TOP}(M,\nu^\infty,\tau^\infty) \\ \cap & & \cap \\ G(M,S\nu^\infty,S\tau^\infty) & \subset & \widetilde{G}(M,S\nu^\infty,S\tau^\infty). \end{array}$$

A k–simplex in $\widetilde{G}(M,S\nu^\infty,S\tau^\infty)$ is a triple $(f,\mathcal{S},\psi)$ where

$$f : \Delta^k \times M \longrightarrow \Delta^k \times M$$

is a homotopy equivalence respecting $d_j\Delta^k \times M$ for $0 \le j \le k$, and

$$\mathcal{S} : p^*S\nu \longrightarrow p^*S\nu \quad \text{and} \quad \psi : p^*S\tau \longrightarrow p^*S\tau$$

are *stable* fiber maps between spherical fibrations which cover f . (Here p is the projection from $\Delta^k \times M$ to M .) The simplex $(f,\mathcal{S},\psi)$ belongs to $G(M,S\nu^\infty,S\tau^\infty)$ if f respects the projection to Δ^k ; to $\widetilde{TOP}(M,\nu^\infty,\tau^\infty)$ if f is homeomorphic and $\mathcal{S}$, ψ are stable bundle maps respecting the sections at infinity; and $TOP(M,\nu^\infty,\tau^\infty)$ is just $G(M,S\nu^\infty,S\tau^\infty) \cap \widetilde{TOP}(M,\nu^\infty,\tau^\infty)$.

5.1. Proposition. The forgetful maps

$$\begin{array}{lcl} \widetilde{G}(M, S\nu^\infty, S\tau^\infty) \longrightarrow \widetilde{G}(M) & , & \widetilde{TOP}(M, \nu^\infty, \tau^\infty) \longrightarrow \widetilde{TOP}(M), \\ G(M, S\nu^\infty, S\tau^\infty) \longrightarrow G(M) & , & TOP(M, \nu^\infty, \tau^\infty) \longrightarrow TOP(M) \end{array}$$

have preferred and compatible sections.

Proof. These forgetful maps are Kan fibrations, so it is more than enough to produce compatible trivializations up to homotopy. To this end, define simplicial sets

$$\begin{array}{ccc} V & \subset & \widetilde{V} \\ \cap & & \cap \\ W & \subset & \widetilde{W} \end{array}$$

as follows. A k–simplex in $\widetilde{W}$ is a map

$$\mu : \Delta^k_+ \wedge \underline{S}^n \longrightarrow \underline{\underline{\underline{Th}}}\,(\Delta^k \times M, p^*\nu)$$

sending $d_j\Delta^k_+ \wedge \underline{S}^n$ to $\underline{\underline{\underline{Th}}}\,(d_j\Delta^k \times M, p^*\nu)$ for $0 \le j \le k$, with the property that

$$(\text{Thom class of } p^*\nu) \cap [\mu]$$

is a fundamental class for $\Delta^k \times M \simeq M$. If μ respects the projection to Δ^k, then it belongs to $W \subset \widetilde{W}$. If μ is a collapse map, which means that it induces an isomorphism

$$(\Delta^k_+ \wedge \underline{S}^n)/\mu^{-1}(*) \longrightarrow \underline{\underline{\underline{Th}}}\,(\Delta^k \times M, p^*\nu),$$

then μ belongs to $\widetilde{V} \subset \widetilde{W}$. Let V be the intersection $\widetilde{V} \cap W$.

The simplicial monoid $\widetilde{G}(M, S\nu^\infty, S\tau^\infty)$ acts on $\widetilde{W}$ by composition: $(f, \mathcal{S}, \psi)\mu = (f, \mathcal{S})_*\mu$. Further, $\widetilde{W}$ has a base point: the collapse map η coming from the given embedding of M in $\mathbb{R}^{n+s}$ with normal bundle ν. Evaluating the action on the base point gives

$$e_1 : \widetilde{G}(M, S\nu^\infty, S\tau^\infty) \longrightarrow \widetilde{W}.$$

Next, let hU be the space of stable trivializations of $S\tau \oplus S\nu$ as a spherical fibration, and let $U \subset hU$ consist of the stable trivializations of $\tau \oplus \nu$ as a fiber bundle. The given trivialization w of $\tau \oplus \nu$ serves as base point for U and hU. Again, $\widetilde{G}(M, S\nu^\infty, S\tau^\infty)$ acts on hU by composition (on the right):

$$z \cdot (f, \mathcal{S}, \psi) = z \cdot (\psi \oplus \mathcal{S}) \longrightarrow \text{ for } z \in hU.$$

(In more detail, hU should be defined as a simplicial set, and z is a simplex in hU .) Evaluating the action on the base point gives

$$e_2 : \widetilde{G}(M, S\nu^\infty, S\tau^\infty) \longrightarrow hU.$$

Now the map

$$(e_1, e_2, ?) : \widetilde{G}(M, S\nu^\infty, S\tau^\infty) \longrightarrow \widetilde{W} \times hU \times \widetilde{G}(M)$$

is a homotopy equivalence by inspection, and this trivializes the first Kan fibration in 5.1, up to homotopy. Suitable restrictions of $(e_1, e_2, ?)$ trivialize the remaining three:

$$\begin{aligned} G(M, S\nu^\infty, S\tau^\infty) &\xrightarrow{\simeq} W \times hU \times G(M), \\ \widetilde{TOP}(M, \nu^\infty, \tau^\infty) &\xrightarrow{\simeq} \widetilde{V} \times U \times \widetilde{TOP}(M), \\ TOP(M, \nu^\infty, \tau^\infty) &\xrightarrow{\simeq} V \times U \times TOP(M). \end{aligned}$$

$\square$

5.2. Notation. We write the sections in 5.1 in the form

$$\begin{aligned} f &\longmapsto (f, d^\perp f, df) && \text{for a simplex } f \text{ in } \widetilde{TOP}(M), \\ f &\longmapsto (f, D^\perp f, Df) && \text{for a simplex } f \text{ in } \widetilde{G}(M), \end{aligned}$$

so that $d^\perp f$ and df are stable bundle maps covering f , while $D^\perp f$ and Df are stable fiber maps covering f .

We should certainly take a closer look at the section

$$TOP(M) \longrightarrow TOP(M, \nu^\infty, \tau^\infty)\ ;\ f \longmapsto (f, d^\perp f, df).$$

5.3. Supplement. Here the tangent microbundle $\underline{\tau}$ of M (see Milnor [22]) can be substituted for τ , and the Newton–Leibniz–Milnor derivative $\underline{d}f$ of f can be substituted for df .

Try to disregard the fact that $\underline{\tau}$ is a microbundle and not an $\mathbb{R}^n$–bundle. The way to justify 5.3 is

(i) to trivialize the microbundle $\underline{\tau} \oplus \nu$, by a trivialization w ;

(ii) to construct a nullhomotopy for the map

$$f \longmapsto w \cdot (\underline{d}f \oplus d^\perp f)$$

from $TOP(M)$ to the space of stable microbundle trivializations of $\underline{\tau} \oplus \nu$. We will do slightly less (in 5.4 and 5.5), but still enough from a homotopy theoretic point of view.

