

SMOOTH PARAMETRIZED TORSION A MANIFOLD APPROACH

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ABSTRACT. We give a construction of a torsion invariant of bundles of smooth manifolds which is based on the work of Dwyer, Weiss and Williams on smooth structures on fibrations.

1. INTRODUCTION

1.1. Recently there has been considerable interest and activity related to the construction and the computations of parametrized torsion invariants. The goal here is the development of a generalization of Reidemeister torsion – which is a classical secondary invariant of CW-complexes – to bundles of manifolds. One approach to this problem was proposed by Bismut and Lott [2]. Another, using parametrized Morse functions resulted from the work of Igusa [6] and Klein [8].

In [5] Dwyer, Weiss, and Williams presented yet another definition of torsion of bundles whose main feature is that it is described entirely in terms of algebraic topology. As a result their construction is quite intuitive. Given a smooth bundle $p: E \rightarrow B$ we can consider the Becker-Gottlieb transfer map $p^!: B \rightarrow Q(E_+)$. If $\rho: M \rightarrow E$ is a locally constant sheaf of R -modules we can construct a map $c_\rho: B \rightarrow K(R)$ which assigns to a point $b \in B$ the point of $K(R)$ represented by the singular chain complex $C_*(F_b, \rho|_{F_b})$ of the fiber of p over b with coefficient in ρ . One of the results of [5] implies that there exists a map $\lambda: Q(E_+) \rightarrow K(R)$ such that the diagram

$$\begin{array}{ccc} & & Q(E_+) \\ & \nearrow p^! & \downarrow \lambda \\ B & \xrightarrow{c_\rho} & K(R) \end{array}$$

commutes up to a preferred homotopy. If the sheaf ρ is such that all chain complexes $C_*(F_b, \rho|_{F_b})$ are acyclic, then the map c_ρ is canonically homotopic to the constant map. Thus we obtain a lift of $p^!$ to the space $\text{Wh}_\rho(E)$ which is the homotopy fiber of λ over the trivial element of $K(R)$. This lift $\tau_\rho(p)$ is the smooth torsion of the bundle p .

In order to see what kind of information about a bundle is carried by its torsion lets assume first that we are given two smooth bundles $p_i: E_i \rightarrow B$, $i = 1, 2$ and a map of bundles $f: E_1 \rightarrow E_2$. Let $A(E_i)$ denote the Waldhausen A -theory of the space E_i . We have the assembly maps $a: Q(E_{i+}) \rightarrow A(E_i)$ (see Section 3) which

fit into a commutative square

$$\begin{array}{ccc} Q(E_{1+}) & \xrightarrow{f_*} & Q(E_{2+}) \\ a \downarrow & & \downarrow a \\ A(E_1) & \xrightarrow{f_*} & A(E_2) \end{array}$$

Let $p_i^A: B \rightarrow A(E_i)$ be the composition $p_i^A = ap_i^!$. One of the implications of [5] is that if f is a fiberwise homotopy equivalence then we can construct a homotopy $\omega_f: B \times I \rightarrow A(E_2)$ joining $f_*p_1^A$ with p_2^A . Moreover, this homotopy can be lifted to a homotopy joining $f_*p_1^!$ with $p_2^!$ in $Q(E_2)$ provided that f is homotopic to a diffeomorphism of bundles. The problem of lifting ω_f defines then an obstruction to replacing f by a diffeomorphism. This obstruction closely resembles the classical Whitehead torsion of a homotopy equivalence of finite CW-complexes.

Lets return now to the case of a single bundle $p: E \rightarrow B$ with fiber M . The map $\lambda: Q(E) \rightarrow K(R)$ factors though the assembly map so we get a commutative diagram

$$\begin{array}{ccc} & & Q(E_+) \\ & p^! \nearrow & \downarrow a \\ & & A(E) \\ & p^A \nearrow & \downarrow \\ B & \xrightarrow{c_\rho} & K(R) \end{array}$$

If we would have a homotopy equivalence $f: M \times B \rightarrow E$ from the product bundle over B to p then the homotopy ω_f would give us a contraction of p^A (and thus also a contraction of c_ρ) to a constant map. The construction of the smooth torsion takes advantage the fact that under some homological conditions the map c_ρ is contractible even if we do not have a homotopy equivalence f . In this case the contraction of c_ρ can be still interpreted as a way of relating the bundle p to the product bundle on the level of the K -theory. Viewed from this perspective the smooth torsion $\tau_\rho(p)$ is an obstruction to the existence of a diffeomorphism between p and the product bundle. In effect we can think of it as a linearized Whitehead torsion. This parallels the way the classical Reidemeister torsion is interpreted in the setting of CW-complexes.

While the idea of the construction of smooth torsion is simple to explain the technical details are rather involved. One of the main problems is that the target of the transfer $p^!$ and the domain of the map λ as described in [5] are different and are only linked by a zigzag of weak equivalences. As a result the torsion of a bundle is not really defined uniquely but rather up to a contractible space of choices. This makes this construction of torsion rather inaccessible to direct computations. In fact, at present there are no examples of bundles for which the smooth torsion does not vanish, even though such examples abound for Bismut-Lott and Igusa-Klein torsions, and intuitively the smooth torsion of Dwyer-Weiss-Williams should be a more delicate invariant. The last sentence points out another problem with the Dwyer-Weiss-Williams construction: at present there are no known results relating it to the other definitions of torsion of smooth bundles.

An additional difficulty with the construction of torsion as described above is that it depends on existence of the local system of coefficients ρ yielding acyclicity of fibers of p . Such systems of coefficient are not easy to construct.

One the goals of this note is to show how these problems can be resolved. First, we substantially simplify the construction of smooth torsion using Waldhausen’s manifold approach to the Q -construction [10], [11]. This idea is not entirely original – in [5] the authors sketch it briefly at the very end of the paper and attribute it to Waldhausen. Our aim, however, is to develop it to the extent which would permit us to study properties of the smooth torsion. In addition we show that smooth torsion can be defined even if the fibers of p are not acyclic, as long as the fundamental group of the base space acts trivially (or even unipotently – see 6.2) on the homology groups of the fibers. This demonstrates that smooth torsion exists for a broad class of bundles.

