

## Bivariant Riemann Roch Theorems

Bruce Williams

ABSTRACT. The goal of this paper is to explain the analogy between certain results in algebraic geometry, namely the Riemann-Roch theorems due to Baum, Fulton, and MacPherson ([BFM2], and [FMac]); and recent results in geometric topology due to Dwyer, Weiss and myself [DWW]. One reason for doing this is that the *bivariant* viewpoint introduced by Fulton-MacPherson in their memoir [FMac], becomes particularly useful in the topological case. In fact we get that the bivariant topological Riemann-Roch theorem has a *converse*.

Both in the algebraic geometry case and in the geometric topology case we'll introduce a pair of functors: one of which is covariant and the other contravariant. However, for certain maps between varieties (or topological spaces) we also get transfer (or Gysin) maps for these functors. The Riemann-Roch theorem in both cases gives natural transformations between the above functors and generalized homology/cohomology theories which are compatible with the transfer maps.

### I. Algebraic Geometry

In this paper variety means quasi-projective variety over  $\mathbb{C}$ . For any variety  $X$ ,  $K_{alg}^0(X)$  is the Grothendieck group of algebraic  $\mathbb{C}$ -vector bundles on  $X$ . Tensor product makes  $K_{alg}^0(X)$  a ring. For any morphism  $f : X \rightarrow Y$ , there is an induced ring homomorphism  $f^* : K_{alg}^0(Y) \rightarrow K_{alg}^0(X)$  given by pulling back bundles. This makes  $K_{alg}^0$  a contravariant functor from varieties to commutative rings.

Let  $K_{top}^0(X)$  denote the Grothendieck group of topological  $\mathbb{C}$ -vector bundles on  $X$  with the classical topology. Under tensor product  $K_{top}^0(X)$  is a ring, and  $K_{top}^0(X) \simeq h^0(X; \mathbb{K}^{top}\mathbb{C})$ . There exists an obvious forgetful natural transformation  $\alpha^0 : K_{alg}^0(X) \rightarrow h^0(X; \mathbb{K}^{top}\mathbb{C})$ .

For any variety  $X$ ,  $K_0^{alg}(X)$  is the Grothendieck group of coherent sheaves of  $\mathcal{O}_X$ -modules. If  $f : X \rightarrow Y$  is a proper morphism, then the map  $f_* : K_0^{alg}(X) \rightarrow K_0^{alg}(Y)$  sends  $[\mathcal{F}]$  to  $\sum_i (-1)^i [(R^i f_* \mathcal{F})]$ . Here  $R^i f_* \mathcal{F}$  is the higher direct image

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sheaf, i.e. the sheaf associated to the presheaf  $U \mapsto H^i(f^{-1}(U); \mathcal{F})$ . Since an algebraic vector bundle can be viewed as a locally free sheaf, we get a group homomorphism  $K_{alg}^0(X) \rightarrow K_0^{alg}(X)$ . If  $X$  is smooth this maps is an isomorphism because any coherent sheaf over a smooth variety has a finite resolution by locally free sheaves. The tensor product of a locally free sheaf and a coherent sheaf is coherent. Thus tensor product yields a cap product  $\cap: K_{alg}^0(X) \otimes K_0^{alg}(X) \rightarrow K_0^{alg}(X)$ .

Since Zariski open set are “so large” it is not true that a smooth variety is locally isomorphic to  $\mathbb{A}^n$ . However a variety  $X$  is *smooth* iff each point  $x$  has an open neighborhood  $U$  and there is an étale map  $U \rightarrow \mathbb{A}^n$ . More generally a morphism  $f: X \rightarrow Y$  is *smooth* iff for each  $x \in X$  there are neighborhoods  $U$  of  $x$ ,  $V$  of  $f(x)$ , with  $f(U) \subset V$ , so that the restriction of  $f$  to  $U$  factors:  $U \xrightarrow{g} \mathbb{A}_V^n \xrightarrow{p} V$ , with  $g$  étale and  $p$  the projection of a trivial vector bundle. (See [FL,p.81] and [Mum,p.41] for background on smooth morphisms.)

A morphism  $f: X \rightarrow Y$  is *perfect* if  $f = p \circ \iota$ , where  $p$  is smooth, and  $\iota$  is a closed embedding such that  $\iota_* \mathcal{O}_X$  can be resolved by a finite complex of locally free sheaves.

When  $f: X \rightarrow Y$  is a *perfect morphism*, we get functorial wrong-way or transfer maps:

$$f_!: K_{alg}^0(X) \rightarrow K_{alg}^0(Y), f^!: K_0^{alg}(Y) \rightarrow K_0^{alg}(X),$$

where for  $f_!$  we have to assume  $f$  is also proper.

For example if  $f$  is a *closed embedding* which is perfect, every locally free sheaf  $F$  on  $X$  has a resolution  $G_\bullet$  on  $Y$ ,

$$f_![F] = \sum (-1)^i [G_i],$$

and for any coherent sheaf  $\mathcal{F}$  on  $Y$ ,

$$f^![\mathcal{F}] = \sum (-1)^{-i} [Tor_i^Y(\mathcal{O}_X, \mathcal{F})],$$

See [F,15.1.8] for the definition of  $f_!$  and  $f^!$  for general perfect morphisms. The maps  $f_!$  and  $f^!$  are also described in section 3 of this paper via bivariant products.

Let  $X^+$  be the 1-point compactification of  $X$ .

#### 1.1. BAUM-FULTON-MACPHERSON RR THEOREM.

There exists a natural transformation  $\alpha_0: K_0^{alg}(X) \rightarrow h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C}) = h_0(X^+; \mathbb{K}^{top}\mathbb{C})$  with the following properties.

- (i:) If  $a \in K_{alg}^0(X)$  and  $b \in K_0^{alg}(X)$ , then  $\alpha_0(a \cap b) = \alpha^0(a) \cap \alpha_0(b)$ .
- (ii:) If  $X$  is smooth, then  $\alpha_0(\mathcal{O}_X)$  is a  $\mathbb{K}^{top}\mathbb{C}$  fundamental class for  $X$ .

Furthermore, if  $f: X \rightarrow Y$  is perfect we get the following two commutative diagrams:

$$\begin{array}{ccc} K_0^{alg}(X) & \xrightarrow{\alpha_o} & h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C}) \\ f^! \uparrow & & f_h^! \uparrow \\ K_0^{alg}(Y) & \xrightarrow{\alpha_o} & h_0^{l.f.}(Y; \mathbb{K}^{top}\mathbb{C}) \end{array}$$

$$\begin{array}{ccc}
 K_{alg}^0(X) & \xrightarrow{\alpha^o} & h^0(X; \mathbb{K}^{top}\mathbb{C}) \\
 f_! \downarrow & & f_!^h \downarrow \\
 K_{alg}^0(Y) & \xrightarrow{\alpha^o} & h^0(Y; \mathbb{K}^{top}\mathbb{C}),
 \end{array}$$

where in the second diagram we need to also assume  $f$  is proper.

