

## EQUIVALENCE OF HIGHER TORSION INVARIANTS

BERNARD BADZIOCH, WOJCIECH DORABIAŁA, JOHN R. KLEIN,  
AND BRUCE WILLIAMS

ABSTRACT. We show that the smooth torsion of bundles of manifolds constructed by Dwyer, Weiss, and Williams satisfies the axioms for higher torsion developed by Igusa. As a consequence we obtain that the smooth Dwyer-Weiss-Williams torsion is proportional to the higher torsion of Igusa and Klein.

## CONTENTS

1. Introduction	1
2. Igusa's axioms	4
3. Waldhausen $K$ -theories	6
4. The smooth torsion of Dwyer-Weiss-Williams	8
5. Additivity of the smooth torsion	15
6. The secondary transfer	21
7. The transfer axiom	27
8. Nontriviality of the smooth torsion	29
Appendix : Homology of chain complexes	31
References	36

## 1. INTRODUCTION

Higher torsion invariants are invariants of smooth fiber bundles of manifolds which generalize the classical Reidemeister torsion of topological spaces. Just as the Reidemeister torsion helps to distinguish spaces which have the same homotopy type but differ in their geometric properties, the purpose of the higher torsion is to aid in classification of smooth bundles up to fiberwise diffeomorphism, especially in the case when bundles are fiberwise homotopy equivalent and thus cannot be set apart using homotopy type invariants.

The idea that the Reidemeister torsion could be generalized to the setting of smooth bundles appeared in the work of Hatcher and Wagoner [13, 20]. The task of constructing higher torsion, however, proved challenging and required prior development of such areas as local index theory, the theory of parametrized Morse functions, and Waldhausen's algebraic  $K$ -theory of spaces.

---

*Date:* 04/30/2009.

These advancements brought in recent years three independent and very distinct constructions of higher torsion. In [4] Bismut and Lott presented an analytic construction. The torsion invariant developed by Igusa and Klein [15, 17] follows from a more geometric, Morse theoretical approach. The most recent of the three is the construction of smooth torsion described by Dwyer, Weiss, and Williams in [9]. Its striking feature is that while it captures some information about the smooth structure of a bundle, it can be described entirely in terms of homotopy theory.

These invariants have already proved to be effective tools for studying smooth bundles. For example, in [14] Igusa demonstrated that the Igusa-Klein torsion can be used to distinguish exotic disc bundles constructed by Hatcher, while Goette [11, 12] obtained results of a similar kind in the realm of the Bismut-Lott torsion. The central question have become, however, how the three notions of higher torsion are related to one another. The analytic torsion of Bismut-Lott and the higher torsion of Igusa-Klein are known to coincide (up to a normalization constant) for several classes of bundles for which both of these invariants are defined; see e.g. [5, 11, 12]. On the other hand, no results have been known comparing them to the smooth torsion of Dwyer-Weiss-Williams.

In order to provide a general framework for settling the comparison problem Igusa developed in [16] an axiomatic approach to higher torsion. He showed that any cohomological higher torsion invariant which is defined on the class of unipotent bundles (2.1) and which satisfies the additivity (2.5) and transfer (2.6) axioms must coincide with the Igusa-Klein torsion up to a couple of scalar constants. In [2] Badzioch, Dorabiała and Williams showed that the Dwyer-Weiss-Williams torsion (which was originally defined in [9] for acyclic bundles) can be extended to the class of all unipotent bundles and that it yields a cohomological invariant of such bundles. The main result of this paper (Theorem 4.11) is that this cohomological invariant satisfies the axioms of Igusa. Combined with the observation that the Dwyer-Weiss-Williams torsion is an exotic invariant (2.9) we obtain

**1.1. Theorem** (cf. Theorem 4.12). *For any  $k > 0$  the smooth cohomological Dwyer-Weiss-Williams torsion of unipotent bundles in degree  $4k$  is proportional to the Igusa-Klein torsion in the same degree.*

Apart from its intrinsic interest, the fact that these two very different constructions yield the same information about smooth bundles has several useful consequences. On the one hand, the Igusa-Klein torsion is more suitable for direct computations than the Dwyer-Weiss-Williams invariant. In fact, while Igusa-Klein torsion has been calculated for many classes of bundles, in Section 8 of this paper we give the first, to our knowledge, examples of bundles for which the Dwyer-Weiss-Williams torsion does not vanish. By our comparison result, however, all computations using Igusa-Klein torsion apply instantly to the smooth torsion of Dwyer-Weiss-Williams.

On the other hand, the approach of Dwyer-Weiss-Williams seems to provide a better understanding of the information carried by torsion of a bundle. Recall, for example, that in the classical setting the Reidemeister torsion is an invariant related to the Whitehead torsion of a homotopy equivalence of spaces. Results of [9] imply that, analogously, one can define a smooth parametrized Whitehead torsion as an invariant associated to fiberwise homotopy equivalence  $f$  of smooth bundles. This invariant vanishes when  $f$  is homotopic to a diffeomorphism of bundles. Moreover, the opposite is also true: if the smooth parametrized Whitehead torsion of  $f$  is trivial then  $f$  (after an appropriate stabilization) is homotopic to a diffeomorphism. The smooth torsion of Dwyer-Weiss-Williams is an absolute invariant associated to the parametrized Whitehead torsion. Viewed from this perspective the Dwyer-Weiss-Williams torsion of a bundle  $p$  can be seen as an obstruction to existence of a diffeomorphism between  $p$  and a product bundle.

Another application of Theorem 1.1 comes as a consequence of Theorem 7.1 of this paper. For unipotent bundles  $p: E \rightarrow B$  and  $q: D \rightarrow E$  the transfer axiom of Igusa (2.6) gives a formula expressing torsion of the bundle  $pq$  in terms of torsions of  $p$  and  $q$ . This formula is known to hold for the Igusa-Klein torsion under the assumption that  $q$  is an oriented linear sphere bundle. Our Theorem 7.1 shows that for the Dwyer-Weiss-Williams torsion the same formula holds in a more general setting, whenever the bundle  $q$  satisfies the assumptions of the Leray-Hirsch theorem (we say then that  $q$  is a Leray-Hirsch bundle, see Definition 6.1). Theorem 1.1 immediately implies that this is true for the Igusa-Klein torsion as well:

**1.2. Theorem.** *For  $k > 0$  let  $t_{4k}^{IK}$  be the Igusa-Klein torsion invariant in the degree  $4k$ . Let  $p: E \rightarrow B$  be a unipotent bundle, and let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F$ . Then we have*

$$t_{4k}^{IK}(pq) = \chi(F)t^{IK}(p) + tr_B^E(t_{4k}^{IK}(q))$$

where  $\chi(F)$  is the Euler characteristic of  $F$  and  $tr_B^E: H^*(E; \mathbb{R}) \rightarrow H^*(B; \mathbb{R})$  is the transfer homomorphism associated to  $p$ .

This property extends computability of the Igusa-Klein invariant.

**1.3. Organization of the paper.** In Section 2 we give a summary of Igusa's axiomatic description of higher torsion of smooth bundles. Since our work on the smooth torsion is heavily dependent on the language of Waldhausen categories we give a brief overview of the relevant notions and construction in Section 3. Section 4 describes the main steps in the construction of the smooth torsion of Dwyer-Weiss-Williams for unipotent bundles. We also state there precisely the main results of this paper (Theorems 4.11 and 4.12). The focus of Section 5 is Theorem 5.1 which describes the additive property of the smooth torsion. As we have already mentioned the transfer axiom of Igusa provides a relationship between the torsion of bundles  $p$ ,  $q$ , and  $pq$ . In Section 6 we show that for the smooth torsion such relationship can be expressed using a "secondary transfer" map (Theorem

6.8) and then, in Section 7, we verify that Igusa's axiom can be derived from our formula. Finally, in Section 8 we demonstrate that the smooth torsion is a non-trivial invariant of bundles.

A part of the construction of the smooth torsion of unipotent bundles, and consequently a part of our arguments, involves the passage from chain complexes to their homology on the level of the algebraic  $K$ -theory. The material related to this topic is gathered in the [Appendix](#) which closes this paper.

#### 1.4. Terminology and notation.

- By a smooth bundle we will understand a smooth submersion  $p: E \rightarrow B$  where  $E$  and  $B$  are smooth compact manifolds. By the Ehresmann fibration theorem [10]  $p$  is then a locally trivial fiber bundle with fiber  $F = p^{-1}(b)$  for  $b \in B$ , and with the group of diffeomorphisms of  $F$  as the structure group.
- All chain complexes and homology groups in this paper are taken with coefficients in  $\mathbb{R}$ , the real numbers.
- Let  $f_0, f_1: X \rightarrow Y$  be maps of topological spaces, and let  $h_0, h_1: X \times I \rightarrow Y$  be two homotopies between  $f_0$  and  $f_1$ . By a homotopy of homotopies we will understand in this situation a map

$$H: X \times I \times I \rightarrow Y$$

such that

$$\begin{cases} H|_{X \times \{i\} \times \{t\}} &= f_i \quad \text{for } i = 0, 1, \text{ and } t \in I \\ H|_{X \times I \times \{j\}} &= h_j \quad \text{for } j = 0, 1 \end{cases}$$

## 2. IGUSA'S AXIOMS

In this section we briefly summarize the main results of the work of Igusa on axioms of higher torsion. We refer to [16] for details.

**2.1. Definition.** Let  $p: E \rightarrow B$  be a smooth bundle such that  $B$  is a connected manifold with a basepoint  $b_0$ . Let  $F$  be the fiber of  $p$  over  $b_0$ . The bundle  $p$  is unipotent if  $H_*(F)$  admits a filtration by graded  $\pi_1 B$ -submodules

$$0 = V_0(F) \subseteq V_1(F) \subseteq \dots \subseteq V_k(F) = H_*(F)$$

such that  $\pi_1 B$  acts trivially on the quotients  $V_i(F)/V_{i-1}(F)$ .

**2.2. Definition.** A characteristic class  $t$  of unipotent bundles in degree  $k$  is an assignment which associates to every unipotent bundle  $p: E \rightarrow B$  a cohomology class  $t(p) \in H^k(B)$  in such way that for any smooth map  $f: B' \rightarrow B$  we have

$$t(p') = f^*t(p)$$

where  $p': f^*E \rightarrow B'$  is the bundle induced from  $p$ .

**2.3.** If  $p: E \rightarrow B$  is a smooth bundle whose fibers are manifolds with a boundary then restricting  $p$  to the union of boundaries of fibers of  $p$  we obtain a new smooth bundle

$$\partial^v p: \partial^v E \rightarrow B$$

which we call the vertical boundary of  $p$ . By [16, Prop. 2.1] if the bundle  $p$  is unipotent then so is  $\partial^v p$ .

**2.4. Definition.** Let  $p: E \rightarrow B$  be a smooth bundle with closed fibers. We will say that  $p$  admits a splitting if there exist smooth subbundles of  $p$

$$p_i: E_i \rightarrow B, \quad i = 0, 1, 2$$

such that  $p_0$  is the vertical boundary of both  $p_1$  and  $p_2$ , and the bundle  $p$  is given by

$$p = p_1 \cup_{p_0} p_2: E_1 \cup_{E_0} E_2 \rightarrow B$$

The splitting of  $p$  is unipotent if  $p_1$  and  $p_2$  are unipotent bundles.

If  $p$  admits a unipotent splitting  $p = p_1 \cup_{p_0} p_2$  then by (2.3) the bundle  $p_0$  is unipotent. Using this fact and the Mayer-Vietoris sequence for the homology of the fiber of  $p$  one can see that in such case  $p$  is a unipotent bundle as well. This motivates the following

**2.5. Definition (Additivity Axiom<sup>1</sup>).** A characteristic class of unipotent bundles  $t$  satisfies the additivity axiom if for any bundle  $p$  with a unipotent splitting  $p = p_1 \cup_{p_0} p_2$  we have

$$t(p) = t(p_1) + t(p_2) - t(p_0)$$

**2.6. Definition (Transfer Axiom).** Let  $p: E \rightarrow B$  be a unipotent bundle, and let  $\xi$  be an  $(n+1)$ -dimensional oriented vector bundle over  $E$  with the associated sphere bundle  $q: S^n(\xi) \rightarrow E$ . Let  $F$  be the fiber of  $q$ . We will say that a characteristic class  $t$  satisfies the transfer axiom if for any  $p, q$  as above we have

$$t(pq) = \chi(F)t(p) + tr_B^E(t(q))$$

where  $\chi(F) \in \mathbb{Z}$  is the Euler characteristic of  $F$  and  $tr_B^E: H^*(E) \rightarrow H^*(B)$  is the Becker-Gottlieb transfer of  $p$ .

The transfer axiom relies on the fact that for bundles  $p, q$  as in (2.6) the bundle  $pq: S^n(\xi) \rightarrow B$  is unipotent. This follows from [16, Prop. 2.1].

**2.7. Definition.** A characteristic class of unipotent bundles is a higher torsion invariant if it satisfies the additivity and transfer axioms.

We can now state the main result of [16].

**2.8. Theorem.** *For any  $k > 0$  the collection of higher torsion invariants in degree  $4k$  has the structure of a 2-dimensional real vector space.*

Igusa shows that the vector space of higher torsion invariants in the degree  $4k$  is spanned by “even” and “odd” parts of the Igusa-Klein torsion  $t_{4k}^{IK}$ . A closer relationship between  $t_{4k}^{IK}$  and other higher torsion invariants can be obtained by restricting attention to the class of exotic invariants.

<sup>1</sup>In [16] Igusa formulates axioms for higher torsion of unipotent bundles with *closed* fibers. In effect his additivity axiom comes in a slightly different (though equivalent) form to the one given here.

**2.9. Definition** ([16, p. 185]). A characteristic class of unipotent bundles  $t$  is exotic if for any unipotent bundle  $p: E \rightarrow B$  and for any oriented linear disc bundle  $q: D \rightarrow E$  we have

$$t(pq) = t(p)$$

Results of [16] imply that exotic higher torsion invariants form a 1-dimensional subspace in the vector space of higher torsion invariants. More precisely we have

**2.10. Theorem** ([16, Thm 9.13]). *If  $t$  is an exotic higher torsion invariant in degree  $4k$  then there exists  $\lambda \in \mathbb{R}$  such that for any unipotent bundle  $p$  we have*

$$t(p) = \lambda \cdot t_{4k}^{IK}(p)$$

### 3. WALDHAUSEN $K$ -THEORIES

In our work on the smooth torsion of Dwyer-Weiss-Williams we will be using extensively the machinery of Waldhausen categories and their  $K$ -theories. In this section we review the basic notions and constructions related to this area. This material is standard and can be found in [23]. Our goal here is to provide a concise summary of its aspects relevant to this paper and to fix the notation.

