

The Segal–Bargmann Transform for Path-Groups

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Let K be a connected Lie group of compact type and let $W(K)$ denote the set of continuous paths in K , starting at the identity and with time-interval $[0, 1]$. Then $W(K)$ forms an infinite-dimensional group under the operation of pointwise multiplication. Let ρ denote the Wiener measure on $W(K)$. We construct an analog of the Segal–Bargmann transform for $W(K)$. Let $K_{\mathbb{C}}$ be the complexification of K , $W(K_{\mathbb{C}})$ the set of continuous paths in $K_{\mathbb{C}}$ starting at the identity, and μ the Wiener measure on $W(K_{\mathbb{C}})$. Our transform is a unitary map of $L^2(W(K), \rho)$ onto the “holomorphic” subspace of $L^2(W(K_{\mathbb{C}}), \mu)$. By analogy with the classical transform, our transform is given by convolution with the Wiener measure, followed by analytic continuation. We prove that the transform for $W(K)$ is nicely related by means of the Itô map to the classical Segal–Bargmann transform for the path-space in the Lie algebra of K . © 1998 Academic Press

1. INTRODUCTION

Let ρ_t be the function on \mathbb{R}^n given by

$$\rho_t(x) = (2\pi t)^{-n/2} e^{-x^2/2t}. \quad (1)$$

Here $x = (x_1, x_2, \dots, x_n)$ and $x^2 = x_1^2 + x_2^2 + \dots + x_n^2$. Clearly, ρ_t has a (unique) analytic continuation to \mathbb{C}^n . The finite-dimensional Segal–Bargmann transform is a map B_t from $L^2(\mathbb{R}^n, \rho_t(x) dx)$ into $\mathcal{H}(\mathbb{C}^n)$, where

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t is a positive parameter, and $\mathcal{H}(\mathbb{C}^n)$ denotes the space of holomorphic functions on \mathbb{C}^n . The map is given by

$$B_t f(z) = \int_{\mathbb{R}^n} \rho_t(z-x) f(x) dx \quad z \in \mathbb{C}^n, \tag{2}$$

where $\rho_t(z-x)$ refers to the analytic continuation of ρ_t .

For $y \in \mathbb{R}^n$, a change of variable shows that

$$B_t f(y) = \int_{\mathbb{R}^n} f(y-x) \rho_t(x) dx \quad y \in \mathbb{R}^n. \tag{3}$$

Either way, $B_t f$ is the analytic continuation to \mathbb{C}^n of the convolution $\rho_t * f$. Since ρ_t is the fundamental solution at the origin of the heat equation $du/dt = \frac{1}{2} \Delta u$, $\rho_t * f$ is simply the heat operator $e^{t\Delta/2}$ applied to f . So

$$B_t f = \text{analytic continuation of } e^{t\Delta/2}(f). \tag{4}$$

Finally, using the explicit formula (1) in (2), we obtain yet another expression for B_t :

$$B_t f(z) = e^{-z^2/2t} \int_{\mathbb{R}^n} e^{z \cdot x/t} f(x) \rho_t(x) dx. \tag{5}$$

Meanwhile, let us define a measure μ_t on \mathbb{C}^n by

$$d\mu_t(z) = (\pi t)^{-n} e^{-|z|^2/t} dz, \tag{6}$$

where here dz refers to $2n$ -dimensional Lebesgue measure on \mathbb{C}^n . The following result was proved by Bargmann [B] with somewhat different normalizations.

For each $t > 0$, the map B_t is an isometric isomorphism of $L^2(\mathbb{R}^n, \rho_t(x) dx)$ onto $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t)$.

We will use the expression $\mathcal{H}L^2(\mathbb{C}^n, \mu_t)$ as shorthand for $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t)$.

There is an infinite-dimensional version of this transform, due to Segal [Se1, Se2, Se3], in which \mathbb{R}^n is replaced by an infinite-dimensional separable real Hilbert space $H_{\mathbb{R}}$, and \mathbb{C}^n is replaced by the corresponding complex Hilbert space $H_{\mathbb{C}} = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$. (See also [BSZ].) Actually, there are several different infinite-dimensional versions, which are formally equivalent, but which differ in their treatment of technical issues.

The chief technical issue is that in the infinite-dimensional setting the measures $d\rho_t = \rho_t(x) dx$ and μ_t do not make sense as measures on $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$. To resolve this, one must either content oneself with “cylinder set measures” on $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$, or else embed $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$ into larger Banach spaces $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ on which ρ_t and μ_t can be made into honest measures. In the latter case, $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$ are the so-called Cameron–Martin subspaces of $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$.