The natural transformation  $\alpha_0: K_0^{alg}(X) \rightarrow h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C})$  is constructed as follows. Choose an embedding of  $X$  in a smooth variety  $W$ , and then a  $C^\infty$  embedding of  $W$  in a sphere  $S^{2n}$ , with normal bundle  $\pi: \mathcal{N} \rightarrow W$ . Then excision, Spanier Whitehead duality, and Bott periodicity yield isomorphisms:

$$h^0(\mathcal{N}, \mathcal{N} - X; \mathbb{K}^{top}\mathbb{C}) \simeq h^0(S^{2n}, S^{2n} - X; \mathbb{K}^{top}\mathbb{C}) \simeq h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C}).$$

Suppose  $S$  is a coherent sheaf on  $X$ , then there exists on the smooth variety  $W$  a finite resolution  $E_\bullet$  of  $S$  by locally free sheaves. Then  $\alpha_0$  sends the element in  $K_0^{alg}(X)$  represented by  $S$  to the element represented by  $\pi^* E_\bullet \otimes \wedge^{\bullet} \pi^* \mathcal{N}^\vee$ , where  $\mathcal{N}^\vee$  is the dual of  $\mathcal{N}$  and  $\wedge^{\bullet} ?$  is the exterior alg. bundle of  $?$ . What is surprising is that this construction is not only independent of the above choices, but is natural for proper morphisms.

See [BFM1, IV.4] for descriptions of  $f_!^h$  and  $f_!^l$  when  $f$  is a complete intersection morphism. In Section 3  $f_!^h$  and  $f_!^l$  are described in general using products of bivariant functors.

See [G] for generalizations to higher algebraic K-theory.

The first paragraph of 1.1 was first proven in [BFM2]. The existence of the two commutative diagrams in (1.1) when  $f$  is a complete intersection morphism was stated in [BFM2, 4.2]. Later in [FMac] stronger results with simpler proofs were given by first proving a bivariant Riemann-Roch theorem (see (3.1) in this paper) and then using bivariant machinery to deduce (1.1). Theorem 2.7 in this paper can be viewed as a topological analogue of (1.1). It is also deduced from a bivariant version, namely (4.2) and (4.3).

## 2. Topological Riemann-Roch

Technical points:

In the topological case our functors will take values in the category of infinite loop spaces. Recall that given a category  $\mathcal{D}$  with cofibrations  $cof\mathcal{D}$  and weak equivalences  $w\mathcal{D}$ , Waldhausen [Wald1] has constructed an infinite loop space  $K(\mathcal{D})$ . Also the functors that we construct in this subsection will appear to be functors on “spaces with base point”, however by using somewhat more complicated descriptions they factor thru the functor which just forgets the base point.

**2.1 Warm-Up.** We’ll first consider a pair of covariant/contravariant functors which are too “naive” for us in that they factor thru the fundamental groupoid functor. Then we’ll introduce a better pair of functors and explain why they are better for our purposes.

Fix a commutative ring  $R$ . Our “naive” covariant functor is

$$\begin{aligned} K(R\pi_1 X) &= K(\text{f.g. projective } R\pi_1 X\text{-modules}) \\ &\simeq K\left(\begin{array}{l} R\pi_1 X\text{-chain complexes which are} \\ \text{homotopy equivalent to f.g. projective} \\ R\pi_1 X\text{-chain complexes} \end{array}\right) \end{aligned}$$

Recall that a chain complex is f.g. if the chain complex is non-zero only in a finite number of degrees and finitely generated in each degree.

Our “naive” contravariant functor is

$$\begin{aligned} K'(R\pi_1 X) &= K(\text{representations of } \pi_1 X \text{ on f.g. projective } R\text{-modules}) \\ &\simeq K\left(\begin{array}{l} \text{representations of } \pi_1 X \text{ on } R\text{-modules} \\ \text{with finite resolutions by f.g. projective} \\ R \text{ modules} \end{array}\right) \\ &\simeq K(\text{locally constant sheaves of f.g. proj. } R \text{ modules on } X) \\ &\simeq K(\text{f.g. proj. } R\text{-bundles on } X \text{ in the sense of Karoubi } [\mathbf{K}]). \end{aligned}$$

**2.2 “Better” Functors.** We replace, the discrete group  $\pi_1 X$ , with the simplicial group  $G(S(X))$ , where  $S$  is the singular complex functor, and  $G$  is Kan’s simplicial loop group functor. We’ll abuse notation and denote  $G(S(X))$  by  $\Omega X$ .

There are two reasons why  $K(R\Omega X)$  and  $K'(R\Omega X)$  are better for us than the above “naive” functors.

- 1: We want transfers not just for finite covering maps, but for *fibrations with fibers homotopy dominated by finite CW complexes*. We’ll call such fibrations *perfect fibrations*. For example a proper, smooth map of complex varieties becomes a perfect fibration when the varieties are given the classical topology.
- 2: Suppose we let  $R$  be the sphere spectrum which is one of the “brave” new commutative rings in the sense of stable homotopy theory (see [Wald2],[EKMM],[Ly],[HSS]). Then  $K(R\Omega X) = \text{Waldhausen’s } A(X)$ , which is important for the study of homeomorphisms and diffeomorphisms of manifolds (see [WW2]). Furthermore,  $A(X) \simeq K(\text{finitely dominated retractive spaces over } X)$ . Similarly, when  $R$  is the sphere spectrum, Waldhausen denotes  $K'(R\Omega X)$  by  $\forall(X)$ , where

$$\begin{aligned} \forall(X) &\simeq K(\text{representations of } \Omega X \text{ on based finitely dominated spaces}) \\ &\simeq K\left(\begin{array}{l} \text{retractive spaces } Y \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{s} \end{array} X \text{ over } X \text{ such} \\ \text{that } r \text{ is a fibration with finitely domi-} \\ \text{nated fibers} \end{array}\right). \end{aligned}$$

Suppose  $R$  is a discrete commutative ring. Then

$$K(R\Omega X) \simeq K_0(R\pi_1(X)) \times \widehat{BGL}^+(R\Omega X),$$

where  $\widehat{GL}$  denotes matrices over  $R\Omega X$  which are invertible over  $R\pi_1(X)$ .

Recall that a  $R$ -chain complex  $\mathcal{C}$  is chain homotopy equivalent to a f.g. projective  $R$ -chain complex iff it is *finitely dominated*, i.e. homotopy finitely dominated by a f.g. free chain complex. If  $\mathcal{C}$  is such a chain complex, we let  $\text{haut}_R(\mathcal{C})$  be the simplicial monoid of chain homotopy automorphisms of  $\mathcal{C}$ . A  $k$ -simplex in  $\text{haut}_R(\mathcal{C})$  is a chain map  $\phi : \mathcal{C} \otimes \mathcal{C}(\Delta^k) \rightarrow \mathcal{C}$  such that  $\phi$  restricted to each  $\mathcal{C} \otimes \mathcal{C}(\text{vertex})$  is a chain homotopy equivalence. (Here  $\mathcal{C}(\Delta^k)$  is the cellular chain complex of the

standard  $k$ -simplex.) A representation of  $\Omega X$  on  $\mathcal{C}$  is a map of simplicial monoids  $\Omega X \rightarrow \text{haut}_R(\mathcal{C})$ . Then

$$\begin{aligned} K'(R\Omega X) &= K(\text{representations of } \Omega X \text{ on f.dominated } R \text{ chain complexes}) \\ &\simeq K(\text{f. dominated chain fibrations over } X \text{ in the sense of Dwyer-Kan [DK]}) \end{aligned}$$