**3.1. Waldhausen categories.** By a Waldhausen category we will understand a category  $\mathcal{C}$  equipped with subcategories of cofibrations and weak equivalences satisfying the conditions of [23, Def. 1.2]. We can turn  $\mathcal{C}$  into a simplicial category in two ways:

- $w\mathcal{S}_\bullet\mathcal{C}$  is the simplicial category obtained by applying to  $\mathcal{C}$  the  $\mathcal{S}_\bullet$ -construction of Waldhausen [23, 1.3];
- $w\mathcal{T}_\bullet\mathcal{C}$  is obtained by applying to  $\mathcal{C}$  Thomason's variant of the  $\mathcal{S}_\bullet$ -construction [23, p. 343]

The objects of the category  $w\mathcal{S}_n\mathcal{C}$  which appears in the  $n$ -th simplicial dimension of  $w\mathcal{S}_\bullet\mathcal{C}$  are sequences of cofibrations in  $\mathcal{C}$ :

$$c_1 \twoheadrightarrow c_2 \twoheadrightarrow \dots \twoheadrightarrow c_n$$

together with implicitly present quotient data. Morphisms in  $w\mathcal{S}_n\mathcal{C}$  are commutative diagrams

$$\begin{array}{ccccccc} c_1 & \twoheadrightarrow & c_2 & \twoheadrightarrow & \dots & \twoheadrightarrow & c_n \\ \downarrow & & \downarrow & & & & \downarrow \\ c'_1 & \twoheadrightarrow & c'_2 & \twoheadrightarrow & \dots & \twoheadrightarrow & c'_n \end{array}$$

with the vertical arrows given by weak equivalences. We also set  $w\mathcal{S}_0\mathcal{C} = \{*\}$ . Notice that  $w\mathcal{S}_1\mathcal{C}$  is just the subcategory of weak equivalences of  $\mathcal{C}$ .

The category  $w\mathcal{T}_n\mathcal{C}$  appearing in the  $n$ -simplicial dimension of  $w\mathcal{T}_\bullet\mathcal{C}$  has as its objects sequences of cofibrations

$$c_0 \twoheadrightarrow c_1 \twoheadrightarrow c_2 \twoheadrightarrow \dots \twoheadrightarrow c_n$$

Morphisms in  $w\mathcal{T}_n\mathcal{C}$  are defined similarly as in  $w\mathcal{S}_n\mathcal{C}$ , but the requirement that vertical morphisms are weak equivalences is replaced by the condition that for every  $i \geq j$  the map

$$c'_i \cup_{c_i} c_j \rightarrow c'_j$$

is a weak equivalence.

**3.2.** Consider the spaces  $\Omega|w\mathcal{S}_\bullet\mathcal{C}|$  and  $\Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$ . By [23] these are weakly equivalent infinite loop spaces, representing different combinatorial models of the  $K$ -theory of the Waldhausen category  $\mathcal{C}$ .

**3.3.** We have the standard maps

$$|w\mathcal{S}_1\mathcal{C}| \times \Delta^1 \rightarrow |w\mathcal{S}_\bullet\mathcal{C}| \quad \text{and} \quad |w\mathcal{T}_1\mathcal{C}| \times \Delta^1 \rightarrow |w\mathcal{T}_\bullet\mathcal{C}|$$

By adjunction they induce canonical maps

$$k: |w\mathcal{S}_1\mathcal{C}| \rightarrow \Omega|w\mathcal{S}_\bullet\mathcal{C}| \quad \text{and} \quad k: |w\mathcal{T}_1\mathcal{C}| \rightarrow \Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$$

As a result given any small category  $\mathcal{D}$  and a functor  $F: \mathcal{D} \rightarrow w\mathcal{S}_1\mathcal{C}$  we obtain a map

$$|\mathcal{D}| \xrightarrow{|F|} |w\mathcal{S}_1\mathcal{C}| \xrightarrow{k} K(\mathbb{R})$$

Analogously, any functor  $F: \mathcal{D} \rightarrow w\mathcal{T}_1\mathcal{C}$  induces a map

$$|\mathcal{D}| \rightarrow \Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$$

Notice that using the map  $k$  we can identify objects of  $\mathcal{C}$  with points in the space  $\Omega|w\mathcal{S}_\bullet\mathcal{C}|$ . Similarly, cofibrations in  $\mathcal{C}$  can be identified with points in  $\Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$ .

In [23] Waldhausen defines the notion of an exact functor of Waldhausen categories. The main property of such functors is that they preserve the  $\mathcal{S}_\bullet$ -construction. The following definition gives a slightly relaxed variant of exactness.

**3.4. Definition.** Let  $\mathcal{C}, \mathcal{D}$  be Waldhausen categories. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is almost exact if it preserves cofibrations and weak equivalences and if for any diagram in  $\mathcal{C}$  of the form

$$c' \leftarrow c \twoheadrightarrow c''$$

the map

$$F(c') \cup_{F(c)} F(c'') \rightarrow F(c' \cup_c c'')$$

is a weak equivalence in  $\mathcal{D}$ .

**3.5.** An almost exact functor  $F$  induces a functor of simplicial categories  $F_\bullet: w\mathcal{T}_\bullet\mathcal{C} \rightarrow w\mathcal{S}_\bullet\mathcal{D}$  where  $F_n: w\mathcal{T}_n\mathcal{C} \rightarrow w\mathcal{S}_n\mathcal{D}$  is given by

$$F_n(c_0 \twoheadrightarrow c_1 \twoheadrightarrow \dots \twoheadrightarrow c_n) = (F(c_1)/F(c_0) \twoheadrightarrow \dots \twoheadrightarrow F(c_n)/F(c_0))$$

Here  $c_i/c_0 := \text{colim}(* \leftarrow c_0 \twoheadrightarrow c_i)$  and  $*$   $\in \mathcal{C}$  is the terminal object. As a consequence  $F$  defines a map

$$\Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|) \rightarrow \Omega|w\mathcal{S}_\bullet\mathcal{D}|$$

If  $F$  is the identity functor on  $\mathcal{C}$  then this map gives the weak equivalence of (3.2).

**3.6. Waldhausen's pre-additivity theorem.** If  $c, c'$  are objects in  $\mathcal{C}$  then the coproduct  $c \sqcup c'$  represents the sum  $c + c'$  in the  $H$ -space structure on  $\Omega|w\mathcal{S}_\bullet\mathcal{C}|$ . Similarly, taking coproducts of cofibrations coincides with addition in  $\Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$ . Waldhausen's additivity theorem [23, Proposition 1.3.2] relates the  $H$ -space structure on the  $K$ -theory of  $\mathcal{C}$  with the cofibration sequences in  $\mathcal{C}$ . In this paper we will use a simplified, combinatorial formulation of this theorem which can be described as follows.

For a Waldhausen category  $\mathcal{C}$  consider the evaluation functors

$$Ev_i: w\mathcal{S}_2\mathcal{C} \rightarrow w\mathcal{S}_1\mathcal{C}, \quad i = 1, 2$$

given by  $Ev_i(c_1 \twoheadrightarrow c_2) := c_i$ . Also, let  $Ev_{12}: w\mathcal{S}_2\mathcal{C} \rightarrow w\mathcal{S}_1\mathcal{C}$  be defined by  $Ev_{12}(c_1 \twoheadrightarrow c_2) := c_2/c_1$ . Passing to the nerves of categories we obtain maps

$$|Ev_i|: |w\mathcal{S}_2\mathcal{C}| \rightarrow |w\mathcal{S}_1\mathcal{C}|, \quad i = 1, 2, 12$$

We have

**3.7. Theorem** ([23, 1.3.3]). *Let  $k$  is the map described in (3.3). There exists a homotopy*

$$\mathcal{U}: |w\mathcal{S}_2\mathcal{C}| \times I \rightarrow \Omega|w\mathcal{S}_\bullet\mathcal{C}|$$

between the maps  $k \circ |Ev_2|$  and  $k \circ |Ev_1| + k \circ |Ev_{12}|$ .

Theorem 3.7 can be equivalently formulated using the  $\mathcal{T}_\bullet$ -construction. In this case the functors  $Ev_i: w\mathcal{T}_2\mathcal{C} \rightarrow w\mathcal{T}_1\mathcal{C}$  are defined by

$$Ev_i(c_0 \twoheadrightarrow c_1 \twoheadrightarrow c_2) := \begin{cases} c_0 \twoheadrightarrow c_i & i = 1, 2 \\ c_1 \twoheadrightarrow c_2 & i = 12 \end{cases}$$

We will call Theorem 3.7 Waldhausen's pre-additivity theorem.

**3.8.** In this paper we will typically use Theorem 3.7 in the following way. Assume that for a small category  $\mathcal{D}$  and a Waldhausen category  $\mathcal{C}$  we have a functor  $F: \mathcal{D} \rightarrow w\mathcal{S}_2\mathcal{C}$  (or  $F: \mathcal{D} \rightarrow w\mathcal{T}_2\mathcal{C}$ ). By (3.3) for  $i = 1, 2, 12$  the compositions  $Ev_i \circ F$  induce maps  $f_i: |\mathcal{D}| \rightarrow \Omega|w\mathcal{S}_\bullet\mathcal{C}|$  (or respectively  $f_i: |\mathcal{D}| \rightarrow \Omega(|w\mathcal{T}_\bullet\mathcal{C}|/|w\mathcal{T}_0\mathcal{C}|)$ ). Then  $\mathcal{U}$  defines a preferred homotopy

$$f_2 \simeq f_1 + f_{12}$$

#### 4. THE SMOOTH TORSION OF DWYER-WEISS-WILLIAMS

Below we review the construction of the Dwyer-Weiss-Williams smooth torsion for unipotent bundles. Our approach is the same as that of [2], although the notation differs in some places.

Since the constructions described in this section are rather technical, it may be useful to keep in mind the following rough idea which motivates them. For any fibration  $p: E \rightarrow B$  with a finitely dominated fiber  $F$  the action of  $\pi_1 B$  on  $H_*(F, \mathbb{R})$  yields a map

$$c_p: B \rightarrow K(\mathbb{R})$$

where  $K(\mathbb{R})$  is the infinite loop space of the algebraic  $K$ -theory of  $\mathbb{R}$ . The real cohomology of  $K(\mathbb{R})$  contains the Borel regulator classes. Pulling back these classes along  $c_p$  we obtain characteristic classes of  $p$ , which take values in the cohomology groups  $H^{4k+1}(B, \mathbb{R})$ . These classes are primary invariants of  $p$ .

If the action of  $\pi_1 B$  on  $H_*(F, \mathbb{R})$  is unipotent (2.1), then we get a preferred homotopy  $\omega_p$  from  $c_p$  to a constant map. This yields a trivialization of our primary characteristic classes.

Also, if  $p$  is a smooth bundle, then  $c_p$  has a factorization  $p^!$  through a space homotopy equivalent to  $\Omega^\infty \Sigma^\infty(S^0)$  which is rationally a discrete space. Therefore the smooth structure on  $p$  determines another trivialization of the primary characteristic classes.

As a consequence, for  $p$  which is both smooth and unipotent we have two trivializations of our primary invariants. Taken together they yield secondary characteristic classes  $t_{4k}^s(p) \in H^{4k}(B, \mathbb{R})$  which are the Dwyer-Weiss-Williams smooth torsion classes.

**4.1. Transfer.** Let  $p: E \rightarrow B$  be a smooth bundle. We will denote by  $\mathcal{S}(B)$  the category of smooth singular simplices of  $B$  [2, 4.1] and by  $|\mathcal{S}(B)|$  the geometric realization of the nerve of  $\mathcal{S}(B)$ . We have a weak equivalence  $B \simeq |\mathcal{S}(B)|$ . As in [2, Sec. 3] by  $\tilde{Q}(E_+)$  we will understand the space obtained by applying the  $\mathcal{T}_\bullet$ -construction (3.1) to the category of partitions of  $E$ . We have a weak equivalence

$$\tilde{Q}(E_+) \simeq \Omega^\infty \Sigma^\infty E_+$$

Following [2, Sec. 4] by  $p^!: |\mathcal{S}(B)| \rightarrow \tilde{Q}(E_+)$  we will denote the Becker-Gottlieb transfer of the bundle  $p$  and by  $\tilde{Q}(p^!): \tilde{Q}(B_+) \rightarrow \tilde{Q}(E_+)$  its extension to  $\tilde{Q}(B_+)$ .

Next, let  $\mathcal{R}^{fd}(E)$  be the category of homotopy finitely dominated retractive spaces over  $E$  with maps of retractive spaces as morphisms. It is a Waldhausen category with cofibrations given by Serre cofibrations and weak equivalences defined as weak homotopy equivalences. Denote

$$A(E) := \Omega(|w\mathcal{T}_\bullet \mathcal{R}^{fd}(E)| / |w\mathcal{T}_0 \mathcal{R}^{fd}(E)|)$$

This is the Waldhausen  $A$ -theory of the space  $E$ . We have the assembly map  $\tilde{a}_E: \tilde{Q}(E) \rightarrow A(E)$  [2, Sec. 3].

Let  $\sigma: \Delta^k \rightarrow B$  be a smooth singular simplex and let

$$\sigma^* E := \lim(\Delta^k \xrightarrow{\sigma} B \xleftarrow{p} E)$$

The space  $\sigma^* E \sqcup E$  is in the obvious way a retractive space over  $E$ . Consider the functor

$$(4-1) \quad F_{p^A}: \mathcal{S}(B) \rightarrow w\mathcal{T}_1 \mathcal{R}^{fd}(E)$$

which assigns to  $\sigma \in \mathcal{S}(B)$  the cofibration  $F_{p^A}(\sigma) := (E \rightarrow E \sqcup \sigma^* E)$ . By (3.3) the functor  $F_{p^A}$  induces a map  $p^A: |\mathcal{S}(B)| \rightarrow A(E)$ . The combinatorial

constructions of the maps  $p^!$  and  $\tilde{a}_E$  in [2] imply that the following diagram commutes:

$$(4-2) \quad \begin{array}{ccc} & & \tilde{Q}(E_+) \\ & \nearrow p^! & \downarrow \tilde{a}_E \\ |\mathcal{S}(B)| & \xrightarrow{p^A} & A(E) \end{array}$$

**4.2. Remark.** We note that commutativity of the diagram (4-2) depends in an essential way on the smooth structure of the bundle  $p$ . Indeed, while all maps appearing in this diagram exist for any fibration  $p: E \rightarrow B$  with finitely dominated fibers, Dwyer has given examples of fibrations where the map  $\tilde{a}_E \circ p^!$  is not homotopic to  $p^A$  (see proof of Theorem F in [18]).

**4.3. Linearization.** By  $\mathcal{C}h^{fd}(\mathbb{R})$  we will denote the category of homotopy finitely dominated chain complexes of  $\mathbb{R}$ -vector spaces. This is a Waldhausen category with degreewise monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. Applying the  $\mathcal{S}_\bullet$ -construction (3.1) we obtain the space

$$K(\mathbb{R}) := \Omega |w\mathcal{S}_\bullet \mathcal{C}h^{fd}(\mathbb{R})|$$

which has the homotopy type of the infinite loop space underlying the algebraic  $K$ -theory spectrum of  $\mathbb{R}$ .