Of particular interest is the case in which $t=1$ and $H_{\mathbb{R}}$ is the space of absolutely continuous functions $X: [0, 1] \rightarrow \mathbb{R}^n$ with $X(0)=0$ and

$$\int_0^1 \left| \frac{dX}{ds} \right|^2 ds < \infty.$$

As the inner product on $H_{\mathbb{R}}$ we take $\langle X, Y \rangle = \int_0^1 X'(s) \cdot Y'(s) ds$. In this case, $W_{\mathbb{R}}$ may be taken to be the space of all continuous functions $X: [0, 1] \rightarrow \mathbb{R}^n$ with $X(0)=0$. There is a well-defined, countably additive measure ρ_1 on $W_{\mathbb{R}}$ which is formally the infinite-dimensional limit of the measures $\rho_t(x) dx$ on \mathbb{R}^n . This measure is in fact the Wiener measure, that is, the distribution of standard Brownian motion in \mathbb{R}^n , starting at the origin and with time interval $[0, 1]$. We may then take $H_{\mathbb{C}}$ and $W_{\mathbb{C}}$ to be the analogs of $H_{\mathbb{R}}$ and $W_{\mathbb{R}}$ with \mathbb{R}^n replaced by \mathbb{C}^n , in which case μ_1 is the Wiener measure for Brownian motion in $\mathbb{C}^n = \mathbb{R}^{2n}$. (More precisely, μ_1 is the distribution of “half-speed” Brownian motion in \mathbb{C}^n , that is, standard Brownian motion with t replaced by $t/2$.)

Now let K be a (finite-dimensional) compact connected Lie group, with Lie algebra \mathfrak{k} . If we pick once and for all an Ad- K -invariant inner product on \mathfrak{k} , this determines a “Laplacian” operator Δ_K on K , and we may define $\rho_t(x)$ to be the fundamental solution at the identity of the equation $du/dt = \frac{1}{2} \Delta_K u$. Let $K_{\mathbb{C}}$ be the complexification of K , which is a certain connected complex Lie group which contains K as a subgroup. (For the definition, see [H1, Ho].)

According to [H1, Proposition 1], ρ_t has a unique analytic continuation to $K_{\mathbb{C}}$. So for each $t > 0$ we may define a map

$$B_t: L^2(K, \rho_t(x) dx) \rightarrow \mathcal{H}(K_{\mathbb{C}})$$

by analogy to (2) in the \mathbb{R}^n case:

$$B_t f(g) = \int_K \rho_t(gx^{-1}) f(x) dx \quad g \in K_{\mathbb{C}}.$$

Here dx denotes Haar measure on K and $\mathcal{H}(K_{\mathbb{C}})$ denotes the space of holomorphic functions on $K_{\mathbb{C}}$.

For $y \in K$, a change of variable gives

$$B_t f(y) = \int_K f(x^{-1}y) \rho_t(x) dx.$$

Either way, $B_t f$ is the analytic continuation to $K_{\mathbb{C}}$ of the convolution $\rho_t * f$, where the convolution is computed with respect to the group structure of K . Since ρ_t is the fundamental solution of the heat equation, we have, as in \mathbb{R}^n ,

$$B_t f = \text{analytic continuation of } e^{t\Delta_K/2}(f).$$

Since there is no simple formula for the heat kernel on K , there is no explicit formula analogous to (5).

According to Theorem 1' of [H1], there is an appropriately defined heat kernel measure μ_t on $K_{\mathbb{C}}$ such that the following result holds.

For each $t > 0$ the map B_t is an isometric isomorphism of $L^2(K, \rho_t(x) dx)$ onto $\mathcal{H}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \mu_t)$.

Driver [D] has extended this result to Lie groups of compact type, a class which contains both compact Lie groups and \mathbb{R}^n , thus allowing the above result and the finite-dimensional classical transform to be treated in a unified way. (See Section 2.1.) We will use $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$ as shorthand for $\mathcal{H}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \mu_t)$. See [H2, H3, H4] for additional information, [DG, Gr1-5, Hi1-2] for related results, and [A, L] for applications.

The purpose of this paper is to construct a version of the Segal-Bargmann transform which applies to a certain infinite-dimensional group, namely, the group of continuous paths in K , starting at the identity. So let K be a connected Lie group of compact type, and let $W(K)$ denote the set of continuous maps from $[0, 1]$ into K , with zero mapping to the identity. Then $W(K)$ forms a group under the operation of pointwise multiplication. Let ρ denote the Wiener measure on $W(K)$ (Section 2.1). Now let $W(K_{\mathbb{C}})$ denote the set of continuous paths from $[0, 1]$ into $K_{\mathbb{C}}$, starting at the identity, and let μ denote Wiener measure on $W(K_{\mathbb{C}})$. (As in the \mathbb{R}^n case, this Wiener measure is "half speed.") Our transform will be an isometric isomorphism of $L^2(W(K), \rho)$ onto the "holomorphic" subspace of $L^2(W(K_{\mathbb{C}}), \mu)$, denoted $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$. See Section 2.2 for the definition of the holomorphic subspace.