**2.3 PROPOSITION.** *If  $R$  is a discrete, regular ring, then the map  $\psi : K'(R\pi_1(X)) \rightarrow K'(R\Omega X)$  induced by  $R\Omega X \rightarrow R\pi_0(\Omega X)$  is a homotopy equivalence.*

**OUTLINE OF PROOF.** For  $i = 0, 1, \dots$  the map  $K'(R\Omega X) \rightarrow K'(R\Omega X)$  induced by the functor which sends a representation  $\rho : \Omega X \rightarrow \text{haut}_R(\mathcal{C})$  to  $H_i(\rho) : \Omega X \rightarrow \pi_0(\text{haut}_R(\mathcal{C})) \rightarrow \text{Aut}_R(H_i(\mathcal{C}))$  factors thru  $\psi$ . Thus we are done if we can show  $id + \sum_i H_{2i+1}(?)$  induces a map homotopy equivalent to  $\sum_i H_{2i}(?)$ . This is proved by applying Waldhausen's additivity theorem [Wald2] to the functorial cofibrations sequences  $Q_k\mathcal{C} \rightarrow P_k\mathcal{C} \rightarrow P_{k-1}\mathcal{C}$ , where  $P_k\mathcal{C}$  is the  $k$ -th Postnikov approximation to  $\mathcal{C}$ . That is  $(P_k\mathcal{C})_i = C_i$  for  $i \leq k$ ,  $(P_k\mathcal{C})_{k+1} = \text{im}(\partial : C_{k+1} \rightarrow C_k)$ , and  $(P_k\mathcal{C})_i = 0$  for  $i > k + 1$ . Notice that  $Q_k\mathcal{C}$  is weak homotopy equivalent to the chain complex concentrated in degree  $k$  and equal to  $H_k(\mathcal{C})$  there.

**WARNING.**  *$R$  regular does not imply that  $K(R\Omega X)$  and  $K(R\pi_1 X)$  are equivalent.*

**2.4 Transfer maps.** If  $p : E \rightarrow B$  is a perfect fibration we get transfer maps,

$$p^! : K(R\Omega B) \rightarrow K(R\Omega E); \quad p_! : K'(R\Omega E) \rightarrow K'(R\Omega B)$$

Examples

- i:) We assume  $R$  is a discrete regular ring,  $B$  is connected, and  $F$  is the fiber of  $p$  over a choice of base point in  $B$ . Recall that  $\pi_1(B)$  acts on the homology of  $F$  with coefficients in any module over  $\pi_1(E)$ . Then  $p_!$  induces a map  $\pi_0(K'(R\pi_1 E)) \rightarrow \pi_0(K'(R\pi_1(B)))$ . Suppose  $[P]$  is an element in  $\pi_0(K'(R\pi_1 E))$ , then

$$p_![P] = \sum_i (-1)^i [H_i(F; P)]$$

Notice that since  $R$  is regular,  $H_i(F; P)$  has a finite resolution by f.g. proj.  $R$ -modules.

- ii: If  $R$  is the sphere spectrum, then  $p^!$  is the map  $A(B) \rightarrow A(E)$  induced by the functor which sends a finitely dominated retractive space over  $B$  to the retractive space over  $E$  gotten by the pullback construction.

Suppose  $X$  is path connected, then for  $i = 0$  or  $1$ ,  $\pi_i(A(X)) \simeq K_i(\mathbb{Z}\pi_1(X))$ . Thus for perfect fibrations we do get transfer maps for  $K_i(\mathbb{Z}\pi_1(X))$  when  $i = 0$  or  $1$ . It is interesting to compare Luck's earlier bivariant construction for these transfer maps with Section 4 of this paper (see [L1],[L2]).

## 2.5 Assembly and Coassembly.

Given a category  $\mathcal{D}$  with cofibration  $\text{cof}\mathcal{D}$ , and weak equivalences  $w\mathcal{D}$ , we get a map  $Bw\mathcal{D} \rightarrow K(\mathcal{D})$ , which Waldhausen observes can be viewed as an analogue of Segal's group completion. (See [S],[Wald1]). For example, if  $\mathcal{D}$  is the category of f.g projective  $R$  modules, with cofibrations given by admissible monomorphisms, and weak equivalences given by isomorphism, then the "group completion" map equals  $\amalg B\text{Aut}_R(M) \rightarrow K(R)$ , where we take the disjoint union over isomorphism

classes of f.g. projective  $R$  modules. Similarly, if  $\mathcal{D}$  is the simplicial category of finitely dominated chain complexes over  $R$ , with cofibrations given by chain maps which are admissible monomorphism in each dimension, and weak equivalences given by chain homotopy equivalences, then the “group completion” map equals,  $\amalg Bhaut_R(\mathcal{C}) \rightarrow K(R)$ , where we take the union over chain homotopy equivalence classes of finitely dominated  $R$ -chain complexes.

Recall Quillen’s map  $\pi_0 K'(R\pi) \rightarrow [B\pi, KR]$ , which sends a representation  $\pi \xrightarrow{\rho} Aut_R(M)$  to  $B\pi \xrightarrow{B\rho} BAut_R(M) \rightarrow K(R)$ .

Similarly,

$$\alpha^*: K'(R\Omega X) \rightarrow \mathbb{H}^\bullet(X; K(R)),$$

sends  $\Omega X \rightarrow haut_R(\mathcal{C})$  to  $X \simeq B\Omega X \rightarrow Bhaut_R(\mathcal{C}) \rightarrow K(R)$ .

Here  $\mathbb{H}^\bullet(X; K(R))$  denotes the function spectrum with homotopy groups the cohomology of  $X$  with coefficients in  $KR$ . For example,  $\pi_*(\mathbb{H}^\bullet(X; \mathbb{K}^{top}\mathbb{C})) \simeq h_*(X, \mathbb{K}^{top}\mathbb{C})$ .

We call  $\alpha^*$  the *coassembly map*, for  $K'$ . It is an example of one of Thomason’s limit problem maps [T]. Also,  $\alpha^*$  is natural on continuous maps.

For any group  $\pi$ , let  $a_*$  be the composition  $B\pi \rightarrow BGL_1(R\pi) \rightarrow K(R\pi)$ . Then we get the following Loday-Waldhausen assembly map [Lo],[Wald1]

$$\mathbb{H}_\bullet(B\pi; K(R)) = \Omega^\infty(B\pi_+ \wedge K(R)) \xrightarrow{a_* \wedge 1} K(R\pi) \wedge K(R) \xrightarrow{m} K(R),$$

where the multiplication map  $m$  is induced by the algebra structure of  $R\pi$  over  $R$ .