Let $\xi : E \to M$ be a microbundle with fibers of dimension s , and with zero section $i : M \to E$. A *weak trivialization* of ξ is a homeomorphism germ

$$z : (E, i(M)) \longrightarrow (M \times \mathbb{R}^s, M \times \{0\})$$

such that $z(i(x)) = (x, 0)$ for all $x \in M$. (Being a germ, z need only be defined in a neighbourhood of $i(M) \subset E$.)

5.4. Example. Let ν be the normal bundle in 5.1. Claim: The given embedding of $E(\nu)$ in $\mathbb{R}^{n+s}$ determines a weak trivialization of $\underline{\tau} \oplus \nu$ (but not an honest microbundle trivialization, as one might expect from the smooth case).

Proof: Look at the embedding

$$M \longrightarrow M \times E(\nu)\ ;\ y \longmapsto (y, i(y))$$

which has two specific normal bundles: one is $\underline{\tau} \oplus \nu$, the other is the tangent microbundle of $E(\nu)$ restricted to the zero section. The tangent microbundle of $E(\nu)$ is trivialized.

Let $\overline{U}$ be the space of stable weak trivializations of $\underline{\tau} \oplus \nu$.

5.5. Lemma. The map from $TOP(M)$ to $\overline{U}$ given by

$$f \longmapsto w(\underline{d}f \oplus d^{\perp} f)$$

is nullhomotopic.

Proof. In the proof of 5.1, we used the space V of collapse maps from $\underline{S}^n$ to $\underline{\underline{\underline{Th}}}\,(M, \nu)$. Such a collapse map is always associated to a unique stable codimension zero embedding

$$E(\nu) \longrightarrow \mathbb{R}^{n+s}$$

(stable in the sense that taking products with $\mathbb{R}^k$ on both sides is allowed). The stable codimension zero embedding in turn determines a stable weak trivialization of $\underline{\tau} \oplus \nu$, by 5.3. This is a point $z \in \overline{U}$, and we have therefore described a map

$$g : V \longrightarrow \overline{U}.$$

Let $TOP(M,\nu^\infty)$ be the space of homeomorphisms from M to itself, covered by stable bundle maps from ν to itself. This acts on the left of V by composition, and evaluating the action on the base point $\eta \in V$ gives

$$e : TOP(M,\nu^\infty) \longrightarrow V.$$

(Compare this with e_1 in the proof of 5.1.) Define

$$s : TOP(M) \longrightarrow TOP(M,\nu^\infty)$$

by $s(f) = (f, d^\perp f)$. The map es is nullhomotopic by the very definition of $d^\perp$. Therefore ges is nullhomotopic. But ges is just the map in 5.4. $\square$

This ends the justification of 5.3. Until further notice we continue to use τ and *df* as defined in 5.1, 5.2.

5.6. Example. Define $\widetilde{G}(M,S\tau^\infty)$ like $\widetilde{G}(M,S\nu^\infty,S\nu^\infty)$, deleting all references to $S\nu$. Let

$$\widetilde{G}(M,\tau^\infty) \subset \widetilde{G}(M,S\tau^\infty)$$

be the simplicial subset consisting of the simplices (f,ψ) where ψ is a stable *bundle map* respecting the sections at infinity. From 5.1 and 5.2, we obtain a commutative square

$$\begin{tikzcd}
\widetilde{TOP}(M) \arrow[r] \arrow[d, "f\longmapsto(f,df)"] & \widetilde{G}(M) \arrow[d, "f\longmapsto(f,Df)"] \\
\widetilde{G}(M,\tau^\infty) \arrow[r] & \widetilde{G}(M,S\tau^\infty)
\end{tikzcd}$$

(where f denotes simplices in $\widetilde{TOP}(M)$ or $\widetilde{G}(M)$). The resulting map between horizontal homotopy fibers is

$$\Delta : \widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau^M) \simeq \text{ map } (M, \Omega(G/TOP)),$$

by definition. (See the introduction.)

5.7. Example. Define $\widetilde{G}(M,S\nu^\infty)$ like $\widetilde{G}(M,S\nu^\infty,S\tau^\infty)$, deleting all references to $S\tau$. Let

$$\widetilde{G}(M,\nu^\infty) \longrightarrow \widetilde{G}(M,S\nu^\infty)$$

be the simplicial subset consisting of the simplices (f,ψ) where ψ is a stable bundle map respecting the sections at infinity. Let $\widetilde{G}^s(M) \subset \widetilde{G}(M)$ be the union of the components containing simple homotopy equivalences.

How should the diagram

$$\begin{tikzcd}
\widetilde{TOP}(M) \arrow[r] \arrow[d, "f\longmapsto(f{,}d^{\perp}f)"] & \widetilde{G}^s(M) \arrow[r] \arrow[d, "f\longmapsto(f{,}D^{\perp}f)"] & \widetilde{\mathcal{S}}(M) \arrow[d] \\
\widetilde{G}(M,\nu^{\infty}) \arrow[r] & \widetilde{G}(M,S\nu^{\infty}) \arrow[r] & \mathrm{map}(M,G/TOP)
\end{tikzcd}$$

be completed? (The rows are fibrations up to homotopy.) Answer: Substitute u from 4.11 for the broken arrow. The resulting diagram will be sufficiently commutative. Comparison with 5.6 shows that Δ in 5.6 equals $-\Omega u$.

5.8. Example. Here we make the identification $\tau = \underline{\tau}$ and $df = \underline{d}f$ of 5.3. Let $G(M,\tau)$ be the space of homotopy equivalences $f : M \longrightarrow M$ covered by bundle map germs from τ to τ (where τ is the tangent microbundle). Let $G(M,S\tau^{\infty})$ be the space of homotopy equivalences $f : M \to M$ covered by stable fiber maps from $S\tau$ to $S\tau$. By 5.1, 5.2, 5.3, we have a square

$$\begin{tikzcd}
TOP(M) \arrow[r] \arrow[d, "f\longmapsto(f{,}df)"] & G(M) \arrow[d, "f\longmapsto(f{,}Df)"] \\
G(M,\tau) \arrow[r] & G(M,S\tau^{\infty})
\end{tikzcd}$$

which commutes up to a preferred homotopy. The induced map of horizontal homotopy fibers is, by definition,

$$\Delta : \mathcal{V}(M) \longrightarrow \mathcal{V}(\tau)$$

from the introduction.

There are bounded versions of 5.1 and 5.3. Let N be a topological manifold equipped with a proper map

$$c : N \longrightarrow \mathbb{R}^i, \text{ for some } i.$$

Let $G^b(M)$ and $TOP^b(M)$ be the spaces (alias simplicial sets) of bounded homotopy equivalences/bounded homeomorphisms

$$f : N \longrightarrow N$$

(so that $c - cf$ has bounded image in $\mathbb{R}^i$). There are block versions $\widetilde{G}^b(M)$ and $\widetilde{TOP}^b(M)$ as usual.