By analogy to the other cases, the transform B is given by convolution with the Wiener measure ρ , followed by analytic continuation, where the convolution is with respect to the group structure on $W(K)$. This simple description of B may be taken literally for cylinder functions, that is, functions which depend only on the value of the path at a finite number of times. We then extend by continuity to general functions.

Consider a cylinder function f on $W(K)$, $f(x) = \psi(x_{t_1}, \dots, x_{t_n})$, where ψ is a function on K^n . Then Bf is also a cylinder function $Bf(g) = \Psi(g_{t_1}, \dots, g_{t_n})$, where Ψ is a holomorphic function on $K_{\mathbb{C}}^n$. Since K^n is itself a Lie group of compact type, it is reasonable to ask whether the map $\psi \rightarrow \Psi$ is a transform of the sort considered in [H1, D]. If $n = 1$ or if K is commutative, then the answer is yes; in all other cases the answer is no. Thus the results of this paper are *not* just an infinite-dimensional limit of [H1, D]. To put it another way, let us think of the Wiener measure as a sort of heat kernel measure on the infinite-dimensional group $W(K)$. Because this measure is not invariant under the adjoint action of $W(K)$, [H1, D] do not encourage us to think that there should be an isometric transform for $W(K)$. Nevertheless, there *is* an isometric transform for $W(K)$, and it *can* be obtained from [H1, D], but not in the obvious way. See the proof of Theorem 3 in Section 3.2.

If $K = \mathbb{R}^n$, then $W(K) = W(\mathbb{R}^n)$, and our transform reduces precisely to the infinite-dimensional classical transform. On the other hand, for any K , the Itô map serves to identify $W(K)$ with $W(\mathfrak{k})$, the space of continuous paths in \mathfrak{k} starting at the origin. Note that $W(\mathfrak{k})$ is nothing but $W(\mathbb{R}^n)$, where $n = \dim \mathfrak{k}$. The Itô map takes the Wiener measure on $W(\mathfrak{k})$ to the Wiener measure on $W(K)$. Similarly, $W(K_{\mathbb{C}})$ may be identified with $W(\mathfrak{k}_{\mathbb{C}})$ by the complex Itô mapping, which takes the Wiener measure on $W(\mathfrak{k}_{\mathbb{C}})$ to the Wiener measure on $W(K_{\mathbb{C}})$. According to Theorems 13 and 17 below, our transform may be computed as follows: identify a function on $W(K)$ with a function on $W(\mathfrak{k})$ by the Itô map, apply the *classical* transform for $W(\mathfrak{k})$, and then interpret the result as a function on $W(K_{\mathbb{C}})$ by the inverse of the complex Itô map.

Thus we have two procedures for computing our transform. The first is direct, in which we convolve with the Wiener measure for $W(K)$ and then analytically continue. The second is indirect, in which we compose with the Itô map, then convolve with the Wiener measure for $W(\mathfrak{k})$ and analytically continue, and then compose with the inverse Itô map. If the Itô map were a group isomorphism, it would be clear that under the Itô map convolution with Wiener measure on $W(K)$ goes over to convolution with Wiener measure on $W(\mathfrak{k})$. Since the Itô map is certainly not an isomorphism (between the commutative group $W(\mathfrak{k})$ and the non-commutative group $W(K)$), it is surprising that the two procedures yield the same result. However, the Itô map is "close" to being a group isomorphism, and this is sufficient. (See Lemmas 21 and 22 in the proof of Theorem 13, and compare with the proof of Theorem 3.)

In the case $t = 1$, Gross and Malliavin [GM1] have shown that the transform B_1 of [H1] can be computed by the following procedure. Given a function ψ on K , construct the function $f(x) = \psi(x_1)$ on $W(K)$. Then compose with the Itô map to get a function on $W(\mathfrak{k})$, and apply the classical

Segal–Bargmann map. Finally compose with the inverse Itô map to get a function on the path-group in $K_{\mathbb{C}}$. They prove that this function is of the form $F(g) = \Psi(g_1)$, and that $\Psi = B_1\psi$. Our transform B is the same as B_1 for functions of the form $f(x) = \psi(x_1)$, since f depends on the value of the path x at only one time. Thus our results can be viewed as an extension of [GM1] to more general f 's. This paper was strongly motivated by the work of Gross and Malliavin. (Technically speaking, Gross and Malliavin work with finite-energy rather than continuous paths on the complex side; see Section 2.5 for the details of the connection between this paper and [GM1].)