If we replace  $\pi_1(X)$  with  $\Omega X$ , we again get an *assembly map*:

$$\alpha_*: \mathbb{H}_\bullet(X; K(R)) \rightarrow K(R\Omega X),$$

When  $R$  is the sphere spectrum, then  $\alpha_*$  becomes the map

$$\alpha_*: \mathbb{H}_\bullet(X; A(*)) \rightarrow A(X).$$

Remarks

- (1:) Compare the direction of these assembly maps, and the Baum-Fulton-MacPherson map  $\alpha_*: K_0^{alg}(X) \rightarrow h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C})$ .
- (2:) If we replace  $K(R)$  with  $\mathbb{K}^{top}\mathbb{C}$ , and  $K(R\Omega X)$  with  $\mathbb{K}^{top}(C_r^* \pi_1 X)$ ; then we get the assembly map in analytic index theory which maps the (Poincare dual of the symbol of an operator) to the (index of the operator in the sense of Mishchenko-Fomenko).(see [R1],[R2][R3])

All of the above assembly maps are special cases of a general construction for homotopy invariant functors due to Quinn (see [WW1]). In particular they are natural with respect to continuous maps.

## 2.6 Homotopy Transfer.

Recall that for any perfect fibration  $p: E \rightarrow B$  and any infinite loop space  $\mathbf{k}$ , Becker-Gottlieb [BeGo] and Dold[DoP],[Clapp] have constructed *homotopy transfer maps*:

$$\begin{aligned} p_!^h: \mathbb{H}^\bullet(E; \mathbf{k}) &\rightarrow \mathbb{H}^\bullet(B; \mathbf{k}), \text{ and} \\ p_h^! : \mathbb{H}_\bullet(B; \mathbf{k}) &\rightarrow \mathbb{H}_\bullet(E; \mathbf{k}) \end{aligned}$$

**2.7 TOPOLOGICAL RR THEOREM.** *If  $p: E \rightarrow B$  is a fiber bundle with fibers compact, topological manifolds (with or without boundary); then we get the following diagrams which commute up to preferred homotopies.*

$$\begin{array}{ccc}
 K'(R\Omega E) & \xrightarrow{\alpha^*} & \mathbb{H}^\bullet(E; K(R)) \\
 p! \downarrow & & p_!^X \downarrow \\
 K'(R\Omega B) & \xrightarrow{\alpha^*} & \mathbb{H}^\bullet(B; K(R)) \\
 \mathbb{H}_\bullet(E; KR) & \xrightarrow{\alpha_*} & K(R\Omega E) \\
 p_\chi^! \uparrow & & p^! \uparrow \\
 \mathbb{H}_\bullet(B; KR) & \xrightarrow{\alpha_*} & K(R\Omega B)
 \end{array}$$

*If  $p$  is a smooth bundle, then the maps  $p_!^X$  and  $p_\chi^!$  are Becker-Gottlieb-Dold transfer maps.*

In Section 4 the maps  $p_!^X$  and  $p_\chi^!$  are defined using bivariant products.

Also in Section 4 the bivariant machinery is used to deduce 2.7 from 4.2 and 4.3. The commutativity of the top diagram in 2.7 is an improved version of [DWW, Theorem 8.18]. The commutativity of the bottom diagram is new.

**Warnings:** There exist examples of perfect fibrations, where the above commutative diagrams do not exist. Furthermore, there exist examples of nonsmoothable fiber bundles where the above diagrams do not commute if we replace  $p_!^X$  and  $p_\chi^!$  with Becker-Gottlieb-Dold transfer maps.

Suppose  $R = \mathbb{C}$ , and let  $c_i : K^{alg}\mathbb{C} \rightarrow K(\mathbb{Q}, 2i - 1)$  be a Borel regulator cohomology class. Then for any pair of flat bundle  $\zeta_1$  and  $\zeta_2$ ,  $c_i(\zeta_1 \oplus \zeta_2) = c_i(\zeta_1) + c_i(\zeta_2)$ . Thus  $c_i$  is a map of infinite loop spaces and commutes with the homotopy transfers.

**2.8 COROLLARY (BISMUT-LOTT[BiLo].)** *Assume  $p: E \rightarrow B$  is a smooth bundle as in the last sentence of 2.7. Then the following diagram commutes*

$$\begin{array}{ccc}
 \pi_0(K'(\mathbb{C}\pi_1 E)) & \xrightarrow{c_i \circ \alpha^*} & H^{2i-1}(E; \mathbb{Q}) \\
 p! \downarrow & & p_!^h \downarrow \\
 \pi_0(K'(\mathbb{C}\pi_1 B)) & \xrightarrow{c_i \circ \alpha^*} & H^{2i-1}(B; \mathbb{Q})
 \end{array}$$

It is particularly striking that this algebraic K-theory result was first proved by Bismut and Lott using analytic methods. (See [Lott] and [Lott2] for extensions of [BiLo].)

The homotopy fiber of  $\alpha_* : \mathbb{H}_\bullet(X; A(*)) \rightarrow A(X)$  is the space of stable topological h-cobordisms based on  $X$ . When  $R$  is the sphere spectrum, the commutativity of the lower diagram in 2.7 induces a transfer map on the homotopy fibers of the assembly maps which is compatible with the geometric transfer for stable topological h-cobordisms (see [BuLa]).

### 3: Bivariant Riemann-Roch Theorems

Let  $f : X \rightarrow Y$  be a map of quasi-projective varieties. A complex  $A^\bullet$  of sheaves of  $\mathcal{O}_X$ -modules is said to be *f-perfect* if there exists a factorization  $f = p \circ \iota$  such

that  $p$  is smooth,  $\iota$  is a closed embedding, and  $\iota_*(A^\bullet)$  is quasi-isomorphic to a bounded complex of locally free sheaves.

Then  $K_{alg}(f: X \rightarrow Y)$  is the ‘‘Grothendieck group’’ of  $f$ -perfect complexes; i.e.  $K_{alg}(f: X \rightarrow Y)$  is the free abelian group on the set of quasi-isomorphism classes of  $f$ -perfect complexes on  $X$ , modulo short exact sequences of such complexes.

**Operations**(see [FMac,II§1])

- Pushforward

Given a pair of maps,  $X \xrightarrow{f} Y \xrightarrow{g} Z$  where  $f$  is proper, we get a group homomorphism  $f_* : K_{alg}(X \xrightarrow{g \circ f} Z) \rightarrow K_{alg}(Y \xrightarrow{g} Z)$ .

- Pullback

Given a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

which is Tor-independent, i.e.  $Tor_i^Y(\mathcal{O}_X, \mathcal{O}_Y) = 0$  for  $i > 0$ . Then we get a map  $g^* : K_{alg}(X \rightarrow Y) \rightarrow K_{alg}(X' \rightarrow Y')$ .

- Products

Given a pair of maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , we get a homomorphism

$$\bullet : K_{alg}(f: X \rightarrow Y) \otimes K_{alg}(g: Y \rightarrow Z) \rightarrow K_{alg}(g \circ f: X \rightarrow Z)$$

Then  $K_{alg}(? \rightarrow ??)$  equipped with these three operations is an example of what Fulton-MacPherson call a *bivariant theory*. (see [FMac,I§2])

The axioms for a bivariant theory state that the product is associative, pushforward and pullback are functorial, product commutes with pushforward and pullback; plus pushforward commutes with pullback. There is one more axiom (the ‘‘projective axiom’’) the which we describe in detail.

Suppose we are given a fiber square as above with  $g$  proper and a map  $h : Y \rightarrow Z$ . Then for any  $\beta \in K(h \circ g : Y' \rightarrow Z)$  and any  $\alpha \in K(f : X \rightarrow Y)$ ;  $g'_*(g^* \alpha \bullet \beta) = \alpha \bullet g_*(\beta)$ .