Choose a topological $\mathbb{R}^s$–bundle ν on N , and a bounded codimension zero embedding

$$j : E(\nu) \hookrightarrow \mathbb{R}^i \times \mathbb{R}^k \qquad \text{for suitable } k$$

(so that the difference between

$$E(\nu) \to N \xrightarrow{c} \mathbb{R}^i \qquad \text{and} \qquad E(\nu) \xrightarrow{j} \mathbb{R}^i \times \mathbb{R}^k \longrightarrow \mathbb{R}^i$$

is a map having bounded image in $\mathbb{R}^i$). Choose also an $\mathbb{R}^q$–bundle τ on M inverse to ν , so that $\tau \oplus \nu$ comes equipped with a trivialization

$$w : \tau \oplus \nu \longrightarrow \varepsilon^{s+q}.$$

5.9. Proposition. The forgetful maps

$$\begin{array}{llll} \widetilde{G}^b(N, S\nu^\infty, S\tau^\infty) \longrightarrow \widetilde{G}^b(N) & , & \widetilde{TOP}^b(N, \nu^\infty, \tau^\infty) \longrightarrow \widetilde{TOP}^b(N) \\ G^b(N, S\nu^\infty, S\tau^\infty) \longrightarrow G^b(N) & , & TOP^b(N, \nu^\infty, \tau^\infty) \longrightarrow TOP^b(N) \end{array}$$

have preferred and compatible sections.

The notation is self–explanatory. Again, we write these sections in the form

$$\begin{array}{ll} f \longmapsto (f, d^\perp f, df) & \text{for a simplex } f \text{ in } \widetilde{TOP}^b(N), \\ f \longmapsto (f, D^\perp f, Df) & \text{for a simplex } f \text{ in } \widetilde{G}^b(N), \end{array}$$

as in 5.2. Again, we take a closer look at the section

$$TOP^b(N) \longrightarrow TOP^b(N, \nu^\infty, \tau^\infty); f \longmapsto (f, d^\perp f, df).$$

5.10. Supplement. Here the tangent microbundle of N can be substituted for τ , and the Newton–Milnor derivative can be substituted for df .

5.11. Notation. For a microbundle ξ on N , let $TOP^b(N, \xi)$ be the space of bounded homeomorphisms from N to N , covered by bundle map germs from ξ to ξ . This can be thought of as a simplicial group, or as a virtual space with group structure (see Part I, section 0).

It is of course possible in 5.9, 5.10, 5.11 to replace

$$c : N \longrightarrow \mathbb{R}^i$$

by a proper map

$$c : N \longrightarrow V$$

where V is some finite dimensional real vector space with inner product. The standard choice for N is

$$N = M \times V \qquad\qquad\qquad (M \text{ as in } 5.1)$$

and c is the projection to V. At this point it is natural to make V into a variable. Let $\mathcal{J}$ be the category of finite dimensional real vector spaces with inner product, with isometric linear maps as morphisms. (See Part I, 1.11 and sections 2, 3.) For an object V in $\mathcal{J}$, let

$$\begin{aligned} E_1(V) &:= TOP^b(M \times V), \\ E_2(V) &:= TOP^b(M \times V \times \mathbb{R}^\infty, \tau \times \varepsilon^V \times \varepsilon^0) \\ &= \lim_i \; TOP^b(M \times V \times \mathbb{R}^i, \tau \times \varepsilon^V \times \varepsilon^0) \end{aligned}$$

where ε^V is the tangent microbundle of V and ε^0 is the zero–dimensional microbundle on $\mathbb{R}^i$ (and τ is the tangent microbundle of M). Both $E_1(V)$ and $E_2(V)$ are continuous functors in the variable V, and

$$\begin{aligned} \Delta : E_1(V) &\longrightarrow E_2(V) \\ f &\longmapsto (f, df) \times \mathrm{id} \end{aligned}$$

(where id is the identity on $\mathbb{R}^\infty$, and df is the N.–L.–M. derivative) is a natural transformation respecting the group structure. Now apply the hyperplane test. (This is summarized in Part I, 3.8, with a small inaccuracy: the source of Φ should be $E(\mathbb{R}^\infty)/E(\mathbb{R}^0)$.) The result is a commutative square

$$\begin{tikzcd} E_1(\mathbb{R}^\infty)/E_1(\mathbb{R}^0) \arrow[r, "\nabla"] \arrow[d, "\Phi"] & E_2(\mathbb{R}^\infty)/E_2(\mathbb{R}^0) \arrow[d, "\Phi"] \\ Q(S_+ \wedge_{Z_2} F_1) \arrow[r, "\nabla"] & Q(S_+ \wedge_{Z_2} F_2) \end{tikzcd}$$

where F_1, F_2 are the coordinate free spectra with involution given by

$$F_i(V) = E_i(V \oplus \mathbb{R})/E_i(V), \text{ for } i = 1, 2.$$

We know the left column from Part I. We know that it is permitted to replace

$$F_1 = \Omega \underline{\underline{Wh}}^{TOP}(M)$$

by its 0–connected Postnikov cover

$$\Omega \underline{\underline{Wh}}s^{TOP}(M)$$

at the price of replacing

$$E_1(\mathbb{R}^\infty)/E_1(\mathbb{R}^0) = TOP^b(M \times \mathbb{R}^\infty)/TOP(M)$$

by the connected space

$${}^{\text{pos}}TOP^b(M \times \mathbb{R}^\infty)/TOP(M) \simeq \widetilde{TOP}(M)/TOP(M) \simeq \widetilde{\mathcal{V}}(M)/\mathcal{V}(M).$$

(See section 4 of Part I.) Note also that

$$\begin{aligned} E_2(\mathbb{R}^\infty)/E_2(\mathbb{R}^0) &= TOP^b(M \times \mathbb{R}^\infty \times \mathbb{R}^\infty, \tau \times \varepsilon^\infty \times \varepsilon^0)/TOP^b(M \times \mathbb{R}^\infty, \tau \times \varepsilon^0) \\ &\simeq TOP^b(M \times \mathbb{R}^\infty, \tau \times \varepsilon^\infty)/TOP^b(M \times \mathbb{R}^\infty, \tau \times \varepsilon^0) \\ &\cong \operatorname{Aut}(\tau \times \varepsilon^\infty)/\operatorname{Aut}(\tau \times \varepsilon^0) \amalg \text{ something} \\ &\simeq \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \amalg \text{ something else },\end{aligned}$$

which we may as well cut back to $\widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)$. Finally, write

$$F_2 =: \Omega \underline{\underline{Wh}}^{TOP}(\tau).$$

The square then turns into one that was promised in the introduction:

$$\begin{tikzcd} \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \arrow[r, "\nabla"] \arrow[d, "\Phi"] & \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \arrow[d, "\Phi"] \\ Q(S_+ \wedge_{Z_2} \Omega \underline{\underline{Whs}}^{TOP}(M)) \arrow[r, "\nabla"] & Q(S_+^\infty \wedge_{Z_2} \Omega \underline{\underline{Wh}}^{TOP}(\tau)). \end{tikzcd}$$

This raises a few questions. Firstly, is the map

$$\nabla : \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \longrightarrow \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)$$

compatible with its namesakes in 5.6 and 5.8? If so, we can compose the maps Φ with the projections

$$\widetilde{\mathcal{V}}(M) \longrightarrow \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \quad \text{and} \quad \widetilde{\mathcal{V}}(\tau) \longrightarrow \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)$$

to obtain the geometric face of the cube in the introduction:

(&)
$$\begin{tikzcd} \widetilde{\mathcal{V}}(M) \arrow[r, "\nabla"] \arrow[d, "\Phi\cdot\text{pr}"] & \mathcal{V}(\tau) \arrow[d, "\Phi\cdot\text{pr}"] \\ Q(S_+ \wedge_{Z_2} \Omega \underline{\underline{Whs}}^{TOP}(M)) \arrow[r, "\nabla"] & Q(S_+^\infty \wedge_{Z_2} \Omega \underline{\underline{Wh}}^{TOP}(\tau)). \end{tikzcd}$$

Secondly, is the map

$$\Phi : \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \longrightarrow Q(S^{\infty}_{+} \wedge_{Z_2} \Omega \underline{\underline{Wh}}^{TOP}(\tau))$$

highly connected like its counterpart

$$\Phi : \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \longrightarrow Q(S^{\infty}_{+} \wedge_{Z_2} \Omega \underline{\underline{Wh}}s^{TOP}(M)) \;?$$

If so, we can in a stable range identify the map of vertical homotopy fibers in (&) with

$$\nabla : \mathcal{V}(M) \longrightarrow \mathcal{V}(\tau)$$

of 5.8. These two questions are answered in 5.12, 13, 14 below. A third question: Why should one write $\Omega \underline{\underline{Wh}}^{TOP}(\tau)$ for F_2 ? This suggests something about automorphisms of τ and their behaviour under stabilization, a kind of concordance theory of τ . Let E_3 be the functor from $\mathcal{J}$ to virtual spaces with group structure given by

$$V \longmapsto \mathrm{Aut}\,(\tau \oplus \varepsilon^V)$$

where ε^V is the trivial (micro–)bundle with fiber V (on M , this time). There are natural inclusions

$$E_3(V) \hookrightarrow TOP(M, \tau \oplus \varepsilon^V) \hookrightarrow TOP^b(M \times V \times \mathbb{R}^{\infty}, \tau \times \varepsilon^V \times \varepsilon^0) = E_2(V),$$

the second one given by taking product with the identity on $V \times \mathbb{R}^{\infty}$. The induced map

$$F_3(V) := E_3(V \oplus \mathbb{R})/E_3(V) \longrightarrow F_2(V)$$

is a homotopy equivalence on base point components, for every V . As spectra, therefore, F_3 and F_2 are homotopy equivalent. Now F_3 deserves to be called $\Omega \underline{\underline{Wh}}^{TOP}(\tau)$.

(But it is F_2 which receives an interesting map from F_1 .)

5.12. Proposition. The diagram

$$\begin{tikzcd}
\mathcal{V}(M) \arrow[r] \arrow[d, "\nabla \text{ from } 5.8"] & \widetilde{\mathcal{V}}(M) \arrow[r] \arrow[d, "\nabla \text{ from } 5.6"] & \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \arrow[d, "\nabla \text{ just constructed}"] \\
\mathcal{V}(\tau) \arrow[r] & \widetilde{\mathcal{V}}(\tau) \arrow[r] & \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)
\end{tikzcd}$$

commutes up to homotopy.

Comment and proof. The three maps labelled ∇ are supposed to make up one morphism from the top row, which is a fibration up to homotopy, to the bottom row, which is another fibration up to homotopy. So in principle it is necessary to produce three homotopies (as in the comment following Part I, 2.7). However, we need not be as explicit as that. Besides, the commutativity of the subdiagram

(1)
$$\begin{tikzcd}
\mathcal{V}(M) \arrow[r] \arrow[d] & \widetilde{\mathcal{V}}(M) \arrow[r] \arrow[d] & \widetilde{\mathcal{V}}(M)/\mathcal{V}(M) \\
\mathcal{V}(\tau) \arrow[r] & \widetilde{\mathcal{V}}(\tau) \arrow[r] & \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau)
\end{tikzcd}$$

can be taken for granted. Compare (1) with

(2)
$$\begin{tikzcd}
TOP(M) \arrow[r] \arrow[d] & \widetilde{TOP}(M) \arrow[r] \arrow[d] & \widetilde{TOP}(M)/TOP(M) \\
\widetilde{TOP}(M,\tau) \arrow[r] & \widetilde{TOP}(M,\tau^\infty) \arrow[r] & \widetilde{TOP}(M,\tau^\infty)/TOP(M,\tau)
\end{tikzcd}$$

where the vertical arrows are given by

$$f \longmapsto (f, df) \qquad \text{as in 5.2.}$$

(The notation is self–explanatory, we hope; there are fibrations

$$\begin{aligned}
\operatorname{Aut}(\tau) &\hookrightarrow \widetilde{TOP}(M,\tau) \longrightarrow \widetilde{TOP}(M), \\
\operatorname{Aut}(\tau^\infty) &\hookrightarrow \widetilde{TOP}(M,\tau^\infty) \longrightarrow \widetilde{TOP}(M),
\end{aligned}$$

and $\operatorname{Aut}(\tau^\infty)$ denotes the group of stable automorphisms of τ .) There is a forgetful morphism from (1) to (2) which is a componentwise homotopy equivalence when restricted to the terms in the right column. (The morphism sends

$$\widetilde{\mathcal{V}}(M) = \text{hofiber}\ (\widetilde{TOP}(M) \longrightarrow \widetilde{G}(M))$$

forgetfully to $\widetilde{TOP}(M)$, and it sends

$$\widetilde{\mathcal{V}}(\tau) = \text{hofiber}\ (\operatorname{Aut}(\tau^\infty) \to \operatorname{Aut}(S\tau))$$

forgetfully to $\widetilde{TOP}(M,\tau^\infty) \supset \operatorname{Aut}(\tau^\infty)$.) It follows that we can prove 5.12 by supplying and identifying the missing arrow in the right column of (2) .

Now the usual bisimplicial arguments show that one can trade (2) for a homotopy equivalent diagram

(3)

$$\begin{tikzcd}
TOP(M) \arrow[r] \arrow[d] & {}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty) \arrow[r] \arrow[d] & {}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty)/TOP(M) \\
{}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty,\tau\times\varepsilon^0) \arrow[r] & {}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty,\tau\times\varepsilon^\infty) \arrow[r] & \frac{{}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty,\tau\times\varepsilon^\infty)}{{}^{\mathrm{pos}}TOP^b(M\times\mathbb{R}^\infty,\tau\times\varepsilon^0)}
\end{tikzcd}$$

where the vertical arrows are now simplicial group homomorphisms. (They are given by

$$f \longmapsto (f, df),$$

where df is the N.–L.–M. derivative.) So the missing arrow in the right column of (3) is easy to supply and identify. □

5.13. Proposition. If $n = \dim(M)$ is greater than four, and if the stabilization maps

$$\Omega^n(TOP_{n+1}/TOP_n) \to \Omega^{n+1}(TOP_{n+2}/TOP_{n+1}) \to \Omega^{n+2}(TOP_{n+3}/TOP_{n+2}) \to \dots$$

are all k–connected, then

$$\Phi : \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \longrightarrow Q(S^\infty_+ \wedge_{Z_2} \Omega \underline{\underline{Wh}}^{TOP}(\tau))$$

is componentwise k–connected. (That is, the homotopy fiber over an arbitrary point in the target is either empty or $(k-1)$–connected.)