Consider the linearization functor

$$(4-3) \quad \Lambda_E^A: \mathcal{R}^{fd}(E) \rightarrow \mathcal{C}h^{fd}(\mathbb{R})$$

which assigns to a retractive space  $X \in \mathcal{R}^{fd}(E)$  the singular chain complex  $C_*(X)$ . This functor is almost exact (3.4) so it induces a map

$$\lambda_E^A: A(E) \rightarrow K(\mathbb{R})$$

The composition

$$\lambda_E := \lambda_E^A \circ \tilde{a}_E: \tilde{Q}(E_+) \rightarrow K(\mathbb{R})$$

is the linearization map of  $E$ .

Next, let

$$C_p: \mathcal{S}(B) \rightarrow w\mathcal{S}_1 \mathcal{C}h^{fd}(\mathbb{R})$$

be the functor given by  $C_p(\sigma) = C_*(\sigma^* E)$ . It induces a map  $c_p: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$ . Notice that we have a canonical isomorphism of chain complexes

$$C_p(\sigma) \cong \Lambda_{E,1}^A \circ F_{p^A}(\sigma)$$

where

$$\Lambda_{E,1}^A: w\mathcal{T}_1 \mathcal{R}^{fd}(E) \rightarrow w\mathcal{S}_1 \mathcal{C}h^{fd}(\mathbb{R})$$

is the functor induced by  $\Lambda_E^A$  (see 3.5). This shows that the following diagram commutes up to a preferred homotopy:

$$\begin{array}{ccc} & & A(E) \\ & \nearrow p^A & \downarrow \lambda_E^A \\ |\mathcal{S}(B)| & \xrightarrow{c_p} & K(\mathbb{R}) \end{array}$$

Combining this with the diagram (4-2) we obtain a diagram

$$\begin{array}{ccc} & & \tilde{Q}(E_+) \\ & \nearrow p^! & \downarrow \tilde{a}_E \\ & & A(E) \\ & \nearrow p^A & \downarrow \lambda_E^A \\ |\mathcal{S}(B)| & \xrightarrow{c_p} & K(\mathbb{R}) \end{array} \quad \lambda_E$$

which is homotopy commutative (via preferred homotopies).

**4.4. Smooth torsion.** In the next definition we use the fact (3.3) that chain complexes in  $\mathcal{C}h^{fd}(\mathbb{R})$  can be identified with points in the space  $K(\mathbb{R})$ .

**4.5. Definition.** For a chain complex  $C \in \mathcal{C}h^{fd}(\mathbb{R})$  the Whitehead space  $\text{Wh}^{\mathbb{R}}(E)_C$  is the homotopy fiber

$$\text{Wh}^{\mathbb{R}}(E)_C := \text{hofib}(\lambda_E: \tilde{Q}(E_+) \rightarrow K(\mathbb{R}))_C$$

We will denote by  $\text{Wh}^{\mathbb{R}}(E)$  the space  $\text{Wh}^{\mathbb{R}}(E)_0$  where  $0 \in \mathcal{C}h^{fd}(\mathbb{R})$  is the zero chain complex. Since  $\lambda_E$  is a map of infinite loop spaces  $\text{Wh}^{\mathbb{R}}(E)$  has a natural infinite loop space structure.

Assume now that  $p: E \rightarrow B$  is a unipotent bundle (2.1) with a basepoint  $b_0 \in B$  and let  $F$  be the fiber of  $p$  over  $b_0$ . Consider the graded vector space  $H_*(F)$  as a chain complex with trivial differentials. By [2, Thm. 6.7] we have a preferred homotopy

$$\omega_p: |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$$

such that  $\omega_p|_{|\mathcal{S}(B)| \times \{0\}} = c_p$  and  $\omega_p|_{|\mathcal{S}(B)| \times \{1\}}$  is the constant map  $*_{H_*(F)}$ , which sends  $|\mathcal{S}(B)|$  to the point  $H_*(F) \in K(\mathbb{R})$ . We will call  $\omega_p$  the algebraic contraction of  $p$ .

**4.6. Definition** ([2, 6.9, 6.10]). The unreduced smooth torsion of a unipotent bundle  $p: E \rightarrow B$  is the map

$$\tilde{\tau}^s(p): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)_{H_*(F)}$$

which is the lift of the Becker-Gottlieb transfer  $p^!$  determined by the algebraic contraction  $\omega_p$ . The (reduced) smooth torsion of  $p$  is the map

$$\tau^s(p): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)$$

obtained by shifting  $\tilde{\tau}^s(p)$  to  $\text{Wh}^{\mathbb{R}}(E)$ .

More precisely, consider the map  $\tilde{p}^!: |\mathcal{S}(B)| \rightarrow \tilde{Q}(E_+)$  obtained by subtracting from  $p^!$  the constant map which sends  $|\mathcal{S}(B)|$  into the point  $p^!(b_0) \in \tilde{Q}(E_+)$ . Also, let  $\tilde{\omega}_p: |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$  be the homotopy which for every  $t \in I$  is given by the substracing from the map  $\omega_p|_{|\mathcal{S}(B)| \times \{t\}}$  the constant map sending  $|\mathcal{S}(B)|$  to  $\omega_p(b_0, t)$ . Then  $\tilde{\omega}_p$  is a homotopy between  $\lambda_E \circ \tilde{p}^!$  and the constant map sending  $|\mathcal{S}(B)|$  to  $0 \in K(\mathbb{R})$ . We will call  $\tilde{\omega}_p$  the reduced algebraic contraction of  $p$ . The reduced torsion  $\tau^s(p)$  is the lift of  $\tilde{p}^!$  determined by  $\tilde{\omega}_p$ .

**4.7. Remark.** The following observation will be useful later on. Assume that for a map

$$f: |\mathcal{S}(B)| \rightarrow \tilde{Q}(E_+)$$

and a chain complex  $C \in \text{Ch}^{fd}(\mathbb{R})$  we have a homotopy

$$\omega: |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$$

between  $\lambda_E f$  and the constant map  $*_C$ . The pair  $(f, \omega)$  defines a map  $\varphi: |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)_C$ . Reducing  $f$  and  $\omega$  the same way which we used above to obtain  $\tilde{p}^!$  and  $\tilde{\omega}_p$  we get a pair  $(\tilde{f}, \tilde{\omega})$  which determines a map  $\tilde{\varphi}: |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)$ . We will call  $\tilde{\varphi}$  the reduction of  $\varphi$ .

Let  $\varphi': |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)_{C'}$  be another map defined by a pair  $(f', \omega')$  and let  $\tilde{\varphi}'$  be the reduction of  $\varphi'$ . Notice that  $\tilde{\varphi}$  and  $\tilde{\varphi}'$  are homotopic maps if the following conditions hold:

- there exists a homotopy  $h: |\mathcal{S}(B)| \times I \rightarrow \tilde{Q}(E_+)$  between  $f$  and  $f'$ ;
- there exists a path  $\gamma$  in  $K(\mathbb{R})$  joining the points  $C$  and  $C'$ ;
- there exists a homotopy of homotopies (1.4) between the concatenation of  $\lambda_E h$  with  $\omega'$ , and the concatenation of  $\omega$  with  $\gamma$  (we interpret here  $\gamma$  as a homotopy  $\gamma: |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$  via constant maps):

$$\begin{array}{ccc} \lambda_E f & \xrightarrow{\lambda_E h} & \lambda_E f' \\ \omega \downarrow & & \downarrow \omega' \\ *_C & \xrightarrow{\gamma} & *_{C'} \end{array}$$

In the above diagram vertices represent maps  $|\mathcal{S}(B)| \rightarrow K(\mathbb{R})$  and edges stand for homotopies of such maps.

**4.8. The algebraic contraction.** Since our arguments later in this paper will rely on the specifics of the construction of the algebraic contraction  $\omega_p$  we will now review the main steps of this construction.

As before, for a unipotent bundle  $p: E \rightarrow B$  let  $b_0$  be the basepoint of  $B$  and let  $F = p^{-1}(b_0)$ . The homotopy  $\omega_p$  is obtained by concatenating the three following homotopies:

- **The homotopy  $\omega_p^{(1)}$ .** Let

$$H_p: \mathcal{S}(B) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

denote the functor which assigns to a singular simplex  $\sigma \in \mathcal{S}(B)$  the homology chain complex  $H_*(\sigma^*E)$ . It induces a map

$$h_p: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$$

The homotopy  $\omega_p^{(1)}$  is the homotopy between the maps  $c_p$  and  $h_p$  described in Remark A.1.2.

- **The homotopy  $\omega_p^{(2)}$ .** For  $\sigma \in \mathcal{S}(B)$  let  $\sigma(0) \in B$  denote the zeroth vertex of  $\sigma$  and let  $F_{\sigma(0)} = p^{-1}(\sigma(0))$ . We have a functor

$$H_p^0: \mathcal{S}(B) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

such that  $H_p^0(\sigma) = H_*(F_{\sigma(0)})$ . Let  $h_p^0: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$  be the map induced by the functor  $H_p^0$ . The isomorphisms of chain complexes

$$H_*(\sigma^*E) \xrightarrow{\cong} H_*(F_{\sigma(0)})$$

define a natural transformations of the functors  $H_p$  and  $H_p^0$ , and thus induce a homotopy  $\omega_p^{(2)}$  between the maps  $h_p$  and  $h_p^0$ .

- **The homotopy  $\omega_p^{(3)}$ .** As before let

$$*_{H_*(F)}: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$$

denote the constant map sending the space  $|\mathcal{S}(B)|$  to the point of  $K(\mathbb{R})$  represented by the chain complex  $H_*(F)$ . If  $p$  is a bundle such that the group  $\pi_1 B$  acts trivially on  $H_*(F)$  then for every  $\sigma \in \mathcal{S}(B)$  we have a canonical isomorphism

$$H_*(F_{\sigma(0)}) \xrightarrow{\cong} H_*(F)$$

These isomorphisms define a homotopy  $\omega_p^{(3)}$  between  $h_p^0$  and  $*_{H_*(F)}$ .

This construction can be generalized to the case when  $p$  is an arbitrary unipotent bundle. We have then canonical isomorphisms of quotients of the filtrations of  $H_*(F_{\sigma(0)})$  and  $H_*(F)$  given by Definition 2.1. The homotopy  $\omega_p^{(3)}$  is obtained using this fact and Waldhausen's pre-additivity theorem (see [2, Proof of Thm 6.7]). We note here that that the homotopy class of  $\omega_p^{(3)}$  does not depend on the choice of the filtration of  $H_*(F)$ .

**4.9. Smooth cohomological torsion.** Recall (Sec. 2) that in the axiomatic setting of Igusa higher torsion is defined as an invariant taking values in the cohomology groups of the base of the bundle. As a consequence in order to verify that Igusa's axioms hold for the smooth torsion one needs first to reduce  $\tau^s$  to a cohomological invariant. This is accomplished as

follows (cf. [2, Sec. 7]). Let  $p: E \rightarrow B$  be a unipotent bundle and let  $\iota_E: \tilde{Q}(E_+) \rightarrow \tilde{Q}(S^0)$  be the augmentation map. Consider the diagram

$$(4-4) \quad \begin{array}{ccccc} & & \mathrm{Wh}^{\mathbb{R}}(E) & \xrightarrow{\bar{\iota}_E} & \mathrm{Wh}^{\mathbb{R}}(*) \\ & \nearrow \tau^s(p) & \downarrow & & \downarrow \\ |\mathcal{S}(B)| & \xrightarrow{p^!} & \tilde{Q}(E_+) & \xrightarrow{\iota_E} & \tilde{Q}(S^0) \\ & & \downarrow \lambda_E & & \downarrow \lambda_* \\ & & K(\mathbb{R}) & \xlongequal{\quad} & K(\mathbb{R}) \end{array}$$

The lower square commutes up to a preferred homotopy, and so  $\iota_E$  induces a map  $\bar{\iota}_E: \mathrm{Wh}^{\mathbb{R}}(E) \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)$ . We have weak equivalences

$$(4-5) \quad \mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}} \simeq \Omega K(\mathbb{R})_{\mathbb{Q}} \simeq K_1(\mathbb{R}) \times \prod_{k>1} K(\mathbb{R}, 4k)$$

where  $\mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$ ,  $\Omega K(\mathbb{R})_{\mathbb{Q}}$  denote rationalizations of  $\mathrm{Wh}^{\mathbb{R}}(*)$  and  $\Omega K(\mathbb{R})$  respectively. The first weak equivalence is a consequence of the fact that  $\tilde{Q}(S^0)$  is rationally a discrete space and that  $\lambda_*$  induces an isomorphism on the level of  $\pi_0$ . The second weak equivalence in (4-5) is given by the Borel regulator maps [8]<sup>2</sup>.

Consider the map

$$(4-6) \quad \bar{\iota}_E \circ \tau^s(p): |\mathcal{S}(B)| \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$$

We claim that the image of this map always lies in the connected component of the identity element of  $\mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$ . This is obvious if  $B = *$ . The general case follows from this fact and naturality of the map (4-6): if  $p: E \rightarrow B$  is a unipotent bundle,  $f: B' \rightarrow B$  is a smooth map and  $p': f^*E \rightarrow B'$  is the bundle induced from  $p$  then we have a homotopy

$$\bar{\iota}_{f^*E} \circ \tau^s(p') \simeq \bar{\iota}_E \circ \tau^s(p) \circ |f|$$

where  $|f|: |\mathcal{S}(B')| \rightarrow |\mathcal{S}(B)|$  is the map induced by  $f$ . The proof of this property is the same as that of [2, Proposition 7.3].

It follows that for a unipotent bundle  $p$  the target of the map  $\bar{\iota}_E \circ \tau^s(p)$  can be identified with  $\prod_{k>1} K(\mathbb{R}, 4k)$ . This gives rise the following

**4.10. Definition.** For a unipotent bundle  $p: E \rightarrow B$  the smooth cohomological torsion of  $p$  is the cohomology class  $t^s(p) \in \bigoplus_{k>1} H^{4k}(B; \mathbb{R})$  represented by the map

$$\bar{\iota}_E \circ \tau^s(p): |\mathcal{S}(B)| \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$$

We use here the canonical identification of cohomology groups of the spaces  $B$  and  $|\mathcal{S}(B)|$ .

<sup>2</sup>We rectify here a mistake in Section 7 of [2] where it was incorrectly stated that  $\mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$  is a connected space.

We are now ready to state the main result of this paper.