Key Properties:

There exist canonical equivalences,

$$K_{alg}(id: X \rightarrow X) \simeq K_{alg}^0(X); K_{alg}(X \rightarrow pt) \simeq K_0^{alg}(X)$$

The products between the bivariant  $K$ -groups not only recover the ring structure of  $K_{alg}^0(X)$  and the module structure of  $K_0^{alg}(X)$ , but also yield maps

$$\mu^* : K_{alg}(X \xrightarrow{f} Y) \rightarrow Hom(K_0^{alg}(Y), K_0^{alg}(X)),$$

$$\mu_* : K_{alg}(X \xrightarrow{f} Y) \rightarrow Hom(K_{alg}^0(X), K_{alg}^0(Y)), \text{ when } f \text{ is proper.}$$

Here  $\mu^*(\theta) : a \mapsto \theta \bullet a$ , and  $\mu_*(\theta) : b \mapsto f_*(b \bullet \theta)$

This yields a particularly nice description of the transfer maps. When  $f$  is perfect  $\mu^*[\mathcal{O}_X] = f^!$ , and when  $f$  is also proper  $\mu_*[\mathcal{O}_X] = f_!$ .

Furthermore, they also give a purely topological construction which associates to any ring spectrum  $\mathbf{k}$ , a sequence of bivariant theories  $h^i(A \rightarrow B; \mathbf{k})$  on continuous maps between topological spaces which can be embedded as closed subsets of euclidean space.

First we recall how a spectrum  $\mathbf{k}$  determines a cohomology theory. For any finite dimensional polyhedral pair,  $(L, B)$  we let  $h^i(L, B; \mathbf{k})$  be the  $i$ -th homotopy group of the function spectrum  $\mathbb{H}^\bullet(L/B; \mathbf{k})$ . Then the Čech method is used to define  $h^i(X, A; \mathbf{k})$  when  $X$  can be embedded as a closed subspace of euclidean space, and  $A$  is an open subspace of  $X$  (see [FMac] and [LR]).

Now suppose  $f: X \rightarrow Y$  is a continuous map and  $\phi: X \rightarrow \mathbb{R}^n$  is a choice of closed embedding. Then the bivariant functor is defined by  $h^i(f: X \rightarrow Y; \mathbf{k}) = h^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n - \text{image}(f \times \phi); \mathbf{k})$  Fulton-MacPherson show that this is independent of the choice of  $\phi$ . Furthermore, they construct pushforward, pullback and product operations which satisfy their axioms for a bivariant theory.

3.1 BIVARIANT RIEMANN-ROCH. (See [FMac, II§1.4] *On the category of quasi-projective varieties, there exists a natural transformation of bivariant theories:*

$$\alpha: K_{alg}(X \rightarrow Y) \rightarrow h^0(X \rightarrow Y; \mathbb{K}^{top}\mathbb{C})$$

The natural transformations:  $\alpha^0: K_{alg}^0(X) \rightarrow h^0(X; \mathbb{K}^{top}\mathbb{C})$ , and  $\alpha_0: K_0^{alg}(X) \rightarrow h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C})$  can both be recovered by applying  $\alpha$  to  $K_{alg}(id: X \rightarrow X)$  and  $K_{alg}(X \rightarrow pt)$ . The transfer maps  $f_!^h$  and  $f_h^!$  are gotten by applying  $\alpha$  to  $[\mathcal{O}_X]$  and then applying the analogues of  $\mu_*$  and  $\mu^*$ . When  $f: X \rightarrow Y$  is a (l.c.i.)= (locally complete intersection) morphism the element  $\alpha([\mathcal{O}_X] \in h^0(X \rightarrow Y; \mathbb{K}^{top}\mathbb{C})$  has a purely homotopy theoretic description.

Theorem 1.1 is an easy consequences of 3.1. Furthermore, the proof of 3.1 is simpler than the original proof of 1.1 in [BFM2] even in the l.c.i. case.

Problem:

When  $X$  is smooth,  $\alpha_*[\mathcal{O}_X] \in h_0^{l.f.}(X; \mathbb{K}^{top}\mathbb{C})$  is a  $\mathbb{K}^{top}\mathbb{C}$ -fundamental class for  $X$ . Give a homotopy theoretic description of  $\alpha_*[\mathcal{O}_X]$  when we drop the smooth assumption. More generally, for a perfect map  $X \rightarrow Y$  which is not a l.c.i. morphism give a homotopy theoretic description of  $\alpha[\mathcal{O}_X] \in h^0(X \rightarrow Y; \mathbb{K}^{top}\mathbb{C})$ .

Ginzburg has proved an equivariant version of 3.1 and shown that it plays a central role in representation theory(see [CG]). In [P-G] Pasual-Gainza extends  $K_{alg}(X \rightarrow Y)$  to higher algebraic K-theory and verifies that the Fulton-MacPherson axioms for a bivariant functor are still satisfied. In [Levy] Roni Levy extends 3.1 to higher algebraic K-theory.

#### 4: Topological Bivariant Riemann-Roch

Before introducing the bivariant functors we recall some background on Waldhausen's definition of the algebraic K-theory of spaces [Wald1].

Let  $\mathcal{R}(X)$  be the category of retractive spaces over the topological space  $X$ .

Thus an object in  $\mathcal{R}(X)$  is a diagram of topological spaces  $W \begin{matrix} \xrightarrow{r} \\ \xleftarrow{s} \end{matrix} X$  such that  $rs = id_X$  and  $s$  is a closed embedding having the homotopy extension property. The morphisms in  $\mathcal{R}(X)$  are continuous maps over and relative to  $X$ . A morphism is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. A morphism is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence.

Let  $\mathcal{R}^{fd}(X)$  be the full subcategory of homotopy finitely dominated retractive spaces over  $X$  (see [DWWI+II, Sec.6] for details). Then  $\mathcal{R}^{fd}(X)$  is a category

with cofibrations and weak equivalences, i.e. a Waldhausen category, and  $A(X)$  is the K-theory of  $\mathcal{R}^{fd}(X)$ .

Let  $Rep_{fd}(X)$  be the full subcategory of  $\mathcal{R}(X)$  where  $W \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} X$  is in  $Rep_{fd}(X)$  if  $r$  is a fibration such that the fibers are homotopy finitely dominated. Then  $Rep_{fd}(X)$  is also a Waldhausen category and  $\mathcal{V}(X)$  is the K-theory of  $Rep_{fd}(X)$ .

Operations on Retractive Spaces

- Pushforward

Given a continuous map  $X \rightarrow X'$  we get a functor  $f_*: \mathcal{R}(X) \rightarrow \mathcal{R}(X')$  which sends  $W \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} X$  to  $W \cup_X X' \begin{smallmatrix} \xrightarrow{r'} \\ \xleftarrow{s'} \end{smallmatrix} X'$

This induces a map  $f_*: A(X) \rightarrow A(X')$ .

- Pullback

Given a continuous map  $X \rightarrow X'$  we get a functor  $f^*: \mathcal{R}(X') \rightarrow \mathcal{R}(X)$  which sends  $W' \begin{smallmatrix} \xrightarrow{r'} \\ \xleftarrow{s'} \end{smallmatrix} X'$  to  $W' \times_{X'} X \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} X$

This induces a map  $f^*: \mathcal{V}(X') \rightarrow \mathcal{V}(X)$ .