Proof. Recall that Φ in 5.13 extends to

$$\Phi : E_2(\mathbb{R}^\infty)/E_2(\mathbb{R}^0) \longrightarrow Q(S^\infty_+ \wedge_{Z_2} F_2)$$

where

$$E_2(\mathbb{R}^\infty)/E_2(\mathbb{R}^0) \simeq \widetilde{\mathcal{V}}(\tau)/\mathcal{V}(\tau) \amalg \text{ something else.}$$

So it suffices to prove that the extended Φ is componentwise k–connected. The hypothesis in 5.13 implies firstly that, for $i \geq 0$, the stabilization maps

$$TOP_{n+i+1}/TOP_{n+i} \longrightarrow \Omega(TOP_{n+i+2}/TOP_{n+i+1})$$

are $(k+i)$–connected (because TOP_{r+1}/TOP_r is $(r-1)$–connected if $r > 4$; see [18], Essay V §5). It implies then that the stabilization maps

$$F_2(\mathbb{R}^i) \longrightarrow \Omega F_2(\mathbb{R}^{i+1}) \longrightarrow \Omega^2 F_2(\mathbb{R}^{i+2}) \longrightarrow \dots$$

are componentwise $(k+i)$–connected. (Recall that

$$F_2(\mathbb{R}^i) = E_2(\mathbb{R}^{i+1})/E_2(\mathbb{R}^i).)$$

Now Φ respects filtrations. In more detail, for each $i \geq 0$ there is a sufficiently commutative diagram

$$\begin{tikzcd}
E_2(\mathbb{R}^i/E_2(\mathbb{R}^0) \arrow[r, "\Phi"] \arrow[d] & Q(S_+^{i-1} \wedge_{Z_2} F_2) \arrow[d] \\
E_2(\mathbb{R}^{i+1})/E_2(\mathbb{R}^0) \arrow[r, "\Phi"] \arrow[d] & Q(S_+^{i} \wedge_{Z_2} F_2) \arrow[d] \\
F_2(\mathbb{R}^i) \arrow[r] & Q\Sigma^i F_2 = \operatorname*{hocolim}_j \Omega^j F_2(\mathbb{R}^{i+j})
\end{tikzcd}$$

whose columns are fibrations up to homotopy. Assuming for induction purposes that the top horizontal arrow is componentwise k–connected, and using the fact that the lower horizontal arrow is componentwise $(k+i)$–connected, we find that the middle horizontal arrow is componentwise k–connected. Letting i tend to infinity completes the proof.

5.14. Remark. With $k=1$ (and $n>4$) the hypothesis of 5.13 is always fulfilled (Theorem 5.2, Essay V, §5 of [18]). Further, Morlet's sliced smoothing theory in [25] (see also 6.4 below) gives fibrations up to homotopy

$$\Omega^{r+2}(O_{r+1}/O_r) \hookrightarrow \Omega^{r+2}(TOP_{r+1}/TOP_r) \longrightarrow \mathcal{C}^{\mathrm{DIFF}}(D^r),$$

with $r>4$, where $\mathcal{C}$ denotes concordance spaces. These are compatible with stabilization. So, if the stabilization maps

$$\mathcal{C}^{\mathrm{DIFF}}(D^n) \longrightarrow \mathcal{C}^{\mathrm{DIFF}}(D^{n+1}) \longrightarrow \mathcal{C}^{\mathrm{DIFF}}(D^{n+2}) \longrightarrow \dots,$$
$$\Omega^{n+2}S^n \longrightarrow \Omega^{n+3}S^{n+1} \longrightarrow \Omega^{n+4}S^{n+2} \longrightarrow \dots$$

are all $(k-2)$–connected, then the hypothesis of 5.13 is satisfied. Therefore, with the work of Igusa [16] and Freudenthal's theorem in mind, we can say that Φ in 5.13 is highly connected.

5.15. Remark. The results of this section remain valid for a compact M with nonempty boundary. There are almost no changes in notation. For example, it is convenient to write $TOP(M)$ and $G(M)$ as before, meaning the spaces of self homeomorphisms and self homotopy equivalences of M which are the identity near M. Similarly $G(M,\tau)$ consists of the homotopy equivalences $f: M \to M$ which are the identity near ∂M, covered by bundle maps $\mathcal{S}: \tau \to \tau$ which are the identity over a neighbourhood of ∂M; and so on. In 5.7, however, it is necessary to replace $\operatorname{map}(M, G/TOP)$ by $\operatorname{map}(M/\partial M, G/TOP)$. (Note that the slash $/$ has two different meanings here.)

6. Concordance Theory

The goal here is to reformultate (not reprove) some of Waldhausen's results in concordance theory until they take the shape of a commutative square

$$\begin{tikzcd}
\Omega \underset{===}{Wh}s^{TOP}(M) \arrow[r, "\nabla"] \arrow[d, "\simeq"] & \Omega \underset{===}{Wh}^{TOP}(\tau) \arrow[d, "\simeq"] \\
\underset{=}{A}s(M)_{\%} \arrow[r, "\text{forget}"] & \underset{=}{A}s(M)^{\%}.
\end{tikzcd}$$

As always, M is a compact manifold and τ is its tangent microbundle. We do not assume that $\partial M = \emptyset$; see 5.15.

The first step is to show that the homotopy types of $\Omega \underset{===}{Wh}s^{TOP}(M)$ and $\Omega \underset{===}{Wh}^{TOP}(\tau)$ only depend on the homotopy type of M . We use the *disk bundle transfer* of [7]. Racell that

$$\mathcal{C}(M) = TOP(M \times [0,1], M \times \{1\})$$

and let

$$\mathcal{C}'(M) = TOP(M \times [0,1], M \times \{1\} \cup \partial M \times [0,1])$$

so that $\mathcal{C}'(M) \supset \mathcal{C}(M)$. For a topological disk bundle

$$p : \overline{M} \longrightarrow M$$

with fiber D^k , let $\mathcal{C}'(p)$ consist of all pairs

$$(f, \overline{f}) \in \mathcal{C}'(M) \times \mathcal{C}(\overline{M})$$

such that $p\overline{f} = fp$. The *disk bundle transfer*

$$p^! : \mathcal{C}(M) \longrightarrow \mathcal{C}(\overline{M})$$

is obtained by inverting two of the homotopy equivalences in the diagram

$$\mathcal{C}(M) \overset{\simeq}{\hookrightarrow} \mathcal{C}'(M) \overset{\simeq}{\longleftarrow} \mathcal{C}'(p) \longrightarrow \mathcal{C}'(M) \overset{\simeq}{\hookleftarrow} \mathcal{C}(\overline{M}).$$

As such it is well defined up to contractible indeterminacy (because inverses e' of homotopy equivalences e are determined up to contractible choice if chosen together with a homotopy $e'e \simeq \mathrm{id}$). It is functorial; that is,

$$(p_1 p_2)^! = p_2^! p_1^!$$

for disk bundles

$$p_1 : \overline{M} \longrightarrow M \qquad \text{and} \qquad p_2 : \overline{\overline{M}} \longrightarrow \overline{M}.$$