While preparing this paper, we received a preprint [Sa2] of a paper of G. Sadasue which overlaps in places with our results. In particular Sadasue constructs an “ S -transform” for functions f on the space of paths in K . Sadasue’s S -transform $S^G f$ coincides with the restriction of our Bf to the real finite-energy path-group. (Compare the integral in our Theorem 11 to the definition of S^G just prior to Prop. 3.1 of [Sa2].) Sadasue then proves our Theorem 13. Moreover, our Theorem 12 was motivated by (but not proved in) [Sa2]. Finally, in connecting our transform with the classical transform, we use a result (Lemma 24) proved in [Sa2, Lem. 5.1].

Nevertheless, our results go beyond Sadasue’s in several respects. In particular, we give an intrinsic construction of a transform that maps into a Hilbert space of holomorphic functions. By contrast, Sadasue’s S^G produces a function defined only on the *real* finite-energy path-group $H(K)$ (P^1G in the notation of [Sa2]). Moreover, Sadasue’s map C^G is *defined* by composing with the Itô map, applying the classical transform and then composing with the inverse Itô map. Our map B is defined directly at the group level, and we give a proof of its isometricity which does not refer to the Itô map.

2. STATEMENT OF RESULTS

2.1. Preliminaries

A Lie group K is said to be of *compact type* if K is locally isomorphic to some compact Lie group \tilde{K} . Thus of course compact groups are of compact type, but also the non-compact group \mathbb{R}^n is of compact type, since it is locally isomorphic to an n -torus. If \mathfrak{k} is the Lie algebra of K , then K is of compact type if and only if there exists an inner product on \mathfrak{k} which is invariant under the adjoint action of K . If K is simply connected and of compact type, then K can be decomposed uniquely as $K = K_1 \times \mathbb{R}^n$, where K_1 is compact and simply connected. See [He, Prop. II.6.6] and [V, Thm. 4.11.7].

So let K be a connected Lie group of compact type and \mathfrak{k} its Lie algebra. Fix once and for all an Ad- K -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} . This inner product determines a bi-invariant Riemannian metric on K , and hence a bi-invariant Laplacian operator Δ_K .

Let

$$W(K) = \{x \in C([0, 1], K) \mid x_0 = e\}$$

be the set of continuous paths in K , starting at the identity. We will let x denote a typical path in $W(K)$ and $x_s \in K$ its value at time s . Note that $W(K)$ forms a group under the operation of pointwise multiplication of paths: $(xy)_s = x_s y_s$.

Let standard Brownian motion in K be the unique stochastic process in K whose infinitesimal generator is $\frac{1}{2}\Delta_K$. The distribution of Brownian motion starting at the identity is a probability measure on $W(K)$, which we will call the *Wiener measure* ρ . More precisely, ρ is a measure on the Borel σ -algebra in $W(K)$, where $W(K)$ is given the topology of uniform convergence. The Wiener measure may also be described in terms of its finite-dimensional distributions, which are given in terms of the heat kernel on K (Section 3.1).

Let

$$H(K) = \left\{ x \in W(K) \mid \begin{array}{l} x \text{ is absolutely continuous,} \\ \text{and } \int_0^1 \left| x_s^{-1} \frac{dx}{ds} \right|^2 ds < \infty \end{array} \right\}.$$

Here $x_s^{-1} dx/ds$ is the derivative of x at time s , pulled back by means of left-translation to $\mathfrak{k} = T_e(K)$. We use matrix group notation, although it is not really necessary to realize K as a matrix group. The norm is computed with respect to our inner product on \mathfrak{k} . The elements of $H(K)$ are called the *finite-energy* paths. It follows from the product rule and the Ad- K -invariance of the inner product on \mathfrak{k} that $H(K)$ is a subgroup of $W(K)$. It is known that $H(K)$ is a set of Wiener measure zero in $W(K)$.

Now let $K_{\mathbb{C}}$ be the complexification of K . Thus $K_{\mathbb{C}}$ is a certain connected complex Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is the complexification of \mathfrak{k} and which contains K as a subgroup. For the definition, see [H1, Sect. 3] or [Ho, Chap. XVII]. We will use the following real-valued inner product on $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$:

$$\langle X_1 + iY_1, X_2 + iY_2 \rangle = \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle$$

for $X_i, Y_i \in \mathfrak{k}$. This inner product is $\text{Ad-}K$ -invariant, but *not* $\text{Ad-}K_{\mathbb{C}}$ -invariant unless K is commutative. This inner product gives rise to a left- (but not right-) invariant Riemannian metric on $K_{\mathbb{C}}$, and hence to a left-invariant Laplacian operator $\Delta_{K_{\mathbb{C}}}$.