- Products

External smash product of retractive spaces

$$\begin{aligned} \mathcal{R}(X) \times \mathcal{R}(X') &\rightarrow \mathcal{R}(X \times X') \\ (W, W') &\mapsto W \times W' \cup_{(X \times W') \cup (W \times X')} X \times X' \end{aligned}$$

This induces products:

$$\begin{aligned} A(X) \wedge A(X') &\rightarrow A(X \times X') \\ \mathcal{V}(X) \wedge \mathcal{V}(X') &\rightarrow \mathcal{V}(X \times X'). \end{aligned}$$

The second pairing is gotten by first applying external smash product and then pullback  $\Delta^*$ , where  $\Delta: X \rightarrow X \times X$  is the diagonal map.

*Bivariant K-theory of Spaces.*

Given a map  $p: E \rightarrow B$  and a point  $b \in B$ , we let  $\mathcal{F}_b(p)$  denote the homotopy fiber of  $p$  over  $b$ . Suppose  $W \begin{smallmatrix} \xrightarrow{r} \\ \xleftarrow{s} \end{smallmatrix} E$  is a retractive space over  $E$ . Then for any  $b \in B$ ,  $\mathcal{F}_b(p \circ r)$  is a retraction space over  $\mathcal{F}_b(p)$ . Then  $\mathcal{R}(p: E \rightarrow B)$  is the full subcategory of  $\mathcal{R}(E)$  consisting of the retractive spaces such that for any  $b \in B$ ,  $\mathcal{F}_b(p \circ r)$  is finitely dominated as a retractive space over  $\mathcal{F}_b(p)$ . Then  $A(p: E \rightarrow B)$  is the K-theory of  $\mathcal{R}(p: E \rightarrow B)$ .

Bivariant Operations

- Pushforward

Given a pair of maps,  $X \xrightarrow{p} Y \xrightarrow{q} Z$  we get an  $\Omega^\infty$  map  $p_*: A(X \xrightarrow{q \circ p} Z) \rightarrow A(Y \xrightarrow{q} Z)$ .

- Pullback

Given a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & f \downarrow \\ Y' & \xrightarrow{g} & Y \end{array}$$

where  $f$  is a fibration. Then we get a  $\Omega^\infty$  map  $A(X \rightarrow Y) \rightarrow A(X' \rightarrow Y')$ .

- Products

Given a pair of maps  $X \xrightarrow{p} Y \xrightarrow{q} Z$ , we get a pairing

$$\bullet: A(p: X \rightarrow Y) \wedge A(q: Y \rightarrow Z) \rightarrow A(q \circ p: X \rightarrow Z)$$

This is gotten by first using  $p$  to pullback the retractive space over  $Y$  to  $X$ . Then we have two retractive spaces over  $X$  and we apply external smash product. Finally pullback using the diagonal map for  $X$ .

The Fulton-MacPherson axioms for a bivariant theory are then satisfied when we view  $A(? \rightarrow ??)$  as taking values in the homotopy category of  $\Omega^\infty$ -spaces.

Key Properties:

There exist canonical equivalences,

$$A(id: X \rightarrow X) \simeq \forall(X); A(X \rightarrow pt) \simeq A(X)$$

The products between the bivariant A-functors not only recover the ring structure of  $\forall(X)$  and the module structure of  $A(X)$ , but also yield maps

$$\begin{aligned} \mu^*: A(f: X \rightarrow Y) &\rightarrow \text{Map}(A(Y), A(X)), \\ \mu_*: A(f: X \rightarrow Y) &\rightarrow \text{Map}(\forall(X), \forall(Y)) \end{aligned}$$

Here  $\mu^*(\theta) : a \mapsto \theta \bullet a$ , and  $\mu_*(\theta) : b \mapsto f_*(b \bullet \theta)$

This yields a particularly nice description of the transfer maps. When  $p: E \rightarrow B$  is perfect, let  $\chi(p) \in A(p: E \rightarrow B)$ , be the vertex given by the retractive space  $E \amalg E \xrightleftharpoons[s]{r} E$ , then  $\mu^*[\chi(p)] = p^!$ , and  $\mu_*[\chi(p)] = p_!$ . We call  $\chi(p)$  the *parametrized Euler characteristic* for the fibration  $p$ .

*Universal Euler Characteristic.* Suppose  $B$  is a connected, CW complex, and  $p: E \rightarrow B$  is a pullback of the universal fibration  $q: BG_*(F) \rightarrow BG(F)$  where  $F$  is the fiber of  $p$ ,  $G(F)$  is the simplicial monoid of homotopy automorphisms of  $F$ , and  $G_*(F)$  is the simplicial monoid of base point preserving homotopy automorphisms. Then  $\chi(p)$  is the pullback of  $\chi(q)$  which we call the *universal parametrized Euler characteristic* for  $F$ .

*$S^1$ -Fibrations.* Suppose  $p: E \rightarrow B$  is a smooth fiber bundle with a compact Lie groups  $G$  as structure group and fiber  $G/H$ . Then Feshbach [F] has shown that if  $N_G(H)/H$  is not discrete, then the Becker-Gottlieb-Dold transfer for  $p$  is trivial. This can easily be reduced to the special case  $G = S^1$  and  $H$  is the trivial subgroup, and it is natural to ask whether the analogous result is true for the A-theory transfer. However, results of Oliver [O2] imply that even in this case the A-transfer can be nontrivial on  $\pi_1$ . Thus the parametrized Euler characteristic for  $ES^1 \rightarrow BS^1$  is nontrivial!

*Frobenius Reciprocity.* Let  $\bar{\chi}(p)$  be the image of  $\chi(p)$  under the map  $p_*: A(E \rightarrow B) \rightarrow A(B \rightarrow B)$ . Then  $p_* \circ p^!: A(B) \rightarrow A(B)$  is homotopic to capping with  $\bar{\chi}(p)$ ,

and  $p_! \circ p^* : \mathbb{V}(B) \rightarrow \mathbb{V}(B)$  is homotopic to cupping with  $\bar{\chi}(p)$ . This is an easy consequence of the axioms. (See [FMac, p.26, G4].)

*Mackey Double Coset Formula.* A key property of the classical Euler characteristic is that if  $X = X_1 \cup_{X_0} X_2$  where  $X_i$  is a finite CW complex for  $i = 0, 1, 2$ ; then  $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_0)$ . Wojciech Dorabiala has recently proved an analogous result for the parametrized Euler characteristic. He has used this to show that if a smooth fiber bundle has a compact Lie groups as structure group, then one gets A-theory analogues of the Mackey Double coset formula and Feshbach's sum formula. (See [F], [L], and [LMSM, IV sec. 6].)

*Generalized Coassembly Maps.*

Since  $A(?)$  is a continuous homotopy invariant functor we can apply  $A(?)$  fiberwise to a fibration  $p: E \rightarrow B$  to get a new fibration  $p_A: A_B(E) \rightarrow B$  where the fiber over  $b \in B$  is  $A(p^{-1}(b))$ . Let  $\Gamma(p_A: A_B(E) \rightarrow B)$  be the space of cross sections.