We also aim to bring the smooth torsion of Dwyer-Weiss-Williams closer to the Bismut-Lott and the Igusa-Klein constructions. The starting point here is the paper [7] of Igusa which describes a set of axioms for torsion of smooth bundles. Igusa shows that any notion of torsion satisfying these axioms must coincide with the Igusa-Klein torsion up to some scalar constants. In Igusa’s setting, torsion is an invariant defined for all smooth unipotent bundles – the condition which as we mentioned above turns out to be satisfied by the smooth torsion. This invariant is supposed to take its values in the cohomology groups $H^{4k}(B, \mathbb{R})$ of the base space of the bundle. We show that the smooth torsion can be reduced to such a cohomological invariant. What remains to be verified is that the cohomology classes we produce satisfy Igusa’s axioms. This is the goal which the present authors in collaboration with John Klein plan to complete in a future paper.

1.2. Organization of the paper. As we have mentioned above our main tool in this paper is the construction of the space $Q(X_+)$ using the language of “partitions” given by Waldhausen in [10], [11]. We start by summarizing this construction in Section 2. In §3 we describe – again following Waldhausen – the assembly map from $Q(X_+)$ to Waldhausen’s A -theory of the space X . While this map is not our main interest here, we will use it in Section 4 to show that a certain map we construct there coincides with the Becker-Gottlieb transfer $p^! : B \rightarrow Q(E_+)$ of a bundle $p : E \rightarrow B$. In Section 5 the assembly map is used again to construct the linearization map $\lambda : Q(B_+) \rightarrow K(R)$. In §6 we describe the construction of torsion for bundles with acyclic fibers and for unipotent bundles. Finally, in §7 we show how the torsion of a bundle $E \rightarrow B$ defines certain cohomology classes in $H^{4k}(B; \mathbb{R})$, for $k > 0$.

2. WALDHAUSEN’S MANIFOLD APPROACH

2.1. Partitions. Let I denote the closed interval $[0, 1]$. For a smooth manifold X with boundary ∂X a partition of $X \times I$ is a triple (M, F, N) where M and N are codimension 0 submanifolds of $X \times I$ such that $M \cup N = X \times I$ and $F = M \cap N$ is a submanifold of codimension 1. Moreover, we assume that M contains $X \times \{0\}$ and is disjoint from $X \times \{1\}$, and finally, that for some open neighborhood U of ∂X in X the intersection of F with $U \times I$ coincides with $U \times \{t\}$ for some $t \in I$. While this description reflects the basic properties of partitions we will make some further technical assumptions which will make it easier to work with them:

- we will assume that the value of t if fixed once for all partitions (say, $t = \frac{1}{3}$);

- we will assume that $X \times [0, \frac{1}{3}] \subseteq M$ for any partition (M, N, F) .

In the language of [10] partitions satisfying the last two conditions are called lower partitions.

While the idea is all components - M, N, F - of a partition should be smooth this condition is too rigid for the constructions we will want to perform. Following Waldhausen [10, Appendix] we will assume that these are just topological manifolds but we also assume that there is a preferred smooth vector field s on $X \times I$ which is normal to F in the following sense. Given any smooth chart of $X \times I$ containing $x \in F$ there are constants $c, C > 0$ such that for all $|r| \leq C$ the distance function satisfies the inequality

$$d(x + rs(x), F) \geq c|r|$$

This just means that the line passing through x and going in the direction of $s(x)$ stays well away from F .

The existence of such a vector field s implies that the manifold F admits a smoothing which is obtained by sliding points of F along the integral curves of s . Moreover, the space of smoothings of F which can be obtained in this way is contractible, so we can think of the quadruple (M, N, F, s) as describing an essentially unique smooth partition. We again list some additional technical assumptions:

- we will assume that on $U \times I$ the vectors of s are the unit vectors pointing upward, in the direction of I ;
- we will also assume that for $x \in X \times \{\frac{1}{3}\}$ the component of $s(x)$ tangent to the interval I is a non-zero vector pointing upward.

Clearly a partition (M, N, F, s) is determined by the manifold M and the vector field s . In order to simplify notation we will write (M, s) instead of (M, N, F, s) .

For a manifold Y we have the notion of a locally trivial family of partitions of $X \times I$ parametrized by Y . By this we mean a pair (\bar{M}, \bar{s}) where $\bar{M} \subseteq X \times I \times Y$ and s is a smooth vector field on $X \times I \times Y$ such that

- for each $y \in Y$ the pair $(\bar{M} \cap X \times I \times \{y\}, \bar{s}|_{X \times I \times \{y\}})$ is a partition of $X \times I \times \{y\}$
- for every $y \in Y$ there is an open neighborhood $y \in V \subseteq Y$, a partition (M, s) of $X \times I$, and a diffeomorphism

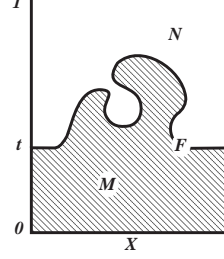
$$\varphi: X \times I \times V \rightarrow X \times I \times V$$

such that $p_V \circ \varphi = p_V$ where $p_V: X \times I \times V \rightarrow V$ is the projection map, φ is the identity map on an appropriate neighborhood of $\partial(X \times I) \times V$, $\varphi(M \times V) = \bar{M} \cap (X \times I \times V)$ and $D\varphi(s) = \bar{s}$.

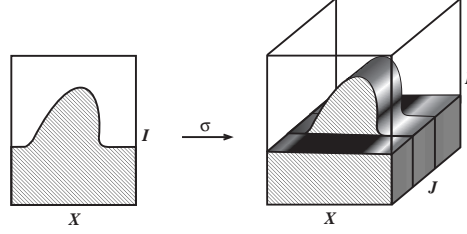
If (\bar{M}, \bar{s}) is a partition of $X \times I$ parametrized by Y and $f: Z \rightarrow Y$ is a smooth function then we obtain the induced partition f^*Y parametrized by Z .

Denote by $\mathcal{P}_k(X)$ the set of all partitions parametrized by the k -simplex Δ^k . These sets can be assembled to form a simplicial set $\mathcal{P}_\bullet(X)$.

2.2. Stabilization. The set $\mathcal{P}_0(X)$, which can be identified with the set of all partitions of $X \times I$, has a partial monoid structure defined as follows: given two partitions (M_1, s) and (M_2, s) in $\mathcal{P}_0(X)$ such that $M_1 \cap M_2 = X \times [0, \frac{1}{3}]$ we set $(M_1, s) + (M_2, s) := (M_1 \cup M_2, s)$. We can extend this definition to $\mathcal{P}_k(X)$ for all $k \geq 0$ to obtain a partial simplicial monoid structure on $\mathcal{P}_\bullet(X)$. In order to define



addition for all partitions (and thus define an H -space structure on $\mathcal{P}_\bullet(X)$) we need to introduce stabilization of partitions (called lower stabilization in [10]). It is a map of simplicial sets $\sigma: \mathcal{P}_\bullet(X) \rightarrow \mathcal{P}_\bullet(X \times J)$ where $J = [0, 1]$. Given a partition $(M, s) \in \mathcal{P}_0(X)$ we set $\sigma(M, s) = (\sigma(M), \sigma(s))$ where $\sigma(M) = X \times J \times [0, \frac{1}{3}] \cup M \times [\frac{1}{3}, \frac{2}{3}]$.