**4.11. Theorem.** *For  $k > 0$  and a unipotent bundle  $p: E \rightarrow B$  let*

$$t_{4k}^s(p) \in H^{4k}(B; \mathbb{R})$$

*be the degree  $4k$  component of  $t^s(p)$ . The invariant  $t_{4k}^s$  is a non-trivial exotic higher torsion invariant of unipotent bundles in degree  $4k$ .*

Combining this with Igusa's Theorem 2.10 we obtain

**4.12. Theorem.** *For each  $k > 0$  there exists  $0 \neq \lambda_{4k} \in \mathbb{R}$  such that for every unipotent bundle  $p: E \rightarrow B$  we have*

$$t_{4k}^s(p) = \lambda_{4k} t_{4k}^{IK}(p)$$

*where  $t_{4k}^{IK}$  is the Igusa-Klein higher torsion invariant in the degree  $4k$ .*

**4.13. Remark.** It would be interesting to know the exact value of the proportionality constant  $\lambda_{4k}$ . Such computation seems within reach, and one should be able accomplish it by a careful analysis of the map  $G/O \rightarrow \Omega \text{Wh}^{\text{diff}}(*)$  constructed by Waldhausen in [22]. We do not attempt it in this paper.

The remainder of this paper is devoted to the proof of Theorem 4.11. The fact the cohomological smooth torsion defines characteristic classes of unipotent bundles was proved in [2, Theorem 7.3]. Also, directly from the constructions in [2, Section 7] it follows that these characteristic classes are exotic<sup>3</sup>. It remains to show that  $t^s$  is a higher torsion invariant i.e. that it satisfies the additivity (2.5) and transfer (2.6) axioms. We verify the additivity formula for  $t^s$  in the next section (see Corollary 5.2). The transfer axiom is the subject of Corollary 7.2. Finally, in Section 8 we show that for every  $k > 0$  there exists a unipotent bundle  $p$  such that  $t_{4k}^s(p) \neq 0$ . This shows that the characteristic classes  $t_{4k}^s$  are non-trivial invariants.

## 5. ADDITIVITY OF THE SMOOTH TORSION

As we have indicated above our goal in this section is to verify that the smooth torsion satisfies an analog of the additivity axiom of Igusa (2.5):

**5.1. Theorem.** *Let  $p: E \rightarrow B$  be a bundle with a unipotent splitting (2.4)*

$$p = p_1 \cup_{p_0} p_2$$

*where  $p_i: E_i \rightarrow B$ . For  $i = 0, 1, 2$  let  $j_{i*}: \text{Wh}^{\mathbb{R}}(E_i) \rightarrow \text{Wh}^{\mathbb{R}}(E)$  be the map induced by the inclusion  $j_i: E_i \hookrightarrow E$ . There exists a homotopy*

$$\tau^s(p) \simeq j_{1*}\tau^s(p_1) + j_{2*}\tau^s(p_2) + \bar{g}^{\text{Wh}^{\mathbb{R}}}$$

*where  $\bar{g}^{\text{Wh}^{\mathbb{R}}}$  is a map representing the homotopy type of  $-j_{0*}\tau^s(p_0)$  in the  $H$ -space structure of  $\text{Wh}^{\mathbb{R}}(E)$ .*

From Theorem 5.1 we immediately obtain

---

<sup>3</sup>This fact was also observed by Igusa [16, p. 185].

**5.2. Corollary.** *The smooth cohomological torsion  $t^s$  satisfies the additivity axiom (2.5).*

As we indicated in Section 4 the smooth torsion  $\tau^s(p)$  of a unipotent bundle  $p: E \rightarrow B$  consists of two main ingredients: the Becker-Gottlieb transfer  $p^!: |\mathcal{S}(B)| \rightarrow \tilde{Q}(E_+)$  and the algebraic contraction  $\omega_p: |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$ . Our first aim is to describe the additive property of the transfer map. Recall the commutative diagram (4-2). We have

**5.3. Theorem.** *Let  $p: E \rightarrow B$  be a smooth bundle with a unipotent splitting*

$$p = p_1 \cup_{p_0} p_2$$

where  $p_i: E_i \rightarrow B$ . Let  $j_{i*}: \tilde{Q}(E_{i+}) \rightarrow \tilde{Q}(E_+)$  and  $j_{i*}: A(E_i) \rightarrow A(E)$  be the maps induced by the inclusions  $j_i: E_i \hookrightarrow E$ .

(i) *There exists a preferred homotopy*

$$\gamma^A: |\mathcal{S}(B)| \times I \rightarrow A(E)$$

between the maps  $p^A$  and  $j_{1*}p_1^A + j_{2*}p_2^A + g^A$ , where  $g^A$  is a map representing the homotopy type of  $-j_{0*}p_0^A$ .

(ii) *There exists a preferred homotopy*

$$\gamma^Q: |\mathcal{S}(B)| \times I \rightarrow \tilde{Q}(E_+)$$

between the maps  $p^!$  and  $j_{1*}p_1^! + j_{2*}p_2^! + g^Q$  where  $g^Q$  is a map representing the homotopy type of  $-j_{0*}p_0^!$ . Moreover, we have  $\tilde{a}_E g^Q = g^A$  and  $\tilde{a}_E \circ \gamma^Q = \gamma^A$ .

**5.4. Remark.** Existence of the homotopy  $\gamma^Q$  is well known; it is in fact one of the properties characterizing the Becker-Gottlieb transfer map given in [3]. The proof of Theorem 5.1 will require, however, a combinatorial description of  $\gamma^A$  and  $\gamma^Q$  given in the proof of Theorem 5.3.

*Proof of Theorem 5.3. Part (i).* Our construction of the homotopy  $\gamma^A$  will parallel the one given in [1, Section 3]. Let  $b: E_0 \times [1, -1] \rightarrow E$  be a fiberwise bicollar neighborhood of  $E_0$  in  $E$ . Thus,  $b$  is a smooth embedding such that  $b(E_0 \times \{0\}) = E_0$ ,  $b(E_0 \times [-1, 0]) \subseteq E_1$ ,  $b(E_0 \times [0, 1]) \subseteq E_2$ , and such that we have a commutative diagram

$$\begin{array}{ccc} E_0 \times [-1, 1] & \xrightarrow{b} & E \\ & \searrow p_0 \circ pr_1 & \swarrow p \\ & & B \end{array}$$

where  $pr_1: E_0 \times [-1, 1] \rightarrow E_0$  is the projection on the first factor. Define

$$E'_1 := E_1 - b(E_0 \times (-1, 0]) \quad \text{and} \quad E'_2 := E_2 - b(E_0 \times [0, 1))$$

Restricting  $p$  to  $E'_i$  we obtain smooth bundles

$$q_i: E'_i \rightarrow B, \quad i = 1, 2$$

Let  $j'_i: E'_i \hookrightarrow E$  be the inclusion map. The bundle  $q_i$  is fiberwise diffeomorphic to  $p_i$  and we have a homotopy

$$j'_{i*}q_i^A \simeq j_{i*}p_i^A$$

In view of this it suffices to construct a homotopy  $\tilde{\gamma}^A$  between the maps  $p^A$  and  $j'_{1*}q_1^A + j'_{2*}q_2^A + g^A$  for an appropriate choice of  $g^A$ .

Recall that the map  $p^A$  is induced by the functor  $F_{p^A}: \mathcal{S}(B) \rightarrow w\mathcal{T}_1\mathcal{R}^{fd}(E)$  (4-1). For  $i = 1, 2$  let

$$\overline{F}_{q_i^A}: \mathcal{S}(B) \rightarrow w\mathcal{T}_1\mathcal{R}^{fd}(E)$$

be the functor which assigns to a simplex  $\sigma \in \mathcal{S}(B)$  the cofibration

$$\overline{F}_{q_i^A}(\sigma) := (E \twoheadrightarrow E \sqcup \sigma^*E'_i)$$

where  $\sigma^*E'_i$  is defined as in (4.1). The maps  $j'_{i*}q_i^A: |\mathcal{S}(B)| \rightarrow A(E)$  are obtained from the functors  $\overline{F}_{q_i^A}$  using (3.3). Likewise, the map  $j_{0*}p_0^A$  comes from the functor  $\overline{F}_{p_0^A}: \mathcal{S}(B) \rightarrow w\mathcal{T}_1\mathcal{R}^{fd}(E)$  given by

$$\overline{F}_{p_0^A}(\sigma) := (E \hookrightarrow E \sqcup \sigma^*E_0)$$

It follows that the map  $j'_{1*}q_1^A + j'_{2*}q_2^A$  is represented by the functor

$$\overline{F}'_{p_1^A} \sqcup \overline{F}'_{p_2^A}: \mathcal{S}(B) \rightarrow w\mathcal{T}_1\mathcal{R}^{fd}(E)$$

where

$$\overline{F}'_{p_1^A} \sqcup \overline{F}'_{p_2^A}(\sigma) = (E \twoheadrightarrow E \sqcup \sigma^*E'_1 \sqcup \sigma^*E'_2)$$

Notice that for every  $\sigma \in \mathcal{S}(B)$  we have a sequence of cofibrations

$$(5-1) \quad E \twoheadrightarrow E \sqcup (\sigma^*E'_1 \sqcup \sigma^*E'_2) \twoheadrightarrow E \sqcup \sigma^*E$$

Consider the functor

$$(5-2) \quad \Gamma^A: \mathcal{S}(B) \rightarrow w\mathcal{T}_2\mathcal{R}^{fd}(E)$$

which assigns to  $\sigma \in \mathcal{S}(B)$  the cofibration sequence (5-1). Applying Waldhausen's pre-additivity theorem (3.7) to  $\Gamma^A$  we obtain a homotopy between  $p^A$  and the map  $j'_{1*}q_1^A + j'_{2*}q_2^A + g^A$  where  $g^A$  is the map induced by the functor

$$G^A: \mathcal{S}(B) \rightarrow w\mathcal{T}_1\mathcal{R}^{fd}(E)$$

given by

$$(5-3) \quad G^A(\sigma) := (E \sqcup (\sigma^*E'_1 \sqcup \sigma^*E'_2) \twoheadrightarrow E \sqcup \sigma^*E)$$

It remains to check that  $g^A$  represents the homotopy type of  $-j_{0*}p_0^A$ . This can be seen by noticing that the cofiber of the cofibration  $G^A(\sigma)$  is isomorphic to the object of  $\mathcal{R}^{fd}(E)$  obtained by applying the suspension functor [23, 1.6] to the cofiber of  $\overline{F}_{p_0^A}(\sigma)$ .

*Part (ii).* As we have mentioned in (4.1) the space  $\tilde{Q}(E_+)$  is constructed by applying the  $\mathcal{T}_\bullet$ -construction to the category of partitions of the manifold  $E$ . The map  $g^Q$  and the homotopy  $\gamma^Q$  can be obtained by arguments paralleling

the ones used in the proof of the part (i), working in the category of partitions instead of  $\mathcal{R}^{fd}(E)$ .  $\square$

*Proof of Theorem 5.1.* We will use the notation introduced in the proof of Theorem 5.3. Using diffeomorphisms between the bundles  $p_i$  and  $q_i$ ,  $i = 1, 2$  we obtain homotopies

$$j_{i*}\tau^s(p_i) \simeq j'_{i*}\tau^s(q_i)$$

Therefore it will suffice to show that we have a homotopy

$$(5-4) \quad \tau^s(p) \simeq j'_{1*}\tau^s(q_1) + j'_{2*}\tau^s(q_2) + \bar{g}^{\text{Wh}^{\mathbb{R}}}$$

A convenient description of the maps  $j'_{i*}\tau^s(q_i)$ ,  $i = 1, 2$  can be obtained as follows. For  $i = 1, 2$  we have a commutative diagram

$$\begin{array}{ccc} \tilde{Q}(E'_{i+}) & \xrightarrow{j'_{i*}} & \tilde{Q}(E_+) \\ & \searrow \lambda_{E'_i} & \swarrow \lambda_E \\ & & K(\mathbb{R}) \end{array}$$

It follows that

$$(5-5) \quad \lambda_E j'_{i*} q_i^! = \lambda_{E'_i} q_i^! = c_{q_i}$$

Let  $F'_i$  be the fiber of the bundle  $q_i$ . Recall (4.6) that the unreduced smooth torsion of  $q_i$  is the map

$$\tilde{\tau}^s(q_i): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E'_i)_{H_*(F'_i)}$$

determined by the transfer  $q_i^!$  and the algebraic contraction  $\omega_{q_i}$ . From the equation (5-5) we obtain that the map  $j_*\tilde{\tau}^s(q_i): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)_{H_*(F'_i)}$  is determined by the pair  $(j'_{i*}q_i^!, \omega_{q_i})$ . The map  $j'_{i*}\tau^s(q_i)$  is then just the reduction of  $j'_*\tilde{\tau}^s(q_i)$  (4.7).

The map  $\bar{g}^{\text{Wh}^{\mathbb{R}}}$  is constructed as follows. Consider the functor

$$G^K: \mathcal{S}(B) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

which assigns to  $\sigma \in \mathcal{S}(B)$  the relative chain complex

$$(5-6) \quad G^K(\sigma) := C_*(\sigma^*E, \sigma^*E'_1 \sqcup \sigma^*E'_2)$$

Let  $g^K: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$  be the map induced by  $G^K$  as in (3.3). Notice that we have

$$G^K = \Lambda_{E,1}^A \circ G^A$$

where  $G^A$  is the functor defined by (5-3) and

$$\Lambda_{E,1}^A: w\mathcal{T}_1\mathcal{R}^{fd}(E) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

is induced by the linearization functor  $\Lambda_E^A$  (4-3), cf. (3.5). This shows that  $g^K = \lambda_E^A \circ g^A$ . Since  $g^A$  has the homotopy type of  $-j_{0*}p_0^A$ ,  $\lambda_E^A$  is a map of

infinite loop spaces, and  $\lambda_E^A j_0^* p_0^A = c_{p_0}$ , it follows that  $g^K$  represents the homotopy type of  $-c_{p_0}$ .

Recall that  $F'_1, F'_2$  are the fibers of  $q_1, q_2$ , respectively, and let  $F$  denote the fiber of  $p$ . One can construct a homotopy  $\omega_g$  from  $g^K$  to a constant map  $*_{H_*(F, F'_1 \sqcup F'_2)}$  following the same steps which we used to obtain the algebraic contraction  $\omega_p$  in (4.8). More precisely, let

$$G^H, G^{H_0} : \mathcal{S}(B) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

be functors given by

$$G^H(\sigma) := H_*(\sigma^* E, \sigma^* E'_1 \sqcup \sigma^* E'_2)$$

and

$$G^{H_0}(\sigma) := H_*(F_{\sigma(0)}, F'_{1,\sigma(0)} \sqcup F'_{2,\sigma(0)})$$

where  $F_{\sigma(0)}, F'_{1,\sigma(0)}, F'_{2,\sigma(0)}$  are the fibers of  $p, q_1, q_2$  respectively taken over the zeroth vertex  $\sigma(0)$  of the singular simplex  $\sigma$ . Let

$$g^H, g^{H_0} : |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$$

be the maps induced by  $G^H$  and  $G^{H_0}$ . A homotopy  $\omega_g^{(1)}$  between  $g^K$  and  $g^H$  can be obtained the same way as the homotopy  $\omega_p^{(1)}$  (cf. Remark A.1.2). The natural isomorphisms

$$H_*(\sigma^* E, \sigma^* E'_1 \sqcup \sigma^* E'_2) \xrightarrow{\cong} H_*(F_{\sigma(0)}, F'_{1,\sigma(0)} \sqcup F'_{2,\sigma(0)})$$

induce a homotopy  $\omega_g^{(2)}$  between  $g^H$  and  $g^{H_0}$ . Finally, since  $\pi_1 B$  acts unipotently on  $H_*(F, F'_1 \sqcup F'_2)$  we also have a homotopy  $\omega_g^{(3)}$  between  $g^{H_0}$  and the constant map  $*_{H_*(F, F'_1 \sqcup F'_2)}$ . Concatenation of the homotopies  $\omega_g^{(1)}, \omega_g^{(2)}, \omega_g^{(3)}$  gives the homotopy  $\omega_g$ .