Let

$$W(K_{\mathbb{C}}) = \{g \in C([0, 1], K_{\mathbb{C}}) \mid g_0 = e\}.$$

We will consider “half-speed” Brownian motion in $K_{\mathbb{C}}$, that is, the unique process whose infinitesimal generator is $\frac{1}{4}\Delta_{K_{\mathbb{C}}}$. The distributions of this process is the *Wiener measure* μ on $W(K_{\mathbb{C}})$. Let the *finite-energy* paths in $W(K_{\mathbb{C}})$ be

$$H(K_{\mathbb{C}}) = \left\{ g \in W(K_{\mathbb{C}}) \mid \begin{array}{l} g \text{ is absolutely continuous,} \\ \text{and } \int_0^1 \left| g_s^{-1} \frac{dg}{ds} \right|^2 ds < \infty \end{array} \right\}.$$

Even through the inner product on $\mathfrak{k}_{\mathbb{C}}$ is not Ad- invariant, it is nevertheless not hard to prove that $H(K_{\mathbb{C}})$ is a subgroup of $W(K_{\mathbb{C}})$.

2.2. The Transform for Paths in K

Our transform will be a isometry of $L^2(W(K), \rho)$ onto the “holomorphic” subspace of $L^2(W(K_{\mathbb{C}}), \mu)$. Define a *cylinder function* on $W(K)$ to be a function of the form $f(x) = \psi(x_{t_1}, \dots, x_{t_n})$, where ψ is a function on K^n . Define cylinder functions on $W(K_{\mathbb{C}})$ similarly. It is not hard to show that f is measurable if and only if ψ is measurable. A function (on $W(K)$ or $W(K_{\mathbb{C}})$) will be called an L^2 cylinder function if it simultaneously square-integrable (with respect to ρ or μ) and a cylinder function. Standard measure-theoretic results imply that the L^2 cylinder functions are dense in L^2 .

We will say that a cylinder function F on $W(K_{\mathbb{C}})$ is *holomorphic* if $F(g) = \Psi(g_{t_1}, \dots, g_{t_n})$ and Ψ is holomorphic on $K_{\mathbb{C}}^n$. Although the same function F on $W(K_{\mathbb{C}})$ may be expressed as a cylinder function in several different ways, it is not hard to see that the notion of holomorphic is independent of the representation.

DEFINITION 1. Let $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ denote the closure in $L^2(W(K_{\mathbb{C}}), \mu)$ of the L^2 holomorphic cylinder functions.

We will refer to $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ as the *holomorphic subspace* of $L^2(W(K_{\mathbb{C}}), \mu)$, even though elements of $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ are not in general holomorphic on $W(K_{\mathbb{C}})$. We will see (Section 2.3) that in a certain sense

functions in $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ are holomorphic on the finite-energy path-group $H(K_{\mathbb{C}})$.

LEMMA 2. *Suppose that f_1 and f_2 are measurable cylinder functions on $W(K)$ and that $f_1(x) = f_2(x)$ for ρ -almost every x . Then for all $y \in W(K)$, $f_1(x^{-1}y)$ and $f_2(x^{-1}y)$ are measurable with respect to x , and $f_1(x^{-1}y) = f_2(x^{-1}y)$ for ρ -almost every x .*

If f is an L^2 cylinder function on $W(K)$, then for all $y \in W(K)$

$$\int_{W(K)} |f(x^{-1}y)| d\rho(x) < \infty.$$

Note that this lemma is certainly not true for general (non-cylinder) functions, which is why it is necessary to define the transform B initially only on cylinder functions. However, a similar result (Theorem 11) holds for general f , but with y restricted to $H(K)$.

THEOREM 3. *Suppose f is an L^2 cylinder function on $W(K)$. Then there exists a unique L^2 holomorphic cylinder function Bf on $W(K_{\mathbb{C}})$ such that the restriction of Bf to $W(K)$ is given by*

$$Bf(y) = \int_{W(K)} f(x^{-1}y) d\rho(x) \quad y \in W(K). \quad (7)$$

Furthermore, every L^2 holomorphic cylinder function F is of the form $F = Bf$ for a unique L^2 cylinder function f .

For any L^2 cylinder function f ,

$$\|f\|_{L^2(W(K), \rho)} = \|Bf\|_{L^2(W(K_{\mathbb{C}}), \mu)}.$$

The map B extends by continuity to an isometric isomorphism of $L^2(W(K), \rho)$ onto $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$.

Note that, in light of the lemma, the integral (7) makes sense for all $y \in W(K)$.