In order to define the vector field $\sigma(s)$, fix a smooth vector field s' on the interval J such that s' is non-zero at the points $\frac{1}{3}$ and $\frac{2}{3}$ and is zero on some neighborhood of ∂J . For $(x, t, t') \in X \times I \times J$ we then set

$$\sigma(s)(x, t, t') := s(x, t) + s'(t)$$

We note that the vector field $\sigma(s)$ does not really satisfy all assumptions of our definition of a partition since it is not a unit vector field pointing in the direction of I when restricted to a neighborhood of $\partial(X \times J) \times I$. This however can be easily fixed.

In a similar way we can define stabilization maps $\sigma: \mathcal{P}_k(X) \rightarrow \mathcal{P}_k(X \times J)$ for all $k > 0$ so that we obtain a simplicial map of $\sigma: \mathcal{P}_\bullet(X) \rightarrow \mathcal{P}_\bullet(X \times J)$. Notice that given any two partitions it is always possible to slide their stabilizations away from each other along J so that the sum is defined. In this way the partial monoid structure becomes a monoid structure on $\text{colim}_m \mathcal{P}_\bullet(X \times I^m)$.

2.3. Group completion. The Waldhausen manifold model for $Q(X_+)$ can be obtained as a group completion of the simplicial monoid $\text{colim}_m \mathcal{P}_\bullet(X \times I^m)$. In order to describe this group completion one can use Thomason's variant of Waldhausen's S_\bullet -construction (see [9], p.343). Let $\mathcal{T}_n \mathcal{P}_0(X)$ denote the category whose objects are $(n+1)$ -tuples $\{(M_i, s_i)\}_{i=0}^n$ such that $(M_i, s_i) \in \mathcal{P}_0(X)$, $s_i = s_j$ for all i, j and that we have inclusions

$$M_0 \subseteq M_1 \subseteq \dots \subseteq M_n$$

In $\mathcal{T}_n \mathcal{P}_0(X)$ we have a unique morphism $\{(M_i, s_i)\} \rightarrow \{(M'_i, s'_i)\}$ if and only if $M'_0 \cap M_i \subseteq M_0$ and $M'_i = M_i \cup M'_0$ for all $i \geq 0$. Analogously, for any $n, k \geq 0$ we can define a category $\mathcal{T}_n \mathcal{P}_k(X)$ whose objects are increasing sequences of length n in $\mathcal{P}_k(X)$. For any fixed n the categories $\mathcal{T}_n \mathcal{P}_k(X)$ assemble to give a simplicial category $\mathcal{T}_n \mathcal{P}_\bullet(X)$. We have functors $d_j: \mathcal{T}_{n+1} \mathcal{P}_\bullet(X) \rightarrow \mathcal{T}_n \mathcal{P}_\bullet(X)$ and $s_j: \mathcal{T}_n \mathcal{P}_\bullet(X) \rightarrow \mathcal{T}_{n+1} \mathcal{P}_\bullet(X)$ which are obtained by removing (or respectively repeating) the j -th element of the sequence $\{(M_i, s_i)\}$. In this way $\mathcal{T}_\bullet \mathcal{P}_\bullet(X)$ becomes a simplicial object in the category of simplicial categories. Let $|\mathcal{T}_\bullet \mathcal{P}_\bullet(X)|$ denote the space obtained by first taking nerve of each of the categories $\mathcal{T}_n \mathcal{P}_\bullet(X)$ and then applying geometric realization to the resulting bisimplicial set. Notice that we have a cofibration $|\mathcal{T}_0 \mathcal{P}_\bullet(X)| \rightarrow |\mathcal{T}_\bullet \mathcal{P}_\bullet(X)|$. One can check that the space $|\mathcal{T}_0 \mathcal{P}_\bullet(X)|$ is contractible, and so we get a weak equivalence

$$|\mathcal{T}_\bullet \mathcal{P}_\bullet(X)| \simeq |\mathcal{T}_\bullet \mathcal{P}_\bullet(X)| / |\mathcal{T}_0 \mathcal{P}_\bullet(X)|$$

By abuse of notation from now on we will denote by $|\mathcal{J}_\bullet \mathcal{P}_\bullet(X)|$ the quotient space on the right. The advantage of this modification is that $|\mathcal{J}_\bullet \mathcal{P}_\bullet(X)|$ has now a canonical choice of a basepoint.

Since everything we did so far behaves well with respect to the stabilization maps we obtain the induced maps of spaces

$$\sigma: |\mathcal{J}_\bullet \mathcal{P}_\bullet(X \times I^m)| \rightarrow |\mathcal{J}_\bullet \mathcal{P}_\bullet(X \times I^{m+1})|$$

Passing to the homotopy colimit we get

2.4. Theorem (Waldhausen[11]). *There is a weak equivalence*

$$\Omega \operatorname{hocolim}_m |\mathcal{J}_\bullet \mathcal{P}_\bullet(X \times I^m)| \simeq Q(X_+)$$

In view of this result from now on by $Q(X_+)$ we will understand the space on the left hand side of the above equivalence.

2.5. Remark. The following observation will be useful for constructing maps into $Q(X_+)$. Notice that we have a map

$$|\mathcal{J}_1 \mathcal{P}_\bullet(X)| \times \Delta^1 \rightarrow |\mathcal{J}_\bullet \mathcal{P}_\bullet(X)|$$

After stabilizing the right hand side and taking the adjoint we obtain a map $|\mathcal{J}_1 \mathcal{P}_\bullet(X)| \rightarrow Q(X_+)$. Thus any map into the nerve of the category $\mathcal{J}_1 \mathcal{P}_\bullet(X)$ naturally yields a map into $Q(X_+)$.

3. THE ASSEMBLY MAP

Waldhausen's motivation for constructing the space $Q(X_+)$ in the way sketched in the last section was to relate this space to $A(X)$ – the A -theory of the space X . Since we will need to use this relationship later in this paper, we now describe a map $a: Q(X_+) \rightarrow A(X)$ which we will call the assembly map.