Notice that

$$g^K = \lambda_E^A g^A = \lambda_E g^Q$$

It follows that the homotopy  $\omega_g$  determines a lift of  $g^Q$ :

$$g^{\text{Wh}^{\mathbb{R}}} : |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)_{H_*(F, F'_1 \sqcup F'_2)}$$

Let  $\bar{g}^{\text{Wh}^{\mathbb{R}}} : |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(E)$  be the map obtained by reducing  $g^{\text{Wh}^{\mathbb{R}}}$ . One can check that  $\bar{g}^{\text{Wh}^{\mathbb{R}}}$  represents the homotopy type of  $-j_0^* \tau^s(p_0)$ .

In order to construct the homotopy (5-4) we will follow the steps described in Remark 4.7. In the notation of (4.7) the map  $\tau^s(p)$  is the reduction of  $\tilde{\tau}^s(p)$  and this last map is determined by the pair  $(p^!, \omega_p)$ . In the same way the map  $j'_{1*} \tau^s(q_1) + j'_{2*} \tau^s(q_2) + \bar{g}^{\text{Wh}^{\mathbb{R}}}$  comes from the pair

$$(j'_{1*} q_1^! + j'_{1*} q_1^! + g^Q, \omega_{q_1} + \omega_{q_2} + \omega_g)$$

Theorem 5.3 provides the homotopy  $\gamma^Q$  between  $p^!$  and  $j'_{1*} q_1^! + j'_{1*} q_1^! + g^Q$ . It follows that to get the homotopy (5-4) it will suffice to show that

- there exists a path  $\gamma$  in  $K(\mathbb{R})$  joining the points  $H_*(F)$  and  $H_*(F'_1) + H_*(F'_2) + H_*(F, F'_1 \sqcup F'_2)$ , and

– there exists a homotopy of homotopies filling the following diagram

$$(5-7) \quad \begin{array}{ccc} c_p & \xrightarrow{\lambda_E \gamma^Q} & c_{q_1} + c_{q_2} + g^K \\ \omega_p \downarrow & & \downarrow \omega_{q_1} + \omega_{q_2} + \omega_g \\ *H_*(F) & \xrightarrow{\gamma} & *H_*(F'_1) + H_*(F'_2) + H_*(F, F'_1 \sqcup F'_2) \end{array}$$

In this diagram every vertex represents a map  $|\mathcal{S}(B)| \rightarrow K(\mathbb{R})$  and every edge represents of homotopy of such maps.

We will first give a combinatorial description of the homotopy  $\lambda_E \gamma^Q$ . Using Theorem 5.3 we get

$$\lambda_E \gamma^Q = \lambda_E^A \tilde{u}_E \gamma^Q = \lambda_E^A \gamma^A$$

Recall that the homotopy  $\gamma^A$  was obtained applying Waldhausen's pre-additivity theorem to the functor  $\Gamma^A$  (5-2). Recall also (4.3) that the map  $\lambda_E^A$  was defined using the linearization functor  $\Lambda_E^A: \mathcal{R}^{fd}(E) \rightarrow \mathcal{C}h^{fd}(\mathbb{R})$  (4-3). Let

$$\Lambda_{E,2}^A: w\mathcal{T}_2\mathcal{R}^{fd}(E) \rightarrow w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R})$$

be the functor induced by  $\Lambda_E^A$  (see 3.5). The composition

$$\Lambda_{E,2}^A \circ \Gamma^A: \mathcal{S}(B) \rightarrow w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R})$$

is the functor which assigns to a singular simplex  $\sigma$  the cofibration of chain complexes

$$\Lambda_{E,2}^A \circ \Gamma^A(\sigma) = (C_*(\sigma^* E'_1 \sqcup \sigma^* E'_2) \twoheadrightarrow C_*(\sigma^* E))$$

Notice that the cofiber of  $\Lambda_{E,2}^A \circ \Gamma^A(\sigma)$  is the chain complex  $G^K(\sigma)$  (5-6). The homotopy  $\lambda_E^A \gamma^A$  is obtained applying Waldhausen's pre-additivity theorem (3.7) to the functor  $\Lambda_{E,2}^A \circ \Gamma^A$ .

Next, recall (4.8) that the algebraic contraction  $\omega_p$  was obtained as a concatenation of homotopies  $\omega_p^{(i)}$ ,  $i = 1, 2, 3$ . Consider the following diagram:

$$(5-8) \quad \begin{array}{ccc} c_p & \xrightarrow{\lambda_E \gamma^Q} & c_{q_1} + c_{q_2} + g^K \\ \omega_p^{(1)} \downarrow & & \downarrow \omega_{q_1}^{(1)} + \omega_{q_2}^{(1)} + \omega_g^{(1)} \\ h_p & \xrightarrow{\gamma^H} & h_{q_1} + h_{q_2} + g^H \\ \omega_p^{(2)} \downarrow & & \downarrow \omega_{q_1}^{(2)} + \omega_{q_2}^{(2)} + \omega_g^{(2)} \\ h_p^0 & \xrightarrow{\gamma^{H_0}} & h_{q_1}^0 + h_{q_2}^0 + g^{H_0} \\ \omega_p^{(3)} \downarrow & & \downarrow \omega_{q_1}^{(3)} + \omega_{q_2}^{(3)} + \omega_g^{(3)} \\ *H_*(F) & \xrightarrow{\gamma} & *H_*(F'_1) + H_*(F'_2) + H_*(F, F'_1 \sqcup F'_2) \end{array}$$

In this diagram every vertex represents a map  $|\mathcal{S}(B)| \rightarrow K(\mathbb{R})$  and every edge stands for a homotopy of such maps. The homotopy  $\gamma^H$  is obtained using the homological additivity  $\mathcal{U}^H$  as described in (A.2.2). It is constructed using the homology long exact sequences associated to the short exact sequences of chain complexes

$$(5-9) \quad C_*(\sigma^*E'_1 \sqcup \sigma^*E'_2) \rightarrow C_*(\sigma^*E) \rightarrow C_*(\sigma^*E, \sigma^*E'_1 \sqcup \sigma^*E'_2)$$

The homotopy  $\gamma^{H_0}$  is obtained in the same manner, while the path  $\gamma$  is just the restriction of  $\gamma^{H_0}$  to the basepoint  $b_0 \in B$ .

In order to obtain a homotopy of homotopies filling the diagram (5-7) it is enough to show that each of the three squares in the diagram (5-8) can be filled with a homotopy of homotopies. In the case of the top square this holds by Lemma A.2.1 (see also A.2.2). Existence of a homotopy of homotopies filling the middle square is trivial. Finally, a homotopy of homotopies fitting in the bottom square exists since the long exact sequence of homology groups induced by the short exact sequence

$$C_*(F'_1 \sqcup F'_2) \rightarrow C_*(F) \rightarrow C_*(F, F'_1 \sqcup F'_2)$$

is a sequence of  $\pi_1 B$ -modules. □

## 6. THE SECONDARY TRANSFER

Our next objective is to develop a formula which, given two suitably good smooth bundles  $p: E \rightarrow B$  and  $q: D \rightarrow E$ , relates the smooth torsion of the bundle  $pq$  to the torsions  $\tau^s(p)$  and  $\tau^s(q)$ . We will do it in this section. In Section 7 we will then verify that on the level of the cohomological torsion our formula yields the transfer axiom (2.6).

While Igusa's transfer axiom assumes that the bundle  $q$  is a linear oriented sphere bundle in our context it will be convenient to work in a more general setting:

**6.1. Definition.** Let  $q: D \rightarrow E$  be a smooth bundle with fiber  $F$ . For  $e \in E$  let  $F_e = p^{-1}(e)$  and let  $i_e: F_e \rightarrow D$  be the inclusion map. We say that  $q$  is a Leray-Hirsch bundle if there exists a homomorphism  $\theta: H^*(F) \rightarrow H^*(D)$  such that for every  $e \in E$  the composition

$$H^*(F) \xrightarrow{\theta} H^*(D) \xrightarrow{i_e^*} H^*(F_e)$$

is an isomorphism.

In other words Leray-Hirsch bundles are smooth bundles which satisfy the Leray-Hirsch Theorem. For our purposes it will be convenient to state this theorem as follows.

**6.2. Theorem** (cf. [19, Sec. 5.7, Theorem 9]). *Let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F$ . There exists an quasi-isomorphism of singular chain complexes*

$$\alpha(q): C_*(D) \rightarrow C_*(E) \otimes H_*(F)$$

Moreover, this quasi-somorphism is natural in the following sense: for any map  $f: E' \rightarrow E$  the pullback diagram

$$\begin{array}{ccc} f^*D & \xrightarrow{\bar{f}} & D \\ q' \downarrow & & \downarrow q \\ E' & \xrightarrow{f} & E \end{array}$$

induces a commutative square

$$\begin{array}{ccc} C_*(f^*D) & \xrightarrow{\bar{f}_*} & C_*(D) \\ \alpha(q') \downarrow & & \downarrow \alpha(q) \\ C_*(E') \otimes H_*(F) & \xrightarrow{f_* \otimes \text{id}} & C_*(E) \otimes H_*(F) \end{array}$$

**6.3. Remark.** Let  $\xi$  be an oriented odd dimensional vector bundle over  $E$  and let  $q: D \rightarrow E$  be the (even dimensional) sphere bundle associated to  $\xi$ . Then  $q$  is a Leray-Hirsch bundle.

Let  $p: E \rightarrow B$  be a unipotent bundle and let  $q: D \rightarrow E$  be a Leray-Hirsch bundle. As we have mentioned above we will want to describe the relationship between the smooth torsions of  $p$ ,  $q$ , and  $pq$ . First, we need to verify that  $\tau^s(pq)$  is defined in this case, i.e. that  $pq$  is a unipotent bundle. This is an immediate consequence of the following

**6.4. Lemma.** Let  $p: E \rightarrow B$  and  $q: D \rightarrow E$  be smooth bundles with fibers  $F_p$  and  $F_q$  respectively, and let  $F_{pq}$  be the fiber of the bundle  $pq: D \rightarrow B$ . Define a  $\pi_1 B$ -module structure on  $H_*(F_p) \otimes H_*(F_q)$  by

$$\gamma \cdot (x \otimes y) := (\gamma \cdot x) \otimes y$$

for  $\gamma \in \pi_1 B$ ,  $x \otimes y \in H_*(F_p) \otimes H_*(F_q)$ . If  $q$  is a Leray-Hirsch bundle then the Leray-Hirsch isomorphism

$$\alpha(q|_{F_{pq}}): H_*(F_{pq}) \rightarrow H_*(F_p) \otimes H_*(F_q)$$

is a  $\pi_1 B$ -equivariant map.

Since the identity map  $\text{id}_E: E \rightarrow E$  defines a unipotent bundle, Lemma 6.4 implies in particular that any Leray-Hirsch bundle is unipotent.

Next, let  $F$  be the fiber of a smooth bundle  $q$ . Consider the functor

$$\otimes H_*(F): \mathcal{C}h^{fd}(\mathbb{R}) \rightarrow \mathcal{C}h^{fd}(\mathbb{R})$$

which assigns to a chain complex  $C$  the complex  $C \otimes H_*(F)$ . This is an exact functor of Waldhausen categories, so it induces a map

$$\otimes H_*(F): K(\mathbb{R}) \rightarrow K(\mathbb{R})$$

We have the following

**6.5. Proposition.** *Let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F$ . The diagram*

$$\begin{array}{ccc} \tilde{Q}(E_+) & \xrightarrow{\tilde{Q}(q^!)} & \tilde{Q}(D_+) \\ \lambda_E \downarrow & & \downarrow \lambda_D \\ K(\mathbb{R}) & \xrightarrow{\otimes_{H_*(F)}} & K(\mathbb{R}) \end{array}$$

*commutes up to a preferred homotopy*

$$\mu_q: \tilde{Q}(E_+) \times I \rightarrow K(\mathbb{R})$$

*Proof.* Consider the diagram

$$(6-1) \quad \begin{array}{ccc} \tilde{Q}(E_+) & \xrightarrow{\tilde{Q}(q^!)} & \tilde{Q}(D_+) \\ \tilde{a}_E \downarrow & & \downarrow \tilde{a}_D \\ A(E) & \xrightarrow{A(q^!)} & A(D) \\ \lambda_E^A \downarrow & & \downarrow \lambda_D^A \\ K(\mathbb{R}) & \xrightarrow{\otimes_{H_*(F)}} & K(\mathbb{R}) \end{array}$$

where  $\tilde{a}_E, \tilde{a}_D$  are the assembly maps and  $\lambda_E^A, \lambda_D^A$  are the  $A$ -theory linearization maps (see 4.3). Recall that we have  $\lambda_E = \lambda_E^A \circ \tilde{a}_E$  and  $\lambda_D = \lambda_D^A \circ \tilde{a}_D$ . The map  $A(q^!)$  is the  $A$ -theory transfer of  $q$ . It is induced by the exact functor  $\mathcal{R}^{fd}(E) \rightarrow \mathcal{R}^{fd}(D)$  which associates to a retractive space  $X \in \mathcal{R}^{fd}(E)$  the space

$$q^* X = \lim(X \rightarrow E \xleftarrow{q} D)$$

Directly from the constructions of [2, Sec. 3] it follows that the upper square in the diagram (6-1) commutes. Consequently, it will suffice to construct a homotopy

$$\mu_q^A: A(E) \times I \rightarrow K(\mathbb{R})$$

which makes the lower square commute. We will then define

$$\mu_q := \mu_q^A \circ (\tilde{a}_E \times \text{id})$$

Notice that the map  $\lambda_D^A \circ A(q^!)$  is induced by the functor

$$F: \mathcal{R}^{fd}(E) \rightarrow \mathcal{C}h^{fd}(\mathbb{R})$$

which assigns to a retractive space  $X \in \mathcal{R}^{fd}(E)$  the singular chain complex  $C_*(q^* X)$ . The map  $\otimes_{H_*(F)} \circ \lambda_E^A$ , on the other hand, comes from the functor

$$G: \mathcal{R}^{fd}(E) \rightarrow \mathcal{C}h^{fd}(\mathbb{R})$$

such that  $G(X) = C(X) \otimes H_*(F)$ .