2.3. The Coherent States and the Restriction Map

The image of the transform B is $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, which is by definition the closure in $L^2(W(K_{\mathbb{C}}), \mu)$ of the L^2 holomorphic cylinder functions. Unfortunately, the functions in $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ are not in general holomorphic on the continuous path-group $W(K_{\mathbb{C}})$, the analyticity on $W(K_{\mathbb{C}})$ being lost in taking the closure. However, analyticity on the finite-energy path-group $H(K_{\mathbb{C}})$ is preserved. That is, if a sequence F_n of holomorphic cylinder functions converges to some F in $L^2(W(K_{\mathbb{C}}), \mu)$ then

the restrictions of the F_n 's to $H(K_C)$ converge pointwise to a holomorphic function, which we will call the "restriction" of F to $H(K_C)$. Note that we cannot interpret "restriction" literally, since $H(K_C)$ is a set of measure zero, and elements of $L^2(W(K_C), \mu)$ are defined only modulo sets of measure zero. We will give a precise definition of "holomorphic" for functions on $H(K_C)$.

If g is a finite-energy path in K_C , then we will prove that evaluation at g , which is a well-defined linear functional on holomorphic cylinder functions, extends by continuity to $\mathcal{H}L^2(W(K_C), \mu)$. By the Riesz theorem, this linear functional is given as the inner product with a unique element χ_g of $\mathcal{H}L^2(W(K_C), \mu)$. We will call the χ_g 's the "coherent states" in $\mathcal{H}L^2(W(K_C), \mu)$ ($g \in H(K_C)$). We will call the vectors $\psi_g = B^{-1}(\chi_g)$ the coherent states in $L^2(W(K), \rho)$. If $g \in H(K)$, then ψ_g is given as the Radon-Nikodym derivative of ρ translated by g , relative to ρ itself.

THEOREM 4. For all $g \in H(K_C)$ and for all L^2 holomorphic cylinder functions F ,

$$|F(g)|^2 \leq c_g \|F\|_{L^2(W(K_C), \mu)}^2,$$

where

$$c_g = \exp \left(\int_0^1 \left| g_t^{-1} \frac{dg}{dt} \right|^2 dt \right).$$

DEFINITION 5. For each $g \in H(K_C)$, let χ_g be the unique element of $\mathcal{H}L^2(W(K_C), \mu)$ such that

$$F(g) = \langle \chi_g, F \rangle$$

for all L^2 holomorphic cylinder functions F . We will refer to the χ_g 's as the *coherent states* in $\mathcal{H}L^2(W(K_C), \mu)$.

DEFINITION 6. For $F \in \mathcal{H}L^2(W(K_C), \mu)$, define the *restriction of F to $H(K_C)$* to be the function RF on $H(K_C)$ given by

$$RF(g) = \langle \chi_g, F \rangle \quad g \in H(K_C).$$

Note that we are adopting the convention that the inner product be linear on the right and conjugate-linear on the left.

We will show in the next section that the restriction map is one-to-one on $\mathcal{H}L^2(W(K_C), \mu)$, which is equivalent to saying that the coherent states span a dense subspace of $\mathcal{H}L^2(W(K_C), \mu)$.

THEOREM 7. For each $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, the restriction RF of F to $H(K_{\mathbb{C}})$ is holomorphic in the sense of [GM1].

Gross and Malliavin's definition of holomorphic is as follows [GM1, Def. 3.3 and Def. 5.3]. Let $H(\mathfrak{k}_{\mathbb{C}})$ denote the Hilbert space of finite-energy paths in $\mathfrak{k}_{\mathbb{C}}$. A function $F: H(\mathfrak{k}_{\mathbb{C}}) \rightarrow \mathbb{C}$ is said to be holomorphic if at each point of $H(\mathfrak{k}_{\mathbb{C}})$ F is Fréchet differentiable and the Fréchet derivative is complex-linear. This is equivalent to requiring that F be locally bounded and that for each $Z^1, Z^2 \in H(\mathfrak{k}_{\mathbb{C}})$, $F(Z^1 + aZ^2)$ be holomorphic as a function of $a \in \mathbb{C}$. (See [HP].)

A function $F: H(K_{\mathbb{C}}) \rightarrow \mathbb{C}$ is then said to be holomorphic if the following two conditions hold. First, we require that for each $g \in H(K_{\mathbb{C}})$, the map $Z \rightarrow F(ge^Z)$ be Fréchet differentiable at $Z=0$ with complex-linear derivative. Second, we require that the Fréchet derivative at $Z=0$ of $Z \rightarrow F(ge^Z)$ be continuous, as a function of g with values in $H(\mathfrak{k}_{\mathbb{C}})^*$. Here e^Z is the pointwise exponential of a finite-energy path in $\mathfrak{k}_{\mathbb{C}}$, which is automatically a finite-energy path in $K_{\mathbb{C}}$.