Notice that roughly speaking  $A(p: E \rightarrow B)$  can be viewed as the K-theory of  $B$ -parametrized families of f. dominated retractive spaces over the fibers of  $p$ . Thus it should not be surprising that we get a *generalized coassembly map*

$$\alpha^*: A(p: E \rightarrow B) \rightarrow \Gamma(p_A: A_B(E) \rightarrow B),$$

which equals the coassembly map in Section 2 when  $p = id_B$ .

We again get that this  $\alpha^*$  is an example of a Thomason limit problem map.

When  $p: E \rightarrow B$  is a perfect fibration, then  $\alpha^*$  sends  $\chi(p)$  to the parametrized Euler characteristic constructed in [DWWI+II, Sec.6].

*Bivariant Theories and Parametrized Spectrum.* The bivariant theory which Fulton-MacPherson associate to a ring spectrum  $\mathbf{k}$  has a particularly nice description when we evaluate it on a map which is a perfect fibration. First we recall the notion of *twisted* generalized cohomology theory.

A *parametrized spectrum*  $\mathcal{E}$  over a space  $B$ , consists of a sequence of retractive spaces over  $B$ ,  $\mathcal{E}_i \xrightarrow{r_i} B$ , with cross sections  $s_i: B \rightarrow \mathcal{E}_i$ ,  $r_i \circ s_i = id$ , plus structure maps  $f_i: \Sigma_B(\mathcal{E}_i) \rightarrow \mathcal{E}_{i+1}$  which are over  $B$  and rel  $B$ . If the retraction maps  $r_i$  are not already fibration, we convert them into fibrations, and let  $\Gamma(r_i: \mathcal{E}_i \rightarrow B)$  be the space of cross sections. The structure map  $f_i$  determines a map  $\bar{f}_i: \Sigma\Gamma(r_i: \mathcal{E}_i \rightarrow B) \rightarrow \Gamma(r_{i+1}: \mathcal{E}_{i+1} \rightarrow B)$ . Thus we get an ordinary spectrum and we let  $\mathbb{H}^\bullet(B; \mathcal{E})$  denote the associated infinite loop space. (see [B] and [ClaPu])

Example:

Suppose  $p: E \rightarrow B$  is a fibration. We get  $p_+: E_{+,B} \rightarrow B$  be the result of adding a disjoint basepoint to each fiber of  $p$ . Then for any ordinary spectrum  $\mathbf{k}$  we let  $\mathcal{E}(p, \mathbf{k})$  be the parametrized spectrum gotten by smashing each fiber of  $p_+$  with  $\mathbf{k}$

**4.1 PROPOSITION.** *Suppose  $p: E \rightarrow B$  is a perfect fibration where  $E$  and  $B$  can be embedded as closed subspaces of euclidean space, and  $\mathbf{k}$  a ring spectrum. Then  $h^i(p: E \rightarrow B; \mathbf{k})$  is canonically isomorphic to the  $i$ -th homotopy group of  $\mathbb{H}^\bullet(B; \mathcal{E}(p, \mathbf{k}))$*

By applying the assembly map for  $A(?)$  fiberwise to  $p: E \rightarrow B$ , we get a map

$$\alpha_*: \mathbb{H}^\bullet(B; \mathcal{E}(p, A(*))) \rightarrow \Gamma(p_A: A_B(E) \rightarrow B)$$

The following result implies 2.7(except for the last sentence in 2.7).

#### 4.2 BIVARIANT TOPOLOGICAL RIEMANN-ROCH THEOREM.

*Suppose  $p: E \rightarrow B$  is a perfect fibration. Consider the following diagram*

$$A(p: E \rightarrow B) \xrightarrow{\alpha^*} \Gamma(p_A: A_B(E)) \rightarrow B \xleftarrow{\alpha_*} \mathbb{H}^\bullet(B; \mathcal{E}(p, A(*)))$$

Recall that  $\chi(p) \in A(p: E \rightarrow B)$ . Then  $p$  is fiber homotopy equivalent to a fiber bundle with fibers topological manifolds with or without boundary iff  $\pi_0(\alpha^*\chi(p)) = \pi_0(\alpha_*\chi^\%(p))$ , for some  $\chi^\%(p) \in \mathbb{H}^\bullet(B; \mathcal{E}(p, A(*)))$ .

The if part of 4.2 is proved in [DWWI+II,8.4]. The converse is proved in [DWWIII].

Suppose in the definition of  $\mu^*$  and  $\mu_*$  we replace  $A(E \rightarrow B)$  with the bivariant theory associated to the ring spectrum  $A(*)$ . Then the maps  $p_\chi^!$  and  $p_\chi^\chi$  in 2.7 are given by  $p_\chi^! = \mu^*(\chi^\%(p))$  and  $p_\chi^\chi = \mu_*(\chi^\%(p))$

Just as the algebraic K-theory transfers for a perfect fibration  $p$  are determined by an element  $\chi(p) \in A(p: E \rightarrow B)$ , the homotopy transfers are determined by an element  $\chi^h(p) \in \mathbb{H}^\bullet(B; \mathcal{E}(p, QS^0))$ .

Let  $u: QS^0 \rightarrow A(*)$  be the unit map. Recall that Waldhausen has shown that the composition  $\mathbb{H}_\bullet(X; QS^0) \xrightarrow{u_*} \mathbb{H}_\bullet(X; A(*)) \xrightarrow{\alpha_*} A(X)$  has a canonical splitting  $s$ . Thus for a fibration  $p: E \rightarrow B$ , the composition  $\mathbb{H}^\bullet(B; \mathcal{E}(p, Q(S^0))) \xrightarrow{u_*} \mathbb{H}^\bullet(B; \mathcal{E}(p, A(*))) \xrightarrow{\alpha_*} \Gamma(p_A: A_B(E)) \rightarrow B$  also has a splitting  $s$ . Furthermore, if  $p$  is perfect, and we apply this splitting to  $\alpha^*\chi(p)$ ; then we get  $\chi^h(p)$  (up to homotopy).

Thus the following result implies the last sentence in 2.7.

**4.3 BIVARIANT SMOOTH RIEMANN-ROCH THEOREM.** *Suppose  $p: E \rightarrow B$  is a perfect fibration. Consider the following diagram*

$$A(p: E \rightarrow B) \xrightarrow{\alpha^*} \Gamma(p_A: A_B(E)) \rightarrow B \xleftarrow{\alpha_* \circ u_*} \mathbb{H}^\bullet(B; \mathcal{E}(p, QS^0))$$

*Then  $p$  is fiber homotopy equivalent to a smooth fiber bundle with fibers smooth manifolds with or without boundary and transition maps diffeomorphisms iff  $\pi_0(\alpha^*\chi(p)) = \pi_0(\alpha_* \circ u_*(\chi^h(p)))$ .*

The if part of 4.3 is proved in [DWWI+II,8.5]. The converse is proved in [DWWIII].

*Untwisted Bivariant K-theory.* Given a pair of spaces  $X, Y$ , we follow Waldhausen(unpublished) and consider the following untwisted bivariant K-theory  $\forall A(X, Y) = A(p_1: X \times Y \rightarrow X)$ , where  $p_1$  is the projection map. Although we need twisted versions to give Riemann-Roch theorems with converses, the untwisted versions can still be used to give transfer maps.