Notice also that for any  $X \in \mathcal{R}^{fd}(E)$  the map  $q^*X \rightarrow X$  is a Leray-Hirsch bundle induced from  $q: D \rightarrow E$ . By Theorem 6.2 we have a quasi-isomorphism

$$\alpha(q^*X \rightarrow X): C_*(q^*X) \rightarrow C_*(X) \otimes H_*(F)$$

By naturality of the quasi-isomorphisms  $\alpha(-)$  this defines a natural transformation

$$\alpha: F \Rightarrow G$$

This natural transformation induces the desired homotopy  $\mu_q^A$ .  $\square$

**6.6. Definition.** Let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F$ . The homotopy  $\mu_q$  defines a map  $\text{Wh}^{\mathbb{R}}(q'): \text{Wh}^{\mathbb{R}}(E) \rightarrow \text{Wh}^{\mathbb{R}}(D)$  such that we get a homotopy commutative diagram

$$\begin{array}{ccc} \text{Wh}^{\mathbb{R}}(E) & \xrightarrow{\text{Wh}^{\mathbb{R}}(q')} & \text{Wh}^{\mathbb{R}}(D) \\ \downarrow & & \downarrow \\ \tilde{Q}(E_+) & \xrightarrow{\tilde{Q}(q')} & \tilde{Q}(D_+) \\ \lambda_E \downarrow & & \downarrow \lambda_D \\ K(\mathbb{R}) & \xrightarrow{\otimes_{H_*(F)}} & K(\mathbb{R}) \end{array}$$

We will call  $\text{Wh}^{\mathbb{R}}(q')$  the secondary transfer of  $q$ .

**6.7. Remark.** The following observation will be useful later on. Let  $F$  be the fiber of a Leray-Hirsch bundle  $q: D \rightarrow E$  and let  $\chi(F) \in \mathbb{Z}$  be the Euler characteristic of  $F$ . Notice that in the infinite loop space structure on  $K(\mathbb{R})$  the map  $\otimes_{H_*(F)}: K(\mathbb{R}) \rightarrow K(\mathbb{R})$  represents multiplication by  $\chi(F)$ . As a consequence we have a homotopy commutative diagram

$$\begin{array}{ccc} \Omega K(\mathbb{R}) & \xrightarrow{\cdot \chi(F)} & \Omega K(\mathbb{R}) \\ \downarrow & & \downarrow \\ \text{Wh}^{\mathbb{R}}(E) & \xrightarrow{\text{Wh}^{\mathbb{R}}(q')} & \text{Wh}^{\mathbb{R}}(D) \end{array}$$

For our purposes the key property of the secondary transfer is given by the following

**6.8. Theorem.** Let  $q: D \rightarrow E$  be a Leray-Hirsch bundle and let  $p: E \rightarrow B$  be a unipotent bundle. The following diagram commutes up to homotopy

$$\begin{array}{ccc} & & \text{Wh}^{\mathbb{R}}(E) \\ & \nearrow \tau^s(p) & \downarrow \text{Wh}^{\mathbb{R}}(q') \\ |\mathcal{S}(B)| & \xrightarrow{\tau^s(pq)} & \text{Wh}^{\mathbb{R}}(D) \end{array}$$

*Proof.* Let  $F_p$ ,  $F_q$  and  $F_{pq}$  be the fibers of the bundles  $p$ ,  $q$ , and  $pq$  respectively. Consider the following homotopies:

- $\mu_q \circ (p^! \times \text{id}_I): |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$  is a homotopy between the maps  $\lambda_D \circ \tilde{Q}(q^!) \circ p^!$  and  $\otimes H_*(F_q) \circ \lambda_E \circ p^!$ . By the construction of  $p^!$  and  $\tilde{Q}(q^!)$  in [2] we have  $\tilde{Q}(q^!) \circ p^! = (pq)^!$  thus

$$\lambda_D \circ \tilde{Q}(q^!) \circ p^! = c_{pq}$$

Also, since  $\lambda_E \circ p^! = c_p$  we obtain

$$\begin{cases} \mu_q \circ (p^! \times \text{id}_I)|_{|\mathcal{S}(B)| \times \{0\}} &= c_{pq} \\ \mu_q \circ (p^! \times \text{id}_I)|_{|\mathcal{S}(B)| \times \{1\}} &= c_p \otimes H_*(F_q) \end{cases}$$

- The homotopy  $\omega_p \otimes H_*(F_q): |\mathcal{S}(B)| \times I \rightarrow K(\mathbb{R})$  is obtained using the algebraic contraction of the bundle  $p$  (4.8). We have

$$\begin{cases} \omega_p \otimes H_*(F_q)|_{|\mathcal{S}(B)| \times \{0\}} &= c_p \otimes H_*(F_p) \\ \omega_p \otimes H_*(F_q)|_{|\mathcal{S}(B)| \times \{1\}} &= *_{H(F_p) \otimes H(F_q)} \end{cases}$$

where  $*_{H(F_p) \otimes H(F_q)}$  is the constant map sending  $|\mathcal{S}(B)|$  to the point  $H(F_p) \otimes H(F_q) \in K(\mathbb{R})$ .

Let  $\psi_{pq}$  denote the homotopy obtained by concatenating these two homotopies. The pair  $((pq)^!, \psi_{pq})$  determines a map

$$\kappa(pq): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(D)_{H_*(F_p) \otimes H_*(F_q)}$$

where as in (4.5) we set

$$\text{Wh}^{\mathbb{R}}(D)_{H_*(F_p) \otimes H_*(F_q)} := \text{hofib}(\lambda_D: \tilde{Q}(D_+) \rightarrow K(\mathbb{R}))_{H_*(F_p) \otimes H_*(F_q)}$$

The map  $\text{Wh}^{\mathbb{R}}(q^!) \circ \tau^s(p)$  is homotopic to the map obtained by reducing  $\kappa(pq)$  i.e. by shifting it to a map  $|\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(D)$  (cf. 4.7). The map  $\tau^s(pq)$  is, in turn, the reduction of the unreduced torsion

$$\tilde{\tau}^s(pq): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(D)_{H_*(F_{pq})}$$

which is determined by the pair  $((pq)^!, \omega_{pq})$ . In order to show that  $\tau^s(pq) \simeq \text{Wh}^{\mathbb{R}}(q^!) \circ \tau^s(p)$  we can now proceed following the steps described in Remark 4.7. We need:

- a homotopy  $(pq)^! \simeq (pq)^!$  - our choice is the trivial one;
- a path in  $K(\mathbb{R})$  joining the points  $H_*(F_{pq})$  and  $H_*(F_p) \otimes H_*(F_q)$ . Such path is defined by the Leray-Hirsch isomorphism

$$\alpha(q|_{F_{pq}}): H_*(F_{pq}) \rightarrow H_*(F_p) \otimes H_*(F_q)$$

- a homotopy of homotopies filling the diagram

$$(6-2) \quad \begin{array}{ccc} c_{pq} & \xlongequal{\quad\quad\quad} & c_{pq} \\ \omega_{pq} \downarrow & & \downarrow \psi_{pq} \\ *_{H_*(F_{pq})} & \xrightarrow{\alpha(q|_{F_{pq}})} & *_{H_*(F_p) \otimes H_*(F_q)} \end{array}$$

To obtain such homotopy of homotopies recall (4.8) that for a unipotent bundle  $p$  the algebraic contraction  $\omega_{pq}$  is the concatenation of homotopies  $\omega_p^{(i)}$ ,  $i = 1, 2, 3$ . Consider the following diagram

$$(6-3) \quad \begin{array}{ccc} c_{pq} & \xlongequal{\quad\quad\quad} & c_{pq} \\ \omega_{pq}^{(1)} \downarrow & & \downarrow \mu_q \circ (p^! \times \text{id}_I) \\ h_{pq} & \xrightarrow{\alpha(q)} & h_p \otimes H_*(F_q) \\ \omega_{pq}^{(2)} \downarrow & & \downarrow \omega_p^{(1)} \otimes H_*(F_q) \\ h_{pq}^0 & \xrightarrow{\alpha^0(q)} & h_p^0 \otimes H(F_q) \\ \omega_{pq}^{(3)} \downarrow & & \downarrow \omega_p^{(2)} \otimes H_*(F_q) \\ H(F_{pq}) & \xrightarrow{\alpha(q|_{F_{pq}})} & H(F_p) \otimes H(F_q) \end{array}$$

In this diagram every vertex represents a map  $|\mathcal{S}(B)| \rightarrow K(\mathbb{R})$ , and every edge represents a homotopy of such maps. Concatenation of the vertical homotopies on the left gives  $\omega_{pq}$ , while concatenating the vertical homotopies on the right we obtain  $\psi_{pq}$ . Recall that the map  $h_{pq}$  is induced by the functor  $H_{pq}: \mathcal{S}(B) \rightarrow w\mathcal{S}_1Ch^{fd}(\mathbb{R})$  given by  $H_{pq}(\sigma) = H_*(\sigma^*D)$ . Since  $q$  is a Leray-Hirsch bundle for every  $\sigma \in \mathcal{S}(B)$  we have the Leray-Hirsch isomorphism  $H_*(\sigma^*D) \xrightarrow{\cong} H_*(\sigma^*E) \otimes H_*(F_q)$ . These isomorphisms define a natural transformation of functors

$$H_{pq} \Rightarrow H_p \otimes H_*(F_q)$$

This natural transformation induces the homotopy  $\alpha(q)$  between the maps  $h_{pq}$  and  $h_p \otimes H(F_q)$ . The homotopy  $\alpha^0(q)$  between  $h_{pq}^0$  and  $h_p^0 \otimes H(F_q)$  is obtained in the same way.

In order to obtain a homotopy of homotopies filling the diagram (6-2) it suffices to show that each of the squares of the diagram (6-3) can be filled by a homotopy of homotopies. Such homotopy of homotopies filling the top square is described in (A.3.2). By the naturality of the Leray-Hirsch isomorphisms the concatenations of  $\alpha(q)$  with  $\omega_p^{(1)} \otimes H_*(F_q)$  and  $\omega_{pq}$  with  $\alpha^0(q)$  actually coincide, so the middle square is trivially filled by a homotopy of homotopies. Finally, existence of homotopy of homotopies filling the bottom square follows directly from the naturality and  $\pi_1 B$ -equivariance of the Leray-Hirsch isomorphism (6.4).  $\square$

## 7. THE TRANSFER AXIOM

The results of the last section can be used to show that the formula of Igusa's transfer axiom (2.6) is satisfied by the smooth cohomological torsion whenever  $p$  is a unipotent bundle and  $q$  is any Leray-Hirsch bundle.

**7.1. Theorem.** *Let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F$  and let  $p: E \rightarrow B$  be a unipotent bundle. We have*

$$t^s(pq) = \chi(F)t^s(p) + tr_B^E(t^s(q))$$

*Proof.* Let  $\text{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$  be the rationalization of the space  $\text{Wh}^{\mathbb{R}}(*)$ , and let

$$\bar{\iota}_D: \text{Wh}^{\mathbb{R}}(D) \rightarrow \text{Wh}^{\mathbb{R}}(S^0)_{\mathbb{Q}}$$

be the map as in (4.9). Recall (4.10) that the cohomology class  $t^s(pq)$  is represented by the map

$$\bar{\iota}_D \tau^s(pq): |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(S^0)_{\mathbb{Q}}$$

Since  $\text{Wh}^{\mathbb{R}}(D)$  is an infinite loop space the map  $\tau^s(q): |\mathcal{S}(E)| \rightarrow \text{Wh}^{\mathbb{R}}(D)$  admits the extension  $\tilde{Q}(E_+) \rightarrow \text{Wh}^{\mathbb{R}}(D)$  which, by abuse of notation, we will also denote by  $\tau^s(q)$ . The cohomology class  $tr_B^E(t^s(q))$  is represented by the map

$$\bar{\iota}_D \tau^s(q)p^!: |\mathcal{S}(B)| \rightarrow \text{Wh}^{\mathbb{R}}(S^0)_{\mathbb{Q}}$$

By Theorem 6.8 we have

$$\bar{\iota}_D \tau^s(pq) \simeq \bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \tau^s(p) \quad \text{and} \quad \bar{\iota}_D \tau^s(q)p^! \simeq \bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \tau^s(\text{id}_E)p^!$$

where  $\tau^s(\text{id}_E)$  is the smooth torsion of the identity bundle  $\text{id}_E: E \rightarrow E$ . It follows that the cohomology class

$$t^s(pq) - tr_B^E(t^s(q))$$

is represented by the map  $\bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \varrho$  where

$$(7-1) \quad \varrho := \tau^s(p) - \tau^s(\text{id}_E)p^!$$

It suffices to show that the map  $\bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \varrho$  represents also the cohomology class  $\chi(F_q)t^s(p)$ .

Consider the diagram

$$\begin{array}{ccccccc} & & \Omega K(\mathbb{R}) & \xrightarrow{\cdot\chi(F)} & \Omega K(\mathbb{R}) & & \\ & \nearrow \tilde{\varrho} & \downarrow j_E & & \downarrow j_D & & \\ |\mathcal{S}(B)| & \xrightarrow{\varrho} & \text{Wh}^{\mathbb{R}}(E) & \xrightarrow{\text{Wh}^{\mathbb{R}}(q^!)} & \text{Wh}^{\mathbb{R}}(D) & \xrightarrow{\bar{\iota}_D} & \text{Wh}^{\mathbb{R}}(S^0)_{\mathbb{Q}} \\ & \searrow \bar{p}^! - \bar{p}^! & \downarrow & & \downarrow & & \\ & & \tilde{Q}(E_+) & \xrightarrow{\tilde{Q}(q^!)} & \tilde{Q}(D_+) & & \end{array}$$

The two pairs of vertical maps are fibration sequences. The lower square in the diagram commutes up to homotopy by the definition of  $\text{Wh}^{\mathbb{R}}(q^!)$  (6.6),

and the upper square is homotopy commutative by Remark 6.7. Notice that both  $\tau^s(p)$  and  $\tau^s(\text{id}_E)p^!$  are lifts of the reduced Becker-Gottlieb transfer map  $\bar{p}^!: |\mathcal{S}(B)| \rightarrow Q(E_+)$ . As a consequence  $\varrho$  is a lift of the contractible map  $\bar{p}^! - \bar{p}^!$ , and so it admits a lift  $\tilde{\varrho}: |\mathcal{S}(B)| \rightarrow \Omega K(\mathbb{R})$ . Homotopy commutativity of the upper square gives

$$\bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \varrho \simeq (\bar{\iota}_D j_D \tilde{\varrho}) \cdot \chi(F)$$

Next, homotopy commutativity of the diagram (4-4) implies that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} \Omega K(\mathbb{R}) & \xlongequal{\quad} & \Omega K(\mathbb{R}) & \xlongequal{\quad} & \Omega K(\mathbb{R}) \\ j_E \downarrow & & \downarrow j_* & & \downarrow j_D \\ \text{Wh}^{\mathbb{R}}(E) & \xrightarrow{\quad \bar{\iota}_E \quad} & \text{Wh}^{\mathbb{R}}(*) & \xleftarrow{\quad \bar{\iota}_D \quad} & \text{Wh}^{\mathbb{R}}(D) \end{array}$$

In particular we have  $\bar{\iota}_E j_E \simeq \bar{\iota}_D j_D$ . This gives

$$(7-2) \quad \bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \varrho \simeq (\bar{\iota}_E j_E \tilde{\varrho}) \cdot \chi(F_q) \simeq (\bar{\iota}_E \varrho) \cdot \chi(F)$$

By the definition (7-1) of the map  $\varrho$  we have

$$\bar{\iota}_E \varrho \simeq \bar{\iota}_E \tau^s(p) - \bar{\iota}_E \tau^s(\text{id}_E)p^!$$

The maps  $\bar{\iota}_E \tau^s(p)$  and  $\bar{\iota}_E \tau^s(\text{id}_E)p^!$  represent, respectively, the cohomology classes  $t^s(p)$  and  $\text{tr}_B^E(t^s(\text{id}_E))$ . Since  $\text{id}_E$  is a trivial bundle, we have  $t^s(\text{id}_E) = 0$ . It follows that  $\bar{\iota}_E \varrho$  represents the class  $t^s(p)$ . Combining this with (7-2) we obtain that the cohomology class  $\chi(F) \cdot t^s(p)$  is represented by the map  $\bar{\iota}_D \text{Wh}^{\mathbb{R}}(q^!) \varrho$ .  $\square$

Theorem 7.1 does not by itself show that Igusa's transfer axiom is satisfied by the smooth cohomological torsion  $t^s$ . Indeed, if  $q$  is an odd dimensional oriented linear sphere bundle then  $q$  need not be a Leray-Hirsch bundle. The statement of Theorem 7.1, however, can be extended to the case where  $q$  any linear oriented sphere bundle.