The simplest way of constructing the space $A(X)$ is to start with the category $\mathcal{R}^{fd}(X)$ whose objects are homotopy finitely dominated retractive spaces over X and whose morphisms are maps of retractive spaces. The category $\mathcal{R}^{fd}(X)$ is a Waldhausen category in the sense of [9, Definition 1.2] with cofibrations given by Serre cofibrations and weak equivalences defined as weak homotopy equivalences. It follows that we can turn it into a simplicial category $\mathcal{J}_\bullet \mathcal{R}^{fd}(X)$ using again the Thomason's variant of Waldhausen's \mathcal{S}_\bullet -construction. We set

$$A(X) := \Omega(|\mathcal{J}_\bullet \mathcal{R}^{fd}(X)|/|\mathcal{J}_0 \mathcal{R}^{fd}(X)|)$$

In order to obtain a direct map from our model of $Q(X_+)$ this construction needs to be modified somewhat. First, for $k \geq 0$ we can construct a category $\mathcal{R}_k^{fd}(X)$ whose objects are locally homotopy trivial families of retractive spaces over X parametrized by the simplex Δ^k . These categories taken together form a simplicial category $\mathcal{R}_\bullet^{fd}(X)$. Analogously as we did in the case of categories of partitions we can define stabilization functors

$$\sigma: \mathcal{R}_\bullet^{fd}(X) \rightarrow \mathcal{R}_\bullet^{fd}(X \times I)$$

in the following way: if Y is a retractive space over X then

$$\sigma(Y) := Y \times \left[\frac{1}{3}, \frac{2}{3} \right] \cup_{X \times \left[\frac{1}{3}, \frac{2}{3} \right]} X \times I$$

Applying the \mathcal{T}_\bullet -construction to $\mathcal{R}_\bullet^{fd}(X)$ we get a bisimplicial category $\mathcal{T}_\bullet \mathcal{R}_\bullet^{fd}(X)$. Define

$$A_p(X) := \operatorname{hocolim}_m |\mathcal{T}_\bullet \mathcal{R}_\bullet^{fd}(X \times I^m)| / |\mathcal{T}_0 \mathcal{R}_0^{fd}(X \times I^m)|$$

Notice that if X is a smooth manifold and if (M, s) is a partition of $X \times I$ then $a(M) := M \cap (X \times [\frac{1}{3}, 1])$ is in an obvious way a retractive space over X . The assignment $(M, s) \mapsto a(M)$ extends to a functor of simplicial categories $a: \mathcal{P}_\bullet(X) \rightarrow \mathcal{R}_\bullet^{fd}(X)$ which commutes with the stabilization functors. This induces a map

$$a: Q(X_+) \rightarrow A_p(X)$$

Since the category $\mathcal{R}^{fd}(X)$ is isomorphic to $\mathcal{R}_0^{fd}(X)$ we have a functor $i: \mathcal{R}^{fd}(X) \rightarrow \mathcal{R}_\bullet^{fd}(X)$ which induces a map $i: A(X) \rightarrow A_p(X)$. We have

3.1. Theorem (Waldhausen, [10, Lemma 5.4]). *The map $i: A(X) \rightarrow A_p(X)$ is a homotopy equivalence.*

While this result says that we are not changing much by replacing $A(X)$ with $A_p(X)$, it will be convenient to have an assembly map whose codomain is $A(X)$. In order to get such a map define $\tilde{Q}(X_+)$ to be the homotopy pullback of the diagram

$$(1) \quad A(X) \longrightarrow A_p(X) \xleftarrow{a} Q(X_+)$$

In view of Theorem 3.1 we have $\tilde{Q}(X_+) \simeq Q(X_+)$, and $\tilde{Q}(X_+)$ comes equipped with a map $\tilde{a}: \tilde{Q}(X_+) \rightarrow A(X)$.

4. TRANSFER

Going back to the diagram on page 1 we see that if we want to describe the smooth torsion of a bundle $p: E \rightarrow B$ we need to construct the transfer map $p^!: B \rightarrow Q(E_+)$ (or rather $p^!: B \rightarrow \tilde{Q}(E_+)$) and the linearization map $\lambda: \tilde{Q}(E_+) \rightarrow K(R)$. We deal with the transfer in this section and with the linearization in the next one.

Let $p: E \rightarrow B$ be a smooth bundle of manifolds with B and E compact. Denote by $T^v E$ the subbundle of the tangent bundle TE consisting of vectors tangent to the fibers of p . By choosing a Riemannian metric on TE we can identify the bundle p^*TB with the subbundle of TE which is the orthogonal complement of $T^v E$. Using this identification given the map $p \times \operatorname{id}: E \times I \rightarrow B \times I$ we can consider the bundle $(p \times \operatorname{id})^*T(B \times I)$ as a subbundle of $T(E \times I)$. As a consequence any section $s: B \times I \rightarrow T(B \times I)$ will define a section $(p \times \operatorname{id})^*s$ of the bundle $T(E \times I)$.

Assume for a moment that fibers of p are closed manifolds. In this case given a partition $(M, s) \in \mathcal{P}_0(B)$ the pair $((p \times \operatorname{id})^{-1}M, (p \times \operatorname{id})^*s)$ defines a partition of $E \times I$, so we get a map of sets $Q(p^!): \mathcal{P}_0(B) \rightarrow \mathcal{P}_0(E)$. Since this map preserves the partial ordering of partitions we in fact obtain a functor $\mathcal{T}_0 \mathcal{P}_0(B) \rightarrow \mathcal{T}_0 \mathcal{P}_0(E)$. In the same way we can define functors $\mathcal{T}_n \mathcal{P}_k(B) \rightarrow \mathcal{T}_n \mathcal{P}_k(E)$ for all $k, n \geq 0$ so that they induce a map

$$Q(p^!): |\mathcal{T}_\bullet \mathcal{P}_\bullet(B)| \rightarrow |\mathcal{T}_\bullet \mathcal{P}_\bullet(E)|$$

Since $Q(p^!)$ is compatible with stabilization we get a map $Q(p^!): Q(B_+) \rightarrow Q(E_+)$.

If fibers of the bundle p are manifolds with boundary we need to modify the above construction slightly so that for a partition $(M, s) \in \mathcal{P}_0(B)$ the element $Q(p^!)(M, s)$ behaves nicely near $\partial E \times I$. This can be done as follows. Let $\partial^v E$ be the subspace

of E consisting of all boundary points of the fibers of p . If F is a fiber of p then $p|_{\partial^v E}: \partial^v E \rightarrow B$ is a subbundle of p with fiber ∂F .