Notice that  $\forall A(*, Y) = A(X)$ , and  $\forall A(X, *) = \forall(X)$ .

Suppose  $p: E \rightarrow B$  is a fibration. If we apply pushforward to  $E \xrightarrow{p \times id} B \times E \rightarrow B$  we get a map  $A(p: E \rightarrow B) \rightarrow \forall A(B, E)$ .

There are untwisted products

$$\forall A(X_1, X_2) \wedge \forall A(X_2, X_3) \rightarrow \forall A(X_1, X_3),$$

which are gotten by first first applying external smash product to get a retractive space over  $X_1 \times X_2 \times X_2 \times X_3$ . Then pullback using  $id \times \Delta_{X_2} \times id$ . Finally pushforward using the projection  $X_1 \times X_2 \times X_3 \rightarrow X_1 \times X_3$ .

Notice that this untwisted product can also be described using the three operations which are part of the full bivariant theory. First apply the following pullback to

the second factor:  $\forall A(X_2, X_3) = A(X_2 \times X_3 \rightarrow X_3) \rightarrow A(X_1 \times X_2 \times X_3 \rightarrow X_1 \times X_3)$ . Then switch the two factors and apply the twisted product  $\bullet$ . Finally pushforward yields a map  $A(X_1 \times X_2 \times X_3 \rightarrow X_1) \rightarrow A(X_1 \times X_3 \rightarrow X_1) = \forall A(X_1, X_3)$ . Similarly for bivariant theory there is an associated untwisted bivariant theory.

The untwisted products then induce maps:

$$\begin{aligned}\nu^*: \forall A(X, Y) &\rightarrow \text{Map}(A(X), A(Y)) \\ \nu_*: \forall A(X, Y) &\rightarrow \text{Map}(\forall(Y), \forall(X)).\end{aligned}$$

If  $p: E \rightarrow B$  is a perfect fibration, then  $p^!$  is gotten by applying  $\nu^*$  to the image of  $\chi(p)$  in  $\forall A(B, E)$ . Similarly,  $\nu_*$  yields  $p_!$ .

This untwisted bivariant theory is analogous to several other untwisted bivariant theories.

*Segal Conjecture:* (See [A],[AGM],[C], and [M].)

Let  $G_1$  and  $G_2$  be finite groups, and let  $\Omega(G_1, G_2)$  be the Grothendieck group of finite  $G_2$ -free  $(G_1 \times G_2)$ -sets. For example,  $\Omega(G, e) = \Omega(G)$ , is the Burnside ring (here  $e$  is the trivial group), and  $\Omega(G_1, G_2)$  is a  $\Omega(G_1)$  module. Let  $I(G_1) =$  the kernel of the map  $\Omega(G_1) \rightarrow \Omega(e)$ . Let  $\hat{\Omega}(G_1, G_2)$  be the completion of  $\Omega(G_1, G_2)$  with respect to  $I(G_1)$ . The analogue of the generalized coassembly map in this case is a map  $\alpha^*: \Omega(G_1, G_2) \rightarrow [BG_{1+}, \Omega^\infty \Sigma^\infty BG_{2+}]$ . The solution to the Segal Conjecture due to Carlsson implies that  $\alpha^*$  extends to an isomorphism  $\alpha^*: \hat{\Omega}(G_1, G_2) \rightarrow [BG_{1+}, \Omega^\infty \Sigma^\infty BG_{2+}]$ .

*Bivariant K-theory of Rings:* (See [O1] and [HTW])

Suppose  $\Lambda$  and  $\Gamma$  are  $R$ -algebras, where  $R$  is a commutative ring. We let  $K_0^R(\Lambda, \Gamma)$  be the Grothendieck group of  $\Gamma - \Lambda$  bimodules which are f.g. projective as left  $\Gamma$ -modules. Notice that  $K_0^R(R, \Gamma) = K_0(R\Gamma)$ . The analogue to  $\nu^*$  is a map  $K_0^R(\Lambda, \Gamma) \rightarrow [K(\Lambda), K(\Gamma)]$ . Jones and Kassel [JK] have constructed a companion bivariant cyclic theory  $HC^*(\Lambda, \Gamma)$ , and Kassel [Kl] has constructed a natural transformation  $ch: K_0^R(\Lambda, \Gamma) \rightarrow HC^*(\Lambda, \Gamma)$ . See also [McC]. The definition of  $K_0^R(\Lambda, \Gamma)$  extends to when  $\Lambda$  and  $\Gamma$  are *simplicial*  $R$ -algebras, and we get a “change of base ring” map  $\pi_o(A(X, Y)) \rightarrow K_0(R\Omega X, R\Omega Y)$ . If  $p: E \rightarrow B$  is a perfect fibration, then the image of  $\chi(p)$  under the composition  $\pi_o(A(p: E \rightarrow B)) \rightarrow \pi_o(A(B, E)) \rightarrow K_0(R\Omega B, R\Omega E)$  induce  $p^!$  and  $p_!$  in (2.7). Schlichtkrull [Schl] has studied transfer maps for topological Hochschild homology applied to fibration of the form  $BH \rightarrow BG$  where  $G$  is finite group and  $H$  is a subgroup. Do there exist transfer maps for topological cyclic homology applied to perfect fibrations? These transfer maps should be compatible with the cyclotomic trace maps.

*Bivariant K-theory of Operator Algebras:*

(see [K], [Hi], [CS], [Co], [Bl], [BB1], [BB2], [Fa], [K], and [Sk])

Given a pair of  $C^*$ -algebras  $A_1$  and  $A_2$  Kasparov has constructed an abelian group  $KK(A_1, A_2)$ ; plus products

$$KK(A_1, A_2) \times KK(A_2, A_3) \rightarrow KK(A_1, A_3),$$

where  $A_3$  is a third  $C^*$ -algebra. This  $KK$ -theory has proven to be an ideal setting for the Atiyah-Singer index theorem plus generalizations, e.g. the longitudinal index theorem for foliations (see [CS], [Co], and [Sk].) In [K] Kasparov applies  $KK$ -theory to the Novikov conjecture. See [Cu] for a companion bivariant cyclic theory.

Suppose  $p: E \rightarrow B$  is a smooth bundle equipped with a fiberwise elliptic operator, e.g. the fiberwise Euler operator, then the “index bundle” is represented by a continuous map  $B \rightarrow \mathbb{K}^t C_r^* \pi_1(E)$ . The index theorem gives a factorization of this thru the Kasparov assembly map  $\mathbb{H}_\bullet(E, \mathbb{K}^{top} \mathbb{C})$ . Notice the analogy between this and the bivariant RR theorems.

It is natural to ask whether there exist twisted versions of the above three untwisted bivariant theories. In particular a twisted version of  $KK$ -theory might yield a strengthened Atiyah-Singer index theorem for families.

As observed in [FMac, 10.2] a twisted version of  $\Omega$  is given by letting  $\Omega(G \xrightarrow{f} H)$  be the Grothendieck group of finite  $G$  sets which are free as  $\text{Ker } f$  sets. Similarly  $K_0^R(RG_1, RG_2)$  comes from a full twisted theory where  $K_0^R(G \xrightarrow{f} H)$  is the Grothendieck group of finitely generated  $RG$  modules which are projective as  $R\text{Ker } f$  modules.

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