**7.2. Corollary.** *Let  $p: E \rightarrow B$  be a unipotent bundle and let  $q: D \rightarrow E$  be an oriented linear sphere bundle with fiber  $S^n$ . We have*

$$t^s(pq) = \chi(S^n)t^s(p) + \text{tr}_B^E(t^s(q))$$

*Proof.* If  $n$  is even the statement follows from Remark 6.3 and Theorem 7.1. Assume then that  $n$  is odd,  $n = 2k - 1$ , and let  $\xi^{2k}$  be the oriented vector bundle such that  $q: D \rightarrow E$  is the sphere bundle associated to  $\xi$ . Let  $\varepsilon^1$  be the 1-dimensional trivial vector bundle over  $E$  and let

$$q': S(\xi \oplus \varepsilon^1) \rightarrow E$$

be the  $2k$ -dimensional sphere bundle associated to  $\xi \oplus \varepsilon^1$ . The bundle  $q'$  admits a splitting

$$q' \cong D(q) \cup_q D(q)$$

where  $D(q)$  is the disc bundle associated to  $\xi$ . Similarly, if  $p: E \rightarrow B$  is a unipotent bundle then we have a splitting

$$pq' \cong (pD(q)) \cup_{pq} (pD(q))$$

Additivity of the smooth cohomological torsion (Corollary 5.1) implies that

$$t^s(pq') = 2t^s(pD(q)) - t^s(pq) \quad \text{and} \quad t^s(q') = 2t^s(D(q)) - t^s(q)$$

Also, since  $t^s$  is an exotic invariant (2.9) we obtain  $t^s(pD(q)) = t^s(p)$  and  $t^s(D(q)) = 0$ . This gives

$$(7-3) \quad t^s(pq') = 2t^s(p) - t^s(pq) \quad \text{and} \quad t^s(q') = -t^s(q)$$

By (6.3) the bundle  $q'$  is a Leray-Hirsch bundle, so applying Theorem 7.1 we get

$$t^s(pq') = \chi(S^{2k}) \cdot t^s(p) + tr_B^E(t^s(q')) = 2t^s(p) + tr_B^E(t^s(q'))$$

Combining this with the equations (7-3) we obtain

$$t^s(pq) = -tr_B^E(t^s(q')) = tr_B^E(t^s(q))$$

Since  $\chi(S^{2k-1}) = 0$  this gives

$$t^s(pq) = \chi(S^{2k-1}) \cdot t^s(p) + tr_B^E(t^s(q))$$

□

## 8. NONTRIVIALITY OF THE SMOOTH TORSION

Recall (4.11) that by  $t_{4k}^s$  we denoted the degree  $4k$  component of the smooth cohomological torsion  $t^s$ . Our results so far show that  $t_{4k}^s$  is an exotic higher torsion invariant of unipotent bundles. Our final goal is to demonstrate that  $t_{4k}^s$  is a non-trivial invariant.

**8.1. Theorem.** *For each  $k > 0$  there exists a smooth bundle  $p: E \rightarrow S^{4k}$  such that  $t_{4k}^s(p) \neq 0$ .*

Let  $BSO$  be the classifying space of oriented virtual vector bundles of dimension 0, and let  $BSG$  the classifying space of 0 dimensional virtual oriented spherical fibrations. We have the  $J$ -homomorphism map

$$J: BSO \rightarrow BSG$$

The proof of Theorem 8.1 will rely on the following

**8.2. Lemma.** *There exists a homotopy commutative diagram*

$$\begin{array}{ccc} BSO & \xrightarrow{\eta} & \tilde{Q}(S^0) \\ J \downarrow & & \downarrow \tilde{a}_* \\ BSG & \xrightarrow{\eta'} & A(*) \end{array}$$

Moreover, if  $p: E \rightarrow B$  is an oriented linear sphere bundle classified by a map  $f: B \rightarrow BSO$  then the diagram

$$\begin{array}{ccc} B & \xrightarrow{p^!} & \tilde{Q}(E_+) \\ f \downarrow & & \downarrow \\ BSO & \xrightarrow{\eta} & \tilde{Q}(S^0) \end{array}$$

commutes up to homotopy.

*Proof.* The first part of the lemma follows from [22], Propositions 3.1 and 3.2. The second part is a consequence of [7], Lemma 2.6.  $\square$

*Proof of Theorem 8.1.* Fix  $k > 0$ . By the Bott periodicity we have  $\pi_{4k}BSO \cong \mathbb{Z}$ . Since the the homotopy group  $\pi_{4k}BSG$  is finite the kernel of the homomorphism

$$J_*: \pi_{4k}(BSO) \rightarrow \pi_{4k}(BSG)$$

contains torsion free elements. Let  $f: S^{4k} \rightarrow BSO$  be a map representing such element in  $\ker J_*$  and let  $p: E \rightarrow S^{4k}$  be a linear oriented sphere bundle classified by  $f$ . We will show that  $t_{4k}^s(p) \neq 0$ .

Consider the homotopy commutative diagram

$$\begin{array}{ccccccc} & & G/O & \longrightarrow & \Omega\mathrm{Wh}^{\mathrm{diff}}(*) & \longrightarrow & \mathrm{Wh}^{\mathbb{R}}(*) \\ & & \downarrow & & \downarrow & & \downarrow \\ S^{4k} & \xrightarrow{\tilde{f}} & BSO & \xrightarrow{\eta} & \tilde{Q}(S^0) & \xlongequal{\quad} & \tilde{Q}(S^0) \\ & \nearrow f & \downarrow J & & \downarrow \tilde{a}_* & & \downarrow \lambda_* \\ & & BSG & \xrightarrow{\eta'} & A(*) & \xrightarrow{\lambda_*^A} & K(\mathbb{R}) \end{array}$$

where each pair of vertical maps is a fibration sequence. By the choice of  $f$  the composition  $J \circ f$  is homotopy trivial, so it admits a lift to  $\tilde{f}: S^{4k} \rightarrow G/O$ . Using Lemma 8.2 one can check that the composition

$$(8-1) \quad |\mathcal{S}(S^{4k})| \xrightarrow{\cong} S^{4n} \xrightarrow{\tilde{f}} G/O \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)$$

is homotopic to the map

$$|\mathcal{S}(S^{4n})| \xrightarrow{\tau^s(p)} \mathrm{Wh}^{\mathbb{R}}(E) \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)$$

As a consequence in order to show that  $t_{4k}^s(p) \neq 0$  we only need to verify that the map (8-1) is rationally non-trivial. By the choice of  $f$  the map  $\tilde{f}$  represents a torsion free element in  $\pi_{4k}(G/O)$ , thus if  $(G/O)_{\mathbb{Q}}$  is the rationalization of  $G/O$  then the map

$$\tilde{f}: S^{4k} \rightarrow (G/O)_{\mathbb{Q}}$$

is not contractible. It is then enough to notice that the map

$$(G/O)_{\mathbb{Q}} \rightarrow \mathrm{Wh}^{\mathbb{R}}(*)_{\mathbb{Q}}$$

yields a monomorphism on the level of homotopy groups. This follows essentially from [6, Theorem 1] and [21, Proposition 2.2].  $\square$

#### APPENDIX : HOMOLOGY OF CHAIN COMPLEXES

In this appendix we gathered some facts related to the passage from chain complexes to their homology on the level of  $K$ -theory which we use throughout the paper.

**A.1. The homotopy  $\Theta$ .** Let  $H: w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R}) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$  be the functor which assigns to each chain complex  $C$  its homology complex  $H_*(C)$ , and let

$$|H|: |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})| \rightarrow |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})|$$

be the map induced by  $H$  on the nerve of  $w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$ . Also, let

$$k: |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})| \rightarrow K(\mathbb{R})$$

be the map as in (3.3). We have

**A.1.1. Lemma** ([2, Lemma 6.8]). *The maps  $k$  and  $k \circ |H|$  are homotopic via a preferred homotopy*

$$\Theta: |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})| \times I \rightarrow K(\mathbb{R})$$

*Proof.* For a chain complex

$$C = (\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0)$$

let  $P_q C$  denote the complex such that  $(P_q C)_i = 0$  for  $i > q + 1$ ,  $(P_q C)_{q+1} = \partial(C_{q+1})$ , and  $(P_q C)_i = C_i$  for  $i \leq q$ . Let  $Q_q C$  be the kernel of the map  $P_q C \rightarrow P_{q-1} C$ . We obtain cofibration sequences of chain complexes

$$Q_q C \rightarrow P_q C \rightarrow P_{q-1} C$$

functorial in  $C$ . Notice that the complex  $Q_q C$  is canonically quasi-isomorphic to its homology complex  $H_*(Q_q C)$ , and that this last complex has only one non-zero module  $H_q(C)$  in degree  $q$ . Let  $P_q: w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R}) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$  be the functor which assigns to  $C \in w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$  the chain complex  $P_q C$ . By Waldhausen's pre-additivity theorem (3.7) the map  $k \circ |P_q|: |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})| \rightarrow K(\mathbb{R})$  is homotopic to the map induced by the assignment

$$C \mapsto P_{q-1} C \oplus H_*(Q_q C)$$

Iterating this argument for each  $q$  we obtain a homotopy

$$\Theta_q: |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})| \times I \rightarrow K(\mathbb{R})$$

between the map  $k \circ |P_q|$  and  $k \circ |H| \circ |P_q|$ . Finally, notice that since chain complexes in  $\mathcal{C}h^{fd}(\mathbb{R})$  are homotopy finitely dominated, for any  $C \in \mathcal{C}h^{fd}(\mathbb{R})$

there is  $q \geq 0$  such that  $P_q C \simeq C$ . This means that on the connected component of  $C$  in  $|w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})|$  we can set  $\Theta := \Theta_q$ .  $\square$

**A.1.2. Remark.** Let  $p: E \rightarrow B$  be a smooth bundle, and let

$$c_p, h_p: |\mathcal{S}(B)| \rightarrow K(\mathbb{R})$$

be the maps defined, respectively, in (4.3) and (4.8). Let  $C_p: \mathcal{S}(B) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$  be the chain complex functor (4.3) and let

$$|C_p|: |\mathcal{S}(B)| \rightarrow |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})|$$

be the map of nerves of categories induced by  $C_p$ . Notice that  $c_p = k \circ |C_p|$  and  $h_p = k \circ |H| \circ |C_p|$ . As a consequence applying Lemma A.1.1 we obtain a homotopy between the maps  $c_p$  and  $h_p$ . This is the homotopy  $\omega_p^{(1)}$  used in (4.8) to construct the algebraic contraction for  $p$ .

**A.2. Homological additivity.** The category  $w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R})$  (3.1) can be identified with the category of sort exact sequences of chain complexes with quasi-isomorphisms of short exact sequences as morphisms. For a short exact sequence

$$S := (0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0)$$

we have then  $Ev_1(S) = A$ ,  $Ev_2(S) = B$ , and  $Ev_{12}(S) = C$  where

$$Ev_i: w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R}) \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

are the functors defined in (3.6). Let

$$\Theta^i: |w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R})| \times I \rightarrow K(\mathbb{R})$$

be the homotopy between  $k \circ |Ev_i|$  and  $k \circ |H| \circ |Ev_i|$  defined by  $\Theta$ . Consider the following diagram

$$(A-2) \quad \begin{array}{ccc} k \circ |Ev_2| & \xrightarrow{\mathcal{U}} & k \circ |Ev_1| + k \circ |Ev_{12}| \\ \Theta^2 \downarrow & & \downarrow \Theta^1 + \Theta^{12} \\ k \circ |H| \circ |Ev_2| & & k \circ |H| \circ |Ev_1| + k \circ |H| \circ |Ev_{12}| \end{array}$$

Here every vertex represents a map  $|w\mathcal{S}_2\mathcal{C}h^{fd}(\mathbb{R})| \rightarrow K(\mathbb{R})$  and each arrow is a homotopy of such maps. The homotopy  $\mathcal{U}$  is given by Waldhausen's pre-additivity theorem (3.7). Concatenation of homotopies appearing in the diagram (A-2) defines a homotopy

$$k \circ |H| \circ |Ev_2| \simeq k \circ |H| \circ |Ev_1| + k \circ |H| \circ |Ev_{12}|$$

This homotopy can be described more directly as follows. Given a short exact sequence of chain complexes

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

consider the associated long exact sequence of homology groups

$$\dots \xrightarrow{\delta_{q+1}} H_q(A) \xrightarrow{f_q} H_q(B) \xrightarrow{g_q} H_q(C) \xrightarrow{\delta_q} H_{q-1}(A) \xrightarrow{f_{q-1}} \dots$$