For $b \in B$ let F_b denote the fiber of p over b . We can find an open neighborhood $U \subseteq E$ in such way that for all $b \in B$ the intersection $U \cap F_b$ is a collar neighborhood of ∂F_b in F_b [1, p.590]. For a partition $(M, s) \in \mathcal{P}_0(M)$ let $Q(p^!)(M) \subseteq E \times I$ be given by

$$Q(p^!)(M) := U \times [0, \frac{1}{3}] \cup ((p \times \text{id})^{-1}(M) \cap ((E \setminus U) \times I))$$

We need then to modify the vector field $(p \times \text{id})^*s$ so that it is normal to $Q(p^!)(M)$. This can be done in a way similar to the one we used to define stabilization of partitions.

In a similar way given a bundle $p: E \rightarrow B$ we can construct maps $A(p^!): A(B) \rightarrow A(E)$ and $A_p(p^!): A_p(B) \rightarrow A_p(E)$. Each of these maps is induced by a functor of categories of retractive spaces which assigns to a retractive space over B its pullback along p (in order to make this compatible with the construction of $Q(p^!)$ we need to modify these pullback slightly in a neighborhood of the boundaries of fibers of p). These three maps induce in turn a map of homotopy pullbacks

$$\tilde{Q}(p^!): \tilde{Q}(B_+) \rightarrow \tilde{Q}(E_+)$$

The map $p^!: B \rightarrow \tilde{Q}(E_+)$ will be obtained as the composition of $\tilde{Q}(p^!)$ with a coaugmentation map $\eta: B \rightarrow \tilde{Q}(B_+)$ which we describe below. For simplicity we will assume first that B is a closed manifold.

Let $\mathcal{S}(B)$ denote the simplicial set of singular simplices of B . It will be convenient to consider it as a simplicial category with identity morphisms only on each simplicial level. We have a weak equivalence $B \simeq |\mathcal{S}(B)|$, so it will suffice to construct a map $|\mathcal{S}(B)| \rightarrow \tilde{Q}(B_+)$. Recall that $\tilde{Q}(B_+)$ was defined as the homotopy pullback of the diagram (1). Notice also that we have a commutative diagram

$$\begin{array}{ccccc} |\mathcal{T}_1 \mathcal{R}^{fd}(B)| & \longrightarrow & |\mathcal{T}_1 \mathcal{R}_\bullet^{fd}(B)| & \longleftarrow & |\mathcal{T}_1 \mathcal{P}_\bullet(B)| \\ \downarrow & & \downarrow & & \downarrow \\ A(B) & \longrightarrow & A_p(B) & \longleftarrow & Q(B_+) \end{array}$$

where the vertical maps are obtained as in Remark 2.5. It will then suffice to define a map from $|\mathcal{S}(B)|$ to the homotopy limit of the upper row of this diagram. This, in turn, can be accomplished by specifying functors $\mathcal{S}(B) \rightarrow \mathcal{T}_1 \mathcal{R}^{fd}(B)$ and $\mathcal{S}(B) \rightarrow \mathcal{T}_1 \mathcal{P}_\bullet(B)$ and a zigzag of natural transformations joining these functors in $\mathcal{T}_1 \mathcal{R}_\bullet^{fd}(B)$. A minor inconvenience here is the fact that $\mathcal{T}_1 \mathcal{P}_\bullet(B)$, $\mathcal{T}_1 \mathcal{R}_\bullet^{fd}(B)$ and $\mathcal{S}(B)$ are simplicial categories while $\mathcal{R}^{fd}(B)$ is not, but we can think about $\mathcal{R}^{fd}(B)$ as of a constant simplicial object in the category of small categories.

We will start then with a diagram of categories

$$\mathcal{T}_1 \mathcal{R}^{fd}(B) \xrightarrow{i} \mathcal{T}_1 \mathcal{R}_\bullet^{fd}(B) \xleftarrow{a} \mathcal{T}_1 \mathcal{P}_\bullet(B)$$

and we will extend it to a diagram

$$\begin{array}{ccccc}
 & & \mathcal{S}(B) & & \\
 & \swarrow \eta_{\mathcal{R}} & \downarrow \eta_{p\mathcal{R}} & \searrow \eta_{\mathcal{P}} & \\
 \mathcal{T}_1\mathcal{R}^{fd}(B) & \xrightarrow{i} & \mathcal{T}_1\mathcal{R}_{\bullet}^{fd}(B) & \xleftarrow{a} & \mathcal{T}_1\mathcal{P}_{\bullet}(B)
 \end{array}$$

such that the two triangles of functors commute up to natural transformations.

The functor $\eta_{\mathcal{R}}: \mathcal{S}(B) \rightarrow \mathcal{T}_1\mathcal{R}^{fd}(B)$ is defined as follows: given a singular simplex $\sigma: \Delta^k \rightarrow B$ consider the retractive space $B \sqcup \Delta^k$ over B . We set $\eta_{\mathcal{R}}(\sigma)$ to be the cofibration of retractive spaces $B \hookrightarrow B \sqcup \Delta^k$.

In order to define the functor $\eta_{\mathcal{P}}: \mathcal{S}(B) \rightarrow \mathcal{T}_1\mathcal{P}_{\bullet}(B)$ fix a Riemannian metric on the tangent bundle TB . Choose $\epsilon > 0$ such that the closed disc bundle TB_{ϵ} consisting of vectors of TM of length $\leq \epsilon$ has the property that the exponential map $\exp: TB_{\epsilon} \rightarrow M$ restricted to each fiber of TB_{ϵ} is a diffeomorphism onto its image. Given a singular simplex $\sigma: \Delta^k \rightarrow B$ consider the induced disc bundle σ^*TB_{ϵ} over Δ^k . The exponential map gives a map of bundles

$$\begin{array}{ccc}
 \sigma^*TB_{\epsilon} & \xrightarrow{\exp} & B \times \Delta^k \\
 & \searrow & \swarrow \\
 & \Delta^k &
 \end{array}$$

which is a fiberwise embedding. Fix numbers a, b such that $\frac{1}{3} < a < b < 1$. We have a fiberwise embedding of fiber bundles over Δ^k :

$$\begin{array}{ccc}
 (\sigma^*TB_{\epsilon} \times [a, b]) \cup (B \times [0, \frac{1}{3}] \times \Delta^k) & \xrightarrow{\quad} & B \times I \times \Delta^k \\
 & \searrow & \swarrow \\
 & \Delta^k &
 \end{array}$$

This fiberwise embedding defines a family of partitions parametrized by Δ^k . We set $\eta_{\mathcal{P}}(\sigma)$ to be the inclusion of families of partitions

$$B \times [0, \frac{1}{3}] \times \Delta^k \hookrightarrow (\sigma^*TB_{\epsilon} \times [a, b]) \cup (B \times [0, \frac{1}{3}] \times \Delta^k)$$