If we think of $\mathfrak{k}_{\mathbb{C}}$ as a commutative Lie group, then we have two apparently different definitions of what it means for a function on $H(\mathfrak{k}_{\mathbb{C}})$ to be holomorphic, the group definition and the Lie algebra definition. These differ only in the requirement that the derivative be continuous, which is automatically true [HP, Thm. 3.17.1], so in fact the definitions are equivalent.

The following result shows that it makes sense to construct a holomorphic function on $H(K_{\mathbb{C}})$ by first defining it on $H(K)$ and then analytically continuing.

THEOREM 8. A holomorphic function on $H(K_{\mathbb{C}})$ is determined by its values on $H(K)$.

DEFINITION 9. For $g \in H(K_{\mathbb{C}})$, we define the *coherent state* $\psi_g \in L^2(W(K), \rho)$ to be

$$\psi_g = B^{-1}(\chi_g).$$

If $y \in W(K)$, the measure $d\rho(xy^{-1})$ will be define by

$$\int_{W(K)} f(x) d\rho(xy^{-1}) = \int_{W(K)} f(xy) d\rho(x),$$

where f is a bounded measurable function on $W(K)$. This notation is supposed to suggest a formal change of variables $x \rightarrow xy$ on the left.

THEOREM 10. *If $y \in H(K)$, then $d\rho(xy^{-1})$ is absolutely continuous with respect to $d\rho(x)$, and the coherent state ψ_y is given by*

$$\psi_y(x) = \frac{d\rho(xy^{-1})}{d\rho(x)}.$$

In particular, $d\rho(xy^{-1})/d\rho(x)$ is ρ -square-integrable as a function of x .

It is easy to obtain this result formally, using the fact that the Wiener measure ρ on $W(K)$ is invariant under the map $x \rightarrow x^{-1}$ (Lemma 20).

For cylinder functions f , the Segal–Bargmann transform Bf is computed as convolution with the Wiener measure, followed by analytic continuation. Unfortunately, this definition does not make sense for general functions f , because Lemma 2 fails in general. However, the original prescription can be taken literally if we compute the restriction of Bf to $H(K_{\mathbb{C}})$.

COROLLARY 11. *If $f \in L^2(W(K), \rho)$ then RBf is the unique holomorphic function on $H(K_{\mathbb{C}})$ whose restriction to $H(K)$ is given by*

$$RBf(y) = \int_{W(K)} f(x^{-1}y) d\rho(x) \quad y \in H(K).$$

Note that the integral is well-defined in light of Theorem 10.

For $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ the next theorem shows that RF can be computed as the convolution of F on the right with the Wiener measure μ . However, we must restrict to the finite-energy path-group in K , because μ is quasi-invariant on the right only under $H(K)$ and not under $H(K_{\mathbb{C}})$. Fortunately, Theorem 8 tells us that computing on $H(K)$ is sufficient. This formula for RF will be used in the proofs of the results of Section 2.4.

Formally, Theorem 12 holds because convolution with the Wiener measure is a sort of heat operator, and holomorphic functions are automatically harmonic, which means that the heat operator acts as the identity.

THEOREM 12. *Suppose $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$. Then for all $y \in H(K)$*

$$RF(y) = \int_{W(K_{\mathbb{C}})} F(gy) d\mu(g) \quad y \in H(K). \tag{8}$$

In particular the integral is well-defined.

In the case $K = \mathbb{R}^n$ (in which case we may take $y \in H(\mathbb{C}^n)$), this formula for the restriction map has been used by H. Sugita [Su1, Su2]. In the general case, the right side of (8) coincides with the definition of the map

S' in [Sa2, Sec. 3]. However, Sadasue does not prove that S' is the identity on holomorphic cylinder functions.

2.4. The Itô Mapping

The purpose of this section is to relate the transform for paths in K with the classical Segal–Bargmann transform for paths in the Lie algebra \mathfrak{k} . The link between the two is given by the Itô map, which we will describe shortly. For the most part, the Itô map serves merely to relate the transform in K to the transform in \mathfrak{k} . In particular, none of the proofs of results in Sections 2.2 or 2.3 relies on the Itô map. However, we have in this section one result whose *statement* does not involve the Itô map, but whose *proof* does, namely, that the restriction map is one-to-one on $\mathcal{HL}^2(W(K_{\mathbb{C}}), \mu)$ (Corollary 14). It would be desirable to find an intrinsic proof of this result. However, the proof of the classical result cannot easily be translated into the group setting, because there are no obvious finite-dimensional subgroups of the path-group $W(K_{\mathbb{C}})$.