Let  $\ker f$  denote the chain complex with trivial differentials given by

$$(\ker f)_q := \ker(f_q: H_q(A) \rightarrow H_q(B))$$

and let  $\ker g$ ,  $\ker \delta$  be chain complexes defined in the analogous way. Notice that we have a short exact sequence

$$0 \rightarrow \ker g \longrightarrow H_*(B) \longrightarrow \ker \delta \rightarrow 0$$

and so the homotopy  $\mathcal{U}$  defines a path in  $K(\mathbb{R})$  between points represented by the complexes  $H_*(B)$  and  $\ker g \oplus \ker \delta$ . By an analogous argument we obtain a path in  $K(\mathbb{R})$  joining the points represented by  $H_*(A) \oplus H_*(C)$  and by the chain complex

$$(\ker f \oplus \ker g) \oplus (\ker \delta \oplus \ker f[-1])$$

where  $\ker f[-1]$  is the complex obtained by shifting grading in  $\ker f$ :

$$(\ker f[-1])_q := (\ker f)_{q-1}$$

Using the additivity homotopy  $\mathcal{U}$  again one can obtain a canonical path in  $K(\mathbb{R})$  between the points represented by  $\ker f \oplus \ker f[-1]$  and the zero chain complex. Combining it with the other paths described above we get a path in  $K(\mathbb{R})$ :

$$\begin{array}{ccc}
 & \ker g \oplus \ker \delta & \\
 & | & \\
 H_*(B) & \nearrow & H_*(A) \oplus H_*(C) \\
 & \ker f \oplus \ker g \oplus \ker \delta \oplus \ker f[-1] & 
 \end{array}$$

Notice that all steps in the construction of this path are functorial, so in fact we obtain in this way a homotopy

$$\mathcal{U}^H: |w\mathcal{S}_2\mathcal{Ch}^{fd}(\mathbb{R})| \times I \rightarrow K(\mathbb{R})$$

between the map  $k \circ |H| \circ |Ev_2|$  and  $k \circ |H| \circ |Ev_1| + k \circ |H| \circ |Ev_{12}|$ . The homotopy  $\mathcal{U}^H$  extends the diagram (A-2):

$$\begin{array}{ccc}
 (A-3) & k \circ |Ev_2| & \xrightarrow{\mathcal{U}} & k \circ |Ev_1| + k \circ |Ev_{12}| \\
 & \Theta^2 \downarrow & & \downarrow \Theta^1 + \Theta^{12} \\
 & k \circ |H| \circ |Ev_2| & \xrightarrow{\mathcal{U}^H} & k \circ |H| \circ |Ev_1| + k \circ |H| \circ |Ev_{12}|
 \end{array}$$

**A.2.1. Lemma.** *There exists a homotopy of homotopies which fills the diagram (A-3).*

A proof of the lemma can be obtained by a fairly straightforward (although tedious) construction of the required homotopy of homotopies.

**A.2.2.** Let  $\mathcal{D}$  be a small category and let

$$F: \mathcal{D} \rightarrow w\mathcal{S}_2\mathcal{Ch}^{fd}(\mathbb{R})$$

be a functor which associates to  $d \in \mathcal{D}$  a short exact sequence

$$F(d) = (0 \rightarrow A(d) \rightarrow B(d) \rightarrow C(d) \rightarrow 0)$$

As before we identify here  $w\mathcal{S}_2\mathcal{Ch}^{fd}(\mathbb{R})$  with the category of short exact sequences in  $\mathcal{Ch}^{fd}(\mathbb{R})$ . We have

$$Ev_1 \circ F = A, \quad Ev_2 \circ F = B, \quad Ev_{12} \circ F = C$$

In this case from (A-3) we obtain a diagram

$$(A-4) \quad \begin{array}{ccc} k \circ |B| & \xrightarrow{\quad \vartheta \quad} & k \circ |A| + k \circ |C| \\ \Theta^2 \downarrow & & \downarrow \Theta^1 + \Theta^{12} \\ k \circ |H| \circ |B| & \xrightarrow{\quad \vartheta^H \quad} & k \circ |H| \circ |A| + k \circ |H| \circ |C| \end{array}$$

Vertices of this diagram represent a maps  $|\mathcal{D}| \rightarrow K(\mathbb{R})$  and edges give homotopies of such maps. Moreover, Lemma A.2.1 shows this diagram can be filled by a homotopy of homotopies. We apply this observation in the proof of Theorem 5.1 as follows. Let  $p: E \rightarrow B$  be a bundle with a unipotent splitting as in the statement (5.1). Take  $\mathcal{D} := \mathcal{S}(B)$  and let  $F: \mathcal{S}(B) \rightarrow w\mathcal{S}_2\mathcal{Ch}^{fd}(\mathbb{R})$  be the functor which assigns to  $\sigma \in \mathcal{S}(B)$  the short exact sequence (5-9). Then the diagram (A-4) is the top square of the diagram (5-8).

**A.3. Tensor products.** Let  $T \in \mathcal{Ch}^{fd}(\mathbb{R})$  be a chain complex with trivial differentials. Consider the functor

$$\otimes T: \mathcal{Ch}^{fd}(\mathbb{R}) \rightarrow \mathcal{Ch}^{fd}(\mathbb{R})$$

which maps a chain complex  $C$  to  $C \otimes T$ . This is an exact functor, so it induces a map

$$K(\otimes T): K(\mathbb{R}) \rightarrow K(\mathbb{R})$$

Restricting the functor  $\otimes T$  to the category  $w\mathcal{S}_1\mathcal{Ch}^{fd}(\mathbb{R})$  and then passing to the nerve we also get a map

$$|\otimes T|: |w\mathcal{S}_1\mathcal{Ch}^{fd}(\mathbb{R})| \rightarrow |w\mathcal{S}_1\mathcal{Ch}^{fd}(\mathbb{R})|$$

Notice that

$$k \circ |\otimes T| = K(\otimes T) \circ k \quad \text{and} \quad k \circ |H| \circ |\otimes T| = K(\otimes T) \circ k \circ |H|$$

As a consequence the homotopy  $K(\otimes T) \circ k \circ |H| \simeq K(\otimes T) \circ k$  can be obtained in two different ways:

- using the map  $K(\otimes T) \circ \Theta$ , or
- using the map  $\Theta \circ (|\otimes T| \times \text{id}_I)$

where  $\Theta$  is the homotopy given by Lemma A.1.1.

**A.3.1. Lemma.** *There exists a homotopy of homotopies between  $K(\otimes T) \circ \Theta$  and  $\Theta \circ (| \otimes T | \times \text{id}_I)$ .*

*Proof.* Let  $C \in \mathcal{C}h^{fd}(\mathbb{R})$ . The homotopy  $K(\otimes T) \circ \Theta$  is obtained applying Waldhausen's pre-additivity theorem to short exact sequences

$$(Q_q C) \otimes T \rightarrow (P_q C) \otimes T \rightarrow (P_{q-1} C) \otimes T$$

while the homotopy  $\Theta \circ (|T| \times \text{id}_I)$  comes from the pre-additivity theorem applied to short exact sequences

$$Q_q(C \otimes T) \rightarrow (P_q C \otimes T) \rightarrow P_{q-1}(C \otimes T)$$

Assume that the chain complex  $T$  is non-zero in only one grading  $n$ . In this case the homotopy of homotopies between  $K(\otimes T) \circ \Theta$  and  $\Theta \circ (|T| \times \text{id}_I)$  comes from isomorphisms of short exact sequences

$$\begin{array}{ccccc} (Q_q C) \otimes T & \longrightarrow & (P_q C) \otimes T & \longrightarrow & (P_{q-1} C) \otimes T \\ \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ Q_{q+n}(C \otimes T) & \longrightarrow & (P_{q+n} C \otimes T) & \longrightarrow & P_{q+n-1}(C \otimes T) \end{array}$$

If  $T$  is an arbitrary complex with trivial differentials then  $T$  is a direct sum of complexes concentrated in a single grading. Using this observation we can reduce the statement of the lemma to the special case considered above.  $\square$

**A.3.2.** Let  $\mathcal{D}$  be a small category and let

$$F, G: \mathcal{D} \rightarrow w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})$$

be functors. Assume that for a some complex  $T \in \mathcal{C}h^{fd}(\mathbb{R})$  with trivial differentials we have a natural transformation of functors

$$\alpha: F \Rightarrow \otimes T \circ G$$

This induces a homotopy  $|\alpha|: |\mathcal{D}| \times I \rightarrow |w\mathcal{S}_1\mathcal{C}h^{fd}(\mathbb{R})|$  between the maps  $|F|$  and  $| \otimes T \circ G |$ . Consider the diagram

$$\begin{array}{ccccc} k \circ |F| & \xrightarrow{k \circ |\alpha|} & k \circ | \otimes T \circ G | & \xlongequal{\quad} & K(\otimes T) \circ k \circ |G| \\ \Theta \circ (|F| \times \text{id}_I) \downarrow & & \Theta \circ (| \otimes T \circ G | \times \text{id}_I) \downarrow & & K(\otimes T) \circ \Theta \circ (|G| \times \text{id}_I) \downarrow \\ k \circ |H| \circ |F| & \xrightarrow{k \circ |H(\alpha)|} & k \circ |H| \circ | \otimes T \circ G | & \xlongequal{\quad} & K(\otimes T) \circ k \circ |H| \circ |G| \end{array}$$

Each vertex of the diagram represents a map  $|\mathcal{D}| \rightarrow K(\mathbb{R})$ , and each edge stands for a homotopy of such maps. We claim that there exists a homotopy of homotopies between the concatenation of  $k \circ |\alpha|$  with  $K(\otimes T) \circ \Theta \circ (|G| \times \text{id}_I)$  and concatenation of  $\Theta \circ (|F| \times \text{id}_I)$  with  $k \circ |H(\alpha)|$ . Indeed, directly from the construction of the homotopy  $\Theta$  one can see that the left square in the diagram can be filled by a homotopy of homotopies. Also, by Lemma A.3.1 we have a homotopy of homotopies filling the right square.

We can apply this observation in the context of the proof of Theorem 6.8 as follows. Let  $p: E \rightarrow B$  be a smooth bundle and let  $q: D \rightarrow E$  be a Leray-Hirsch bundle with fiber  $F_q$ . Take  $\mathcal{D} := \mathcal{S}(B)$ . Let  $F := C_{pq}$  and  $G := C_p$  be the chain complex functors defined in (4.3). Also, let  $T := H_*(F_q)$ . We have a natural transformation

$$\alpha: C_{pq} \Rightarrow \otimes H_*(F_q) \circ C_p$$

given by the Leray-Hirsch quasi-isomorphism. Consider the diagram (6-3). The homotopy  $\mu_q \circ (p^! \times \text{id}_I)$  in that diagram coincides with  $k \circ |\alpha|$  while the homotopies  $\omega_{pq}^{(1)}$  and  $\omega_p^{(1)} \otimes H_*(F_q)$  can be identified with, respectively,  $\Theta \circ (|F| \times \text{id}_I)$  and  $K(\otimes T) \circ \Theta \circ (|G| \times \text{id}_I)$ . Finally, the homotopy  $\alpha(q)$  in (6-3) is the same as  $k \circ |H(\alpha)|$ . As a consequence the homotopy of homotopies described above fills the top square in the diagram (6-3) in the proof of Theorem 6.8.

## REFERENCES

- [1] B. Badzioch and W. Dorabiała. Additivity for parametrized topological Euler characteristic and Reidemeister torsion. *K-Theory*, 38(1):1–22, 2007.
- [2] B. Badzioch, W. Dorabiała, and B. Williams. Smooth parametrized torsion - a manifold approach. *Advances in Mathematics*, 221(2):660–680, 2007.
- [3] J. C. Becker and R. E. Schultz. Axioms for bundle transfers and traces. *Mathematische Zeitschrift*, 227:583–605, 1998.
- [4] J. M. Bismut and J. Lott. Flat vector bundles, direct images and higher real analytic torsion. *J. Amer. Math. Soc.*, 8(2):291–363, 1995.
- [5] Jean-Michel Bismut and Sebastian Goette. Families torsion and Morse functions. *Astérisque*, (275), 2001.
- [6] M. Bökstedt. The rational homotopy type of  $\Omega\text{Wh}^{\text{Diff}}(*)$ . In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 25–37. Springer, Berlin, 1984.
- [7] M. Bökstedt and F. Waldhausen. The map  $BSG \rightarrow A(*) \rightarrow QS^0$ . In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 418–431. Princeton Univ. Press, Princeton, NJ, 1987.
- [8] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.
- [9] W. G. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta Math.*, 190(1):1–104, 2003.
- [10] C. Ehresmann. Sur les espaces fibrés différentiables. *C. R. Acad. Sci. Paris*, 224:1611–1612, 1947.
- [11] S. Goette. Morse theory and higher torsion invariants I. *preprint*, [arXiv:math.DG/0111222](https://arxiv.org/abs/math/0111222), 2001.
- [12] S. Goette. Morse theory and higher torsion invariants II. *preprint*, [arXiv:math.DG/0305287](https://arxiv.org/abs/math/0305287), 2003.
- [13] A. Hatcher and J. Wagoner. *Pseudo-isotopies of compact manifolds*. Société Mathématique de France, Paris, 1973. Astérisque, No. 6.
- [14] K. Igusa. Parametrized Morse theory and its applications. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 643–651, Tokyo, 1991. Math. Soc. Japan.
- [15] K. Igusa. *Higher Franz-Reidemeister torsion*, volume 31 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.

- [16] K. Igusa. Axioms for higher torsion invariants of smooth bundles. *J. Topology*, 1(1):159–186, 2007.
- [17] J. R. Klein. Higher Reidemeister torsion and parametrized Morse theory. In *Proceedings of the Winter School “Geometry and Physics” (Srní, 1991)*, volume 30 of *Rend. Circ. Mat. Palermo (2) Suppl.*, pages 15–20, 1993.
- [18] J. R. Klein and B. Williams. The refined transfer, bundle structures, and algebraic  $K$ -theory. *J. Topology*, to appear.
- [19] E. H. Spanier. *Algebraic topology*. McGraw-Hill, 1966.
- [20] J. B. Wagoner. Diffeomorphisms,  $K_2$ , and analytic torsion. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 23–33. Amer. Math. Soc., Providence, R.I., 1978.
- [21] F. Waldhausen. Algebraic  $K$ -theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.
- [22] F. Waldhausen. Algebraic  $K$ -theory of spaces, a manifold approach. In *Current trends in algebraic topology, Part 1 (London, Ont., 1981)*, volume 2 of *CMS Conf. Proc.*, pages 141–184. Amer. Math. Soc., Providence, R.I., 1982.
- [23] F. Waldhausen. Algebraic  $K$ -theory of spaces. In *Algebraic and Geometric Topology (New Brunswick, NJ, 1983)*, volume 1126 of *Lecture Notes in Mathematics*, pages 318–419. Springer-Verlag, 1985.

DEPARTMENT OF MATHEMATICS, UNIVERSITY AT BUFFALO, SUNY, BUFFALO, NY  
*E-mail address:* badzioch@buffalo.edu

DEPARTMENT OF MATHEMATICS, PENN STATE ALTOONA, ALTOONA, PA  
*E-mail address:* wud2@psu.edu

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI  
*E-mail address:* klein@math.wayne.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, IN  
*E-mail address:* williams.4@nd.edu