Now remember that $\Omega^2 \underline{\underline{\underline{Wh}}}^{TOP}(M)$, as an ordinary spectrum, has i–th term

$$\Omega(TOP^b(M \times \mathbb{R}^{i+1})/TOP^b(M \times \mathbb{R}^i)) \simeq \mathcal{C}^b(M \times \mathbb{R}^i).$$

See Part I, 1.8. The structure maps

$$\mathcal{C}^b(M \times \mathbb{R}^i) \longrightarrow \Omega\mathcal{C}^b(M \times \mathbb{R}^{i+1})$$

are compositions

$$\mathcal{C}^b(M \times \mathbb{R}^i) \xrightarrow{\text{stabilization}} \mathcal{C}^b(M \times [0,1] \times \mathbb{R}^i) \simeq \Omega\mathcal{C}^b(M \times \mathbb{R}^{i+1}).$$

See Part I, 1.5 and 1.12. For a disk bundle

$$p : \overline{M} \longrightarrow M$$

as above, we have disk bundle transfers

$$p^! : \mathcal{C}^b(M \times \mathbb{R}^i) \longrightarrow \mathcal{C}^b(\overline{M} \times \mathbb{R}^i) \qquad \text{for all } i \geq 0,$$

by a straightforward generalization of the case $i = 0$. These disk bundle transfers commute with the stabilization maps

$$\mathcal{C}^b(M \times \mathbb{R}^i) \to \mathcal{C}^b(M \times [0,1] \times \mathbb{R}^i) \text{ and } \mathcal{C}^b(\overline{M} \times \mathbb{R}^i) \to \mathcal{C}^b(\overline{M} \times [0,1] \times \mathbb{R}^i)$$

because stabilization is also a disk bundle transfer (for the trivial disk bundle with fiber $[0,1]$). They commute with the Anderson–Hsiang homotopy equivalences

$$\begin{aligned} \mathcal{C}^b(M \times [0,1] \times \mathbb{R}^i) &\simeq \Omega\mathcal{C}^b(M \times \mathbb{R}^{i+1}), \\ \mathcal{C}^b(\overline{M} \times [0,1] \times \mathbb{R}^i) &\simeq \Omega\mathcal{C}^b(\overline{M} \times \mathbb{R}^{i+1}) \end{aligned}$$

by inspection, and so combine to give a map of spectra

$$p^! : \Omega^2 \underline{\underline{\underline{Wh}}}^{TOP}(M) \longrightarrow \Omega^2 \underline{\underline{\underline{Wh}}}^{TOP}(\overline{M}).$$

This map is a homotopy equivalence. By the functoriality of disk bundle transfers, it is sufficient to prove this when the disk bundle in question is trivial (with fiber D^k), because any disk bundle is stably invertible with respect to Whitney sum. Using functoriality once more, we can also assume that the k in D^k is equal to 1, and then the claim is obvious.

Given homotopy equivalent manifolds M_1 and M_2 (both compact), let $\overline{M}_1$ and $\overline{M}_2$ be the total spaces of the normal disk bundles of M_1 and M_2 in a high dimensional euclidean space. The homotopy equivalence $M_1 \simeq M_2$ can be realized by a codimension zero embedding of $\overline{M}_1$ and $\overline{M}_2$. The complement of $\overline{M}_1$ in $\overline{M}_2$ is an h–cobordism from $\partial\overline{M}_1$ to $\partial\overline{M}_2$. Therefore

$$\underline{\underline{Wh}}^{TOP}(M_1) \simeq \underline{\underline{Wh}}^{TOP}(\overline{M}_1) \overset{\simeq}{\hookrightarrow} \underline{\underline{Wh}}^{TOP}(\overline{M}_2) \simeq \underline{\underline{Wh}}^{TOP}(M_2),$$

and the 1–connected covers $\underline{\underline{Wh}}s^{TOP}(M_1)$, $\underline{\underline{Wh}}s^{TOP}(M_2)$ will also be homotopy equivalent.

A similar argument establishes the homotopy invariance of $\Omega\underline{\underline{Wh}}^{TOP}(\tau^M)$, with respect to M . The following abbreviation is useful.

6.1. Notation. For a manifold N equipped with a proper map $c : N \to \mathbb{R}^i$ and a microbundle $\xi : E \to N$, let

$$\mathcal{C}^b(N, \xi)$$

be the space of pairs $(f, \mathcal{S})$ where f is a bounded homeomorphism from $N \times [0,1]$ to itself, and $\mathcal{S}$ is a bundle map germ from $\xi \times \varepsilon^1 : E \times \mathbb{R} \longrightarrow N \times [0,1]$ to itself which covers f . Two boundary conditions are understood:

(1) f and $\mathcal{S}$ are the identity in/over a neighbourhood of $N \times \{0\} \cup \partial N \times [0,1]$;

(2) over $N \times \{1\}$, the germ $\mathcal{S}$ has the form

$$\mathcal{S}_1 \times \mathrm{id} : \xi \times \varepsilon^1 \longrightarrow \xi \times \varepsilon^1$$

for some bundle map germ $\mathcal{S}_1$ from ξ to ξ .

Write $\tau = \tau^M$. The spectrum $\Omega^2\underline{\underline{Wh}}^{TOP}(\tau)$, as an ordinary spectrum, can be described as having i–th term

$$\mathcal{C}^b(M \times \mathbb{R}^i \times \mathbb{R}^\infty, \tau \times \varepsilon^i \times \varepsilon^0) = \lim_k \, \mathcal{C}^b(M \times \mathbb{R}^i \times \mathbb{R}^k, \tau \times \varepsilon^i \times \varepsilon^0).$$

Using this description it is easy to associate to a disk bundle

$$p : \overline{M} \longrightarrow M$$

a disk bundle transfer

$$p^! : \Omega^2\underline{\underline{Wh}}^{TOP}(\tau) \longrightarrow \Omega^2\underline{\underline{Wh}}^{TOP}(\overline{\tau}),$$

where $\overline{\tau}$ is the tangent microbundle of $\overline{M}$. Again, the disk bundle transfer is a homotopy equivalence.

Continuing in this way, we can describe

$$\Omega\nabla : \Omega^2 \underline{\underline{\underline{Wh}}}^{TOP}(M) \longrightarrow \Omega^2 \underline{\underline{\underline{Wh}}}^{TOP}(\tau)$$

as an ordinary map with i–th term

$$\begin{aligned} \mathcal{C}^b(M \times \mathbb{R}^i) &\longrightarrow \mathcal{C}^b(M \times \mathbb{R}^i \times \mathbb{R}^\infty, \tau \times \varepsilon^i, \varepsilon^0) \\ f &\longmapsto (f \times \mathrm{id}, df) \end{aligned}$$

where id is the identity on $\mathbb{R}^\infty$ and df is the N.–L.–M. derivative. It is not hard to see that $\Omega\nabla$ commutes with disk bundle transfers (up to a preferred homotopy).