Next, we need to define the functor $\eta_{p\mathcal{R}}: \mathcal{S}(B) \rightarrow \mathcal{T}_1\mathcal{R}_{\bullet}^{fd}(B)$. For a singular simplex $\sigma: \Delta^k \rightarrow B$ we have the map

$$\text{id}_{\Delta^k} \sqcup \text{pr}_{\Delta^k}: \Delta^k \sqcup B \times \Delta^k \rightarrow \Delta^k$$

For each $t \in \Delta^k$ the fiber of this map over t is in a natural way a retractive space over B , so we can think of $\Delta^k \sqcup B \times \Delta^k$ as of a family of retractive spaces over B parametrized by Δ^k . The functor $\eta_{p\mathcal{R}}$ is given by the assignment

$$\eta_{p\mathcal{R}}(\sigma) := (B \times \Delta^k \hookrightarrow \Delta^k \sqcup B \times \Delta^k)$$

In order to describe a natural transformation from $\eta_{p\mathcal{R}}$ to $i\eta_{\mathcal{R}}$ notice that for $\sigma: \Delta^k \rightarrow B$ we obtain $i\eta_{\mathcal{R}}(\sigma)$ by taking the retractive space $\eta_{\mathcal{R}}(\sigma) = \Delta^k \sqcup B$ and multiplying it by Δ^k which makes it into a retractive space over B parametrized by Δ^k . The natural transformation is then defined by the maps

$$\eta_{p\mathcal{R}}(\sigma) = \Delta^k \sqcup B \times \Delta^k \rightarrow (\Delta^k \times \Delta^k) \sqcup (B \times \Delta^k) = (\Delta^k \sqcup B) \times \Delta^k = i\eta_{\mathcal{R}}(\sigma)$$

which restrict to the identity map on $B \times \Delta^k$ and which send $x \in \Delta^k$ to $(x, x) \in \Delta^k \times \Delta^k$. The natural transformation from $a\eta_{\mathcal{P}}$ to $\eta_{\mathcal{P}\mathcal{R}}$ is easy to define.

If B is a manifold with a boundary we need to modify this construction somewhat so that the values of the functor $\eta_{\mathcal{P}\bullet}$ are still partitions. This can be done by choosing an open collar neighborhood U of the boundary of B . On $B \setminus U$ the map η can be now defined exactly as before. We then extend it to B by composing it with the retraction $B \rightarrow B \setminus U$.

4.1. Proposition. *If $p: E \rightarrow B$ is a smooth fibration then the map*

$$p^! := \tilde{Q}(p^!) \circ \eta: B \rightarrow Q(E_+)$$

is the Becker-Gottlieb transfer of p .

Proof. The composition $\tilde{a} \circ p^!$ is homotopic to $\chi^h(p)$ - the homotopy Euler characteristic of the bundle p as defined in [4]. By [4, Thm. 3.12] we obtain that $\text{tr} \circ \chi^h(p)$ is the Becker-Gottlieb transfer where $\text{tr}: A(E) \rightarrow \tilde{Q}(E_+)$ is the Waldhausen's trace map [11]. By [11] we have $\text{tr} \circ \tilde{a} \sim \text{id}_{\tilde{Q}(E_+)}$, so

$$\text{tr} \circ \chi^h(p) \sim \text{tr} \circ \tilde{a} \circ p^! \sim p^!$$

□

5. LINEARIZATION

For a ring R let $\mathcal{C}h(R)$ denote the category of finitely homotopy dominated chain complexes of projective R -modules. The category $Ch(R)$ can be equipped with a Waldhausen model category structure with degreewise monomorphisms as cofibrations and quasi isomorphisms as weak equivalences. Applying Waldhausen's \mathcal{S}_\bullet -construction we obtain a simplicial category $\mathcal{S}_\bullet Ch(R)$. The associated space $K(R) = \Omega(|\mathcal{S}_\bullet Ch(R)|)$ is homotopy equivalent to the infinite loop space underlying the K -theory spectrum of the ring R .

Let X be a space and let $\rho: \mathcal{M} \rightarrow X$ be a locally constant sheaf of finitely generated projective R -modules. As in [5, p.40] we notice that we have a functor $\lambda_\rho^{\mathcal{R}}: \mathcal{R}^{fd}(X) \rightarrow Ch(R)$ which assigns to every retractive space $Y \in \mathcal{R}^{fd}(X)$ the relative singular chain complex of $C_*(Y, X, \rho)$ with coefficients in ρ . This functor induces a map $\lambda_\rho^{\mathcal{R}}: A(X) \rightarrow K(R)$. Recall that for a smooth manifold X we constructed the assembly map $\tilde{a}: \tilde{Q}(X_+) \rightarrow A(X)$. By the linearization map $\lambda_\rho: \tilde{Q}(X) \rightarrow K(R)$ we will understand the composition $\lambda_\rho^{\mathcal{R}} \circ \tilde{a}$.

6. SMOOTH TORSION

We are now in position to define smooth Reidemeister torsion of a bundle of manifolds. We will do it under two different sets of assumptions, one replicating the conditions of [5], and the other conforming to the axiomatic setup of Igusa [7]. We note, however, that the idea underlying both constructions is essentially the same: given a bundle $p: E \rightarrow B$ and a sheaf of R -modules $\rho: \mathcal{M} \rightarrow E$ we have constructed maps $p^!: B \rightarrow \tilde{Q}(E_+)$ and $\lambda_\rho: \tilde{Q}(E_+) \rightarrow K(R)$. Consider the composition

$$\lambda_\rho \circ p^!: |\mathcal{S}(B)| \rightarrow K(R)$$

Under certain conditions on the bundle p and the sheaf ρ this map is homotopic to a constant map via a preferred homotopy. As a consequence we obtain a lift of $p^!$

to a map $\tau_\rho^s(p): B \rightarrow \text{hofib}(\tilde{Q}(E_+) \rightarrow K(R))$. This lift is the smooth torsion of the bundle p .