6.2. Conclusion. For any disk bundle $p : \overline{M} \longrightarrow M$, the maps

$$\begin{aligned} &\nabla : \Omega \underline{\underline{\underline{Wh}}}s^{TOP}(M) \longrightarrow \Omega \underline{\underline{\underline{Wh}}}^{TOP}(\tau) \\ \text{and} \quad &\nabla : \Omega \underline{\underline{\underline{Wh}}}s^{TOP}(\overline{M}) \longrightarrow \Omega \underline{\underline{\underline{Wh}}}^{TOP}(\overline{\tau}) \end{aligned}$$

are homotopy equivalent.

¿From now on, we can assume that M is smooth and framed. (If not, replace M by the total space of its normal disk bundle in some euclidean space.) In this situation we may compare ∇ with its smooth version ∇^{DIFF}.

6.3. Proposition. The square of spectra

$$\begin{tikzcd}
\Omega \underline{\underline{\underline{Wh}}}s^{\mathrm{DIFF}}(M) \arrow[r, "\nabla^{\mathrm{DIFF}}"] \arrow[d] & \Omega \underline{\underline{\underline{Wh}}}^{\mathrm{DIFF}}(\tau) \arrow[d] \\
\Omega \underline{\underline{\underline{Wh}}}s^{TOP}(M) \arrow[r, "\nabla"] & \Omega \underline{\underline{\underline{Wh}}}^{TOP}(\tau)
\end{tikzcd}$$

is a homotopy pushout square (alias homotopy pullback square).

This comes out of Morlet's sliced smoothing theory, which we now review. Write Vect(τ) for the space of vector bundle structures on the microbundle τ. (Define it as a simplicial set: a k–simplex is a vector bundle structure on $p^*\tau$, where p is the projection from $\Delta^k \times M$ to M.) If M is smooth, then Vect(τ) has a base point w, and there is a map

$$\begin{aligned} e_M : TOP(M) &\longrightarrow \mathrm{Vect}\,(\tau) \\ f &\longmapsto f_* w. \end{aligned}$$

Its restriction to $\mathrm{DIFF}(M)$ is zero.

6.4. Fact. (Morlet [25]; Burghelea–Lashof [8]; Kirby–Siebenmann [18], Essay V). The inclusion

$$\mathrm{DIFF}(M) \hookrightarrow \mathrm{hofiber}\,(e_M)$$

is a homotopy equivalence if $\dim(M) \neq 4$.

There is a relative version which we do not state. It implies comparison results for concordance spaces, since

$\mathcal{C}(M) = TOP(M \times [0,1], M \times \{1\})$ and $\mathcal{C}^{\mathrm{DIFF}}(M) = \mathrm{DIFF}(M \times [0,1], M \times \{1\})$.

Proof of 6.3. Note first that

$$\Omega \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau) \simeq \mathrm{map}(M/\partial M, \Sigma^n \underline{S}^0) \simeq M_+ \wedge \underline{S}^0$$

$$\text{and} \quad \Omega \underline{\underline{Wh}}^{TOP}(\tau) \simeq \mathrm{map}(M/\partial M, \Sigma^n \underline{\underline{A}}(*)) \simeq M_+ \wedge \underline{\underline{A}}(*),$$

where $\Sigma^n \underline{S}^0$ and $\Sigma^n \underline{\underline{A}}(*)$ arfe certain spectra with i–th term

$$O_{n+i+1}/O_{n+i} \qquad \text{and} \qquad TOP_{n+i+1}/TOP_{n+i},$$

respectively. This is rather easy to see if M is a compact codimension zero submanifold of $\mathbb{R}^n$, and so by 6.2 (and a smooth version of 6.2) it is true in general. Now the inclusion of $\underline{S}^0$ in $\underline{\underline{A}}(*)$ is 1–connected. (With the above geometric definition of $\underline{\underline{A}}(*)$, this is a consequence of Thm. 5.2 in Essay V, §5 of [18].) It follows that the map

$$\Omega \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau) \hookrightarrow \Omega \underline{\underline{Wh}}^{TOP}(\tau)$$

is 1–connected. Moreover, $\Omega \underline{\underline{Whs}}^{\mathrm{DIFF}}(M)$ and $\Omega \underline{\underline{Whs}}^{TOP}(M)$ are 0–connected. To prove 6.3, therefore, it is enough to show that

$$\begin{tikzcd}
Q\Omega^2 \underline{\underline{Whs}}^{\mathrm{DIFF}}(M) \arrow[r] \arrow[d] & Q\Omega^2 \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau) \arrow[d, "V"] \\
Q\Omega^2 \underline{\underline{Whs}}^{TOP}(M) \arrow[r] & Q\Omega^2 \underline{\underline{Wh}}^{TOP}(\tau)
\end{tikzcd}$$

is a homotopy pullback square. But this is the limit over i of commutative squares

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{DIFF}}(M \times D^i) \arrow[r, "\Omega\nabla^{\mathrm{DIFF}}"] \arrow[d] & \mathcal{C}^{\mathrm{DIFF,b}}(M \times D^i \times \mathbb{R}^\infty, \tau \times \varepsilon^i \times \varepsilon^0) \arrow[d] \\
\mathcal{C}(M \times D^i) \arrow[r, "\Omega\nabla"] & \mathcal{C}^b(M \times D^i \times \mathbb{R}^\infty, \tau \times \varepsilon^i \times \varepsilon^0).
\end{tikzcd}$$

For $i+n>4$, these squares are homotopy pullback squares: The relative version of 6.4 describes one of the vertical homotopy fibers (the one on the left), and the other one is obvious. Note that

$$\mathcal{C}^b(M \times D^i \times \mathbb{R}^\infty) \;\simeq\; * \;\simeq\; \mathcal{C}^{\mathrm{DIFF,b}}(M \times D^i \times \mathbb{R}^\infty).$$

□

6.5. Digression. Another useful consequence of 6.4 is that

$$\begin{tikzcd}
\mathcal{V}^{\mathrm{DIFF}}(M) \arrow[r, "\nabla^{\mathrm{DIFF}}"] \arrow[d] & \mathcal{V}^{\mathrm{DIFF}}(\tau) \arrow[d] \\
\mathcal{V}(M) \arrow[r, "\nabla"] & \mathcal{V}(\tau)
\end{tikzcd}$$

is a homotopy pullback square if $\dim(M) \neq 4$. Again, 6.4 describes one of the vertical homotopy fibers, and the other one is obvious.

6.6. Proposition. The map

$$\nabla^{\mathrm{DIFF}} : \Omega \underline{\underline{Whs}}^{\mathrm{DIFF}}(M) \longrightarrow \Omega \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau)$$

is nullhomotopic.

Sketch proof (extracted from the work of Goodwillie [13]). We can assume that M is a codimension zero compact submanifold of $\mathbb{R}^n$. We shall in fact prove that

$$\Omega\nabla^{\mathrm{DIFF}} : \Omega^2 \underline{\underline{Wh}}^{\mathrm{DIFF}}(M) \longrightarrow \Omega^2 \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau)$$

is nullhomotopic. (Note the disappearance of the decoration s.) Think of $\Omega^2 \underline{\underline{Wh}}^{\mathrm{DIFF}}(M)$ as a spectrum with i–th term

$$\mathcal{C}^{\mathrm{DIFF,b}}(M \times \mathbb{R}^i)$$

and think of

$$\Omega^2 \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau) \qquad \simeq \qquad \underline{\underline{\mathrm{map}}}\,(M/\partial M, \Omega \underline{S}^n)$$

as having i–th term

$$\mathrm{map}\,(M/\partial M, \Omega S^{n+i}).$$

Here it is convenient to interpret ΩS^{n+i} as the space of paths in S^{n+i} with endpoints distinct from $(0,0,\ldots,-1)$. Then on i–th terms, $\Omega\nabla^{\mathrm{DIFF}}$ is the map

$$z_i : \mathcal{C}^{\mathrm{DIFF,b}}(M \times \mathbb{R}^i) \longrightarrow \mathrm{map}\,(M/\partial M, \Omega S^{n+i})$$

whose adjoint

$$\mathcal{C}^{\mathrm{DIFF,b}}(M \times \mathbb{R}^i) \times M \times [0,1] \longrightarrow S^{n+i}$$

sends (f,x,t) to

$$f'(x,0,t)/\|f'(x,0,t)\|$$

where

$$f' : M \times \mathbb{R}^i \times [0,1] \longrightarrow \mathbb{R}^n \times \mathbb{R}^i \times \mathbb{R}$$

is the derivative of f in the $[0,1]$–direction.

Let P be the space of all pairs (t_1,t_2) in $[0,1]^2$ such that $t_1 \geq t_2$. Identify $[0,1]$ with a subspace of P by

$$t \longmapsto (t,t)$$

and let $\beta P \subset P$ consist of all pairs (t_1,t_2) with $t_1 = 1$ or $t_2 = 0$. Let $\widehat{\Omega} S^{n+i}$ be the space of maps from P to S^{n+i} whose restriction to βP avoids $(0,0,\dots,0,-1)$. This is contractibel and comes with a forgetful map

$$\widehat{\Omega} S^{n+i} \longrightarrow \Omega S^{n+i}; g \longmapsto g_{|[0,1]}$$

(restriction of maps to $[0,1] \subset P$). The map z_i above has a lift

$$\mathcal{C}^{\mathrm{DIFF,b}}(M \times \mathbb{R}^i) \longrightarrow \mathrm{map}\,(M/\partial M, \widehat{\Omega} S^{n+i})$$

whose adjoint

$$\mathcal{C}^{\mathrm{DIFF,b}}(M \times \mathbb{R}^i) \times M \times P \longrightarrow S^{n+i}$$

sends $(f,x,(t_1,t_2))$ to

$$\begin{cases} (f(x,0,t_1) - f(x,0,t_2))/\|f(x,0,t_1) - f(x,0,t_2)\| & \text{if } t_1 > t_2 \\ f'(x,0,t_1)/\|f'(x,0,t)\| & \text{if } t_1 = t_2. \end{cases}$$

It follows that z_i is nullhomotopic for all $i \geq 0$. We leave it to the reader to check that the nullhomotopies are compatible. □

We apologize for the unsystematic use of the labels ∇ and $\Omega\nabla$. Our policy is to use just one label, such as ∇ , for all maps generated by a particular idea, but in the proofs of 6.3 and 6.6 we did not stick to it.

Now the time has come to quote directly from Waldhausen. This is not easy! The master sometimes makes a sport of hiding his best results, for example in [41]. Other useful references are [42] and [43].

Fix a smooth compact M as in 6.3 and 6.6. We need the assembly map

$$M_+ \wedge \underline{\underline{A}}s(*) \longrightarrow \underline{\underline{A}}s(M)$$

and the composition

$$M_+ \wedge \underline{S}^0 \xrightarrow{u} M_+ \wedge \underline{\underline{A}}s(*) \xrightarrow{\text{assembly}} \underline{\underline{A}}s(M)$$

where u is induced by the unit map from $\underline{S}^0$ to $\underline{\underline{A}}s(*)$. (The unit map embeds the K–theory of finite sets in the K–theory of finite CW–spectra.) Disregarding earlier conventions, we write

$$\underline{\underline{A}}s(M)/(M_+ \wedge \underline{\underline{A}}s(*)) \qquad \text{and} \qquad \underline{\underline{A}}s(M)/(M_+ \wedge \underline{S}^0)$$

for the cofibers.

6.7. Theorem (Waldhausen, expecially [41]). The forgetful map

$$\Omega(\underline{\underline{A}}s(M)/(M_+ \wedge \underline{S}^0)) \longrightarrow \Omega(\underline{\underline{A}}s(M)/(M_+ \wedge \underline{\underline{A}}s(*)))$$

is homotopy equivalent to the forget map

$$\Omega \underline{\underline{Whs}}^{\mathrm{DIFF}}(M) \longrightarrow \Omega \underline{\underline{Whs}}^{TOP}(M).$$

6.8. Theorem (Waldhausen [41], [43]; see also Vogell [40]). The connecting map

$$\Omega(\underline{\underline{A}}s(M)/(M_+ \wedge \underline{S}^0)) \longrightarrow M_+ \wedge \underline{S}^0$$

is nullhomotopic.

Implicit in 6.7 is the claim that the change from decoration h to decoration s causes no problems (since Waldhausen uses h). This was mentioned in Part II; a proof in Part III was promised. But as we are in Part III now, it seems wiser to put it off to Part V. With the fibration theorem 1.6.4 in [42], it is an exercise (hint due to R. Thomason).

We establish our reformulation by comparing two commutative squares:

(1)

$$\begin{tikzcd}
\Omega \underline{\underline{Whs}}^{\mathrm{DIFF}}(M) \arrow[r, "\nabla"] \arrow[d, "\text{forget}"] & \Omega \underline{\underline{Wh}}^{\mathrm{DIFF}}(\tau) \arrow[d, "\text{forget}"] \\
\Omega \underline{\underline{Whs}}^{TOP}(M) \arrow[r, "\nabla"] & \Omega \underline{\underline{Wh}}^{TOP}(\tau)
\end{tikzcd}$$

from 6.3, and

(2)

$$\begin{tikzcd}
\Omega(\underline{\underline{A}}s(M)/(M_+ \wedge \underline{S}^0)) \arrow[r] \arrow[d] & M_+ \wedge \underline{S}^0 \arrow[d, "u"] \\
\Omega(\underline{\underline{A}}s(M)/(M_+ \wedge \underline{\underline{A}}s(*)) \arrow[r] & M_+ \wedge \underline{\underline{A}}s(*)
\end{tikzcd}$$

where the horizontal maps are connecting maps. The two diagrams obtained from (1) and (2) by deleting the lower right corners are homotopy equivalent, by 6.6, 6.8 and 6.7. But now both (1) and (2) are homotopy pullback squares (the first by 6.3), so they will still be homotopy equivalent if the lower right corners are kept. In particular, the bottom rows of (1) and (2) are homotopy equivalent. In the shorthand notation used in the introduction, the bottom row of (2) becomes

$$\underline{\underline{A}}s(M)_{\%} \xrightarrow{\text{forget}} \underline{\underline{A}}s(M)^{\%}.$$

$\square$

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