6.1. Acyclicity. Assume that we are given a sheaf ρ such that for any $b \in B$ we have $H_*(F_b; \rho) = 0$ where F_b is the fiber of p over b . Notice that the map $\lambda_\rho \circ p^!$ comes from a functor $\mathcal{S}(B) \rightarrow Ch(R)$ which assigns to each simplex $\sigma: \Delta^k \rightarrow B$ the relative chain complex $C_*(\sigma^*E \sqcup E, E; \rho)$ where σ^*E denotes the pullback of the diagram

$$\Delta^k \xrightarrow{\sigma} B \xleftarrow{p} E$$

Vanishing of homology groups of the fibers of p implies that all these chain complexes are acyclic, and thus the maps $C_*(\sigma^*E \sqcup E, E; \rho) \rightarrow 0$ are all quasi isomorphisms and define a natural transformation to the functor which sends all simplices of $\mathcal{S}(B)$ to the zero chain complex. On the level of spaces this natural transformation defines a homotopy $\omega_\rho: |\mathcal{S}(B)| \times I \rightarrow K(R)$. Denote by $\text{Wh}_\rho(E)$ the homotopy fiber of the linearization map λ_ρ taken over the basepoint of $K(R)$ represented by the zero chain complex. The smooth torsion of the bundle p is the map $\tau_\rho^s(p): |\mathcal{S}(B)| \rightarrow \text{Wh}_\rho(E)$ determined by the transfer $p^!$ together with the homotopy ω_ρ .

6.2. Unipotent bundles. Let \mathbb{F} be a field and let $p: E \rightarrow B$ be a bundle such that B is a connected manifold with a basepoint $b_0 \in B$. Assume that the fundamental group $\pi_1(B, b_0)$ acts trivially on the homology $H_*(F_{b_0}; \mathbb{F})$ of the fiber over b_0 .

Consider the map $\lambda_\rho: \tilde{Q}(E_+) \rightarrow K(\mathbb{F})$ associated with the constant sheaf of 1-dimensional vector spaces over \mathbb{F} . In this case the composition $\lambda_\rho \circ p^!$ assigns to a simplex σ the chain complex $C_*(\sigma^*E \sqcup E, E; \mathbb{F})$. We will construct a sequence of homotopies joining this map with the constant function which maps the whole space $|\mathcal{S}(B)|$ to $H_*(F_{b_0}; \mathbb{F})$, which we will consider as a chain complex with the trivial differentials.

Let $wCh(\mathbb{F})$ denote the subcategory of $Ch(\mathbb{F})$ with the same objects as $Ch(\mathbb{F})$, but with quasi-isomorphisms as morphisms. We have the canonical map

$$k: |wCh(\mathbb{F})| \rightarrow K(\mathbb{F})$$

Notice that the map $\lambda_\rho \circ p^!$ admits a factorization

$$\begin{array}{ccc} |\mathcal{S}(B)| & \xrightarrow{\lambda_\rho \circ p^!} & K(\mathbb{F}) \\ & \searrow & \nearrow k \\ & |wCh(\mathbb{F})| & \end{array}$$

We have

6.3. Lemma ([5, Prop. 6.6]). *Let $H: |wCh(\mathbb{F})| \rightarrow K(\mathbb{F})$ be the map which assigns to each chain complex C its homology complex $H_*(C)$. There is a preferred homotopy $k \simeq H$*

Proof. For a chain complex

$$C = (\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0)$$

Let $P_q C$ denote the complex such that $(P_q C)_i = 0$ for $i > q + 1$, $(P_q C)_{q+1} = \partial(C_{q+1})$, and $(P_q C)_i = C_i$ for $i \leq q$. Let $Q_q C$ be the kernel of the map $P_q C \rightarrow$

$P_{q+1}C$. We obtain cofibration sequences

$$Q_qC \rightarrow P_qC \rightarrow P_{q-1}C$$

functorial in C . Notice that the complex Q_qC is quasi-isomorphic to its homology complex $H_*(Q_qC)$ and this last complex has only one non-zero module $H_q(C)$ in the degree q . By Waldhausen's additivity theorem we obtain that the map $P_q: |wCh(\mathbb{F})| \rightarrow K(\mathbb{F})$ which assigns to $C \in wCh(\mathbb{F})$ the chain complex P_qC is homotopic to the map $H_q: |wCh(\mathbb{F})| \rightarrow K(\mathbb{F})$ sending C to $P_{q-1}C \oplus H_*(Q_qC)$. Iterating this argument we see that for each q the map P_q is homotopic to the map which assigns to a complex C the chain complex $\bigoplus_{i=0}^q H_i(C)$. Since $C = \lim_q P_qC$ we obtain the statement of the lemma. \square

As a consequence of the lemma we get a homotopy between $\lambda_\rho \circ p^!$ and the map which assigns to a simplex σ the homology chain complex $H_*(\sigma^*E \sqcup E, E; \mathbb{F})$. Since this last chain complex is isomorphic to the chain complex $H_*(\sigma^*E; \mathbb{F})$ we obtain a homotopy from $\lambda_\rho \circ p^!$ to the map represented by a functor $v: \mathcal{S}(B) \rightarrow Ch(R)$ given by $v(\sigma) = H_*(\sigma^*E; \mathbb{F})$.

Next, let $v_0: \mathcal{S}(B) \rightarrow Ch(R)$ denote the functor this assigns to each singular simplex σ the complex $H_*(F_{\sigma(0)}; \mathbb{F})$ where $F_{\sigma(0)}$ is the fiber of the bundle p taken over the zero vertex of σ . The isomorphisms $H_*(F_{\sigma(0)}; \mathbb{F}) \rightarrow H_*(\sigma^*E; \mathbb{F})$ form a natural transformation of functors v and v_0 . Finally, given any point $b \in B$ choose a path joining this point to the basepoint b_0 . Lifting it to the space E we can produce a homotopy equivalence $F_b \rightarrow F_{b_0}$. The map which it induces on the homology groups will not depend on the choice of the path by our assumption the $\pi_1(B)$ acts trivially on the homology of the fibers. The maps $H_*(F_\sigma(0); \mathbb{F}) \rightarrow H_*(F_{b_0}; \mathbb{F})$ yield the natural transformation from v_0 to the constant functor.

On the level of spaces the natural transformations of functors we described above define a homotopy from the map $\lambda_\rho \circ p^!: |\mathcal{S}(B)| \rightarrow K(\mathbb{F})$ to the constant map which maps $|\mathcal{S}(B)|$ to the point of $K(\mathbb{F})$ represented by the chain complex $H_*(F_{b_0}; \mathbb{F})$. This homotopy taken together with the transfer map $p^!: |\mathcal{S}(B)| \rightarrow \tilde{Q}(E_+)$ defines a map $\tilde{\tau}_{\mathbb{F}}(p): |\mathcal{S}(B)| \rightarrow \text{hofib}(\tilde{Q}(E_+) \rightarrow K(\mathbb{F}))_{H_*(F_{b_0}; \mathbb{F})}$. We will call this element the unreduced Reidemeister torsion of the bundle p . The obvious inconvenience of this definition is that changing a basepoint in B changes the target of the map $\tilde{\tau}_{\mathbb{F}}(p)$. This can be fixed by shifting this map so it takes values in the space $\text{Wh}_{\mathbb{F}}(E) := \text{hofib}(\tilde{Q}(E_+) \rightarrow K(\mathbb{F}))_0$ – the homotopy fiber taken over the zero chain complex. Since both $\tilde{Q}(E_+)$ and $K(\mathbb{F})$ are infinite loop spaces this shift can be accomplished by subtracting the element $p^!(b_0)$ from the map $p^!$ and subtracting $H_*(F_{b_0}; \mathbb{F})$ from the contracting homotopy $|\mathcal{S}(B)| \times I \rightarrow K(\mathbb{F})$. One could make it more explicit by constructing models for inverses of elements in $\tilde{Q}(E_+)$ and $K(\mathbb{F})$. This is not hard to do. The new map $\tau_{\mathbb{F}}(p): |\mathcal{S}(B)| \rightarrow \text{Wh}_{\mathbb{F}}(E)$ is the (reduced) torsion of p .

The above construction can be also carried out under more general conditions which conform to the setting of [7]. We will say that a smooth bundle $p: E \rightarrow B$ is unipotent if the homology groups $H_*(F_{b_0}; \mathbb{F})$ admit a filtration

$$0 = V_0(F_{b_0}) \subseteq V_1(F_{b_0}) \subseteq \dots \subseteq V_k(F_{b_0}) = H_*(F_{b_0}; \mathbb{F})$$

such that $\pi_1(B)$ acts trivially on the quotients V_i/V_{i-1} . In this case consider the functor $v_0: \mathcal{S}(B) \rightarrow Ch(\mathbb{F})$ defined above. Waldhausen's additivity theorem implies that the map $v: |\mathcal{S}(B)| \rightarrow K(R)$ is canonically homotopic to the map which assigns to a simplex σ the direct sum $\bigoplus V_i(F_{\sigma(0)})/V_{i-1}(F_{\sigma(0)})$. Triviality of the action of

the fundamental group of B on the quotients $V_i(F_{\sigma(0)})/V_{i-1}(F_{\sigma(0)})$ implies that we can construct a map $\tau_{\mathbb{F}}: |\mathcal{S}(B)| \rightarrow \text{Wh}_{\mathbb{F}}(E)$ similarly as before.

7. CHARACTERISTIC CLASSES

As we mentioned at the beginning of this paper the torsion invariants of smooth bundles constructed by Igusa-Klein and Bismut-Lott are constructed as certain cohomology classes associated to the bundle. More precisely, for a bundle $p: E \rightarrow B$ its torsion in both of these settings is an element of $\bigoplus_{k>0} H^{4k}(B; \mathbb{R})$. Our final goal in this note is to show that the construction of torsion described above also gives rise to an element of $\bigoplus_{k>0} H^{4k}(B; \mathbb{R})$ which brings it on a common ground with the other notions of torsion.

Let then $p: E \rightarrow B$ be a bundle with a unipotent action of $\pi_1(B)$ on the homology of the fiber $H_*(F_{b_0}; \mathbb{R})$, so that the torsion $\tau_{\mathbb{R}}(p): |\mathcal{S}(B)| \rightarrow \text{Wh}_{\mathbb{R}}(E)$ is defined. Consider an embedding $i: E \rightarrow D^N$ where D^N is a closed disc in \mathbb{R}^M for some large $M > 0$. Let NE be a closed tubular neighborhood of E in D^N . Considering NE as a disc bundle over E we obtain a transfer map $\tilde{Q}(E_+) \rightarrow \tilde{Q}(NE_+)$. Also, since the construction of \tilde{Q} is functorial with respect to codimension 0 embeddings of manifolds the inclusion $NE \hookrightarrow D^M$ induces a map $\tilde{Q}(NE) \rightarrow \tilde{Q}(D^M)$. Composing it with the transfer of the bundle $NE \rightarrow E$ we obtain a map $\tilde{Q}(E_+) \rightarrow \tilde{Q}(D^M_+)$. Consider the diagram

$$\begin{array}{ccc}
 & \text{Wh}_{\mathbb{R}}(E) \dashrightarrow \text{Wh}_{\mathbb{R}}(D^M) & \\
 \nearrow \tau_{\mathbb{R}}(p) & \downarrow & \downarrow \\
 B & \xrightarrow{p'} \tilde{Q}(E_+) \longrightarrow \tilde{Q}(D^M_+) & \\
 & \downarrow \lambda_{\mathbb{R}} & \swarrow \lambda_{\mathbb{R}} \\
 & K(\mathbb{R}) &
 \end{array}$$

One can check that the lower triangle commutes up to a preferred choice of homotopy, so that we obtain a map of homotopy fibers $\text{Wh}_{\mathbb{R}}(E) \rightarrow \text{Wh}_{\mathbb{R}}(D^M)$.

Now, consider the fibration sequence

$$\Omega K(\mathbb{R}) \rightarrow \text{Wh}_{\mathbb{R}}(D^M) \rightarrow \tilde{Q}(D^M_+) \rightarrow K(\mathbb{R})$$

Since this is a fibration of infinite loop spaces after applying the rationalization functor we obtain a new fibration sequence.

$$\Omega K(\mathbb{R})_{\mathbb{Q}} \rightarrow \text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}} \rightarrow \tilde{Q}(D^M_+)_{\mathbb{Q}} \rightarrow K(\mathbb{R})_{\mathbb{Q}}$$

Since the homotopy groups of $\pi_i \tilde{Q}(D^M_+) \cong \pi_i Q(S^0)$ are torsion for $i > 0$ the space $\tilde{Q}(D^M_+)$ is homotopically discrete. It follows that every connected component of $\text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}$ is weakly equivalent to the space $\Omega K(\mathbb{R})_{\mathbb{Q}}$. On the other hand by [3] we have a weak equivalence

$$K(\mathbb{R})_{\mathbb{Q}} \simeq \mathbb{Z} \times \prod_{k>0} K(\mathbb{R}, 4k+1)$$

Let $\text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}^B$ denote the connected component of $\text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}$ which is a target of the map $B \rightarrow \text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}$. By the observation above we have $\text{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}^B \simeq$

$\prod_{k>0} K(\mathbb{R}, 4k)$, and so the homotopy class of the map $B \rightarrow \mathrm{Wh}_{\mathbb{R}}(D^M)_{\mathbb{Q}}$ determines an element in $\bigoplus_{k>0} H^{4k}(B; \mathbb{R})$.

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