Note that the Lie algebra \mathfrak{k} is a (commutative) Lie group of compact type under the operation of vector addition, and that $\mathfrak{k}_{\mathbb{C}}$ is the complexification in the group sense of \mathfrak{k} . Thus all of the constructs and results of this paper are applicable with K and $K_{\mathbb{C}}$ replaced by \mathfrak{k} and $\mathfrak{k}_{\mathbb{C}}$. In particular $H(\mathfrak{k})$, $W(\mathfrak{k})$, $H(\mathfrak{k}_{\mathbb{C}})$, and $W(\mathfrak{k}_{\mathbb{C}})$ are all defined in the obvious way. To avoid confusion, we will let p denote the Wiener measure on $W(\mathfrak{k})$, and m the “half speed” Wiener measure on $W(\mathfrak{k}_{\mathbb{C}})$. We will let $S: L^2(W(\mathfrak{k}), p) \rightarrow \mathcal{HL}^2(W(\mathfrak{k}_{\mathbb{C}}), m)$ denote the Segal–Bargmann transform for paths in the Lie algebra. The restriction map from $\mathcal{HL}^2(W(\mathfrak{k}_{\mathbb{C}}), m)$ to the space of holomorphic functions on $H(\mathfrak{k}_{\mathbb{C}})$ will still be denoted R . Note that the map S_1 of [GM1] is the same as what we would here call RS .

We have four versions of the Itô map, one defined on each of $H(\mathfrak{k})$, $W(\mathfrak{k})$, $H(\mathfrak{k}_{\mathbb{C}})$, and $W(\mathfrak{k}_{\mathbb{C}})$. The Itô map for $H(\mathfrak{k})$ will be denoted θ , and it is defined as follows. For $X \in H(\mathfrak{k})$, let x be the unique solution to the differential equation

$$dx_t = x_t dX_t$$

with $x_0 = e$. Equivalently,

$$x_t^{-1} \frac{dx}{dt} = \frac{dX}{dt}.$$

Then $x = \theta(X)$. This map takes $H(\mathfrak{k})$ injectively onto $H(K)$, and the inverse map is given by

$$X_t = \int_0^t x_s^{-1} \frac{dx}{ds} ds.$$

The Itô map $\tilde{\theta}$ for $W(K)$ is defined as the solution to the Stratonovich stochastic differential equation

$$dx_t = x_t \circ dX_t.$$

This map is defined for p -almost every path X_t in $W(\mathfrak{f})$, and it takes $W(\mathfrak{f})$ injectively onto $W(K)$, modulo a set of measure zero. The map is measurable and is measure preserving from $(W(\mathfrak{f}), p)$ onto $(W(K), \rho)$. Formally, θ is the restriction of $\tilde{\theta}$ to $H(\mathfrak{f})$, but this has no precise meaning, since $H(\mathfrak{f})$ is a set of p -measure zero.

We may analogously define the Itô maps $\theta_{\mathbb{C}}$ on $H(\mathfrak{f}_{\mathbb{C}})$ and $\tilde{\theta}_{\mathbb{C}}$ on $W(\mathfrak{f}_{\mathbb{C}})$. Then $\theta_{\mathbb{C}}$ takes $H(\mathfrak{f}_{\mathbb{C}})$ injectively onto $H(K_{\mathbb{C}})$, and $\tilde{\theta}_{\mathbb{C}}$ is a measure-preserving map of $(W(\mathfrak{f}_{\mathbb{C}}), m)$ onto $(W(K_{\mathbb{C}}), \mu)$. In both the real and complex cases, a tilde indicates the stochastic version of the map. The map θ is the restriction of $\theta_{\mathbb{C}}$ to $H(\mathfrak{f})$. According to [GM1, Thm. 5.7], a function F on $H(K_{\mathbb{C}})$ is holomorphic if and only if $F \circ \theta_{\mathbb{C}}$ is holomorphic on $H(\mathfrak{f}_{\mathbb{C}})$. (Gross and Malliavin prove this only for the case K compact, but the proof is the same if K is of compact type.)

Since $\tilde{\theta}$ is measure-preserving and (essentially) one-to-one and onto, composition with $\tilde{\theta}$ is an isometry of $L^2(W(K), \rho)$ onto $L^2(W(\mathfrak{f}), p)$. Similarly, composition with $\tilde{\theta}_{\mathbb{C}}$ is an isometry of $L^2(W(K_{\mathbb{C}}), \mu)$ onto $L^2(W(\mathfrak{f}_{\mathbb{C}}), m)$. The main technical result of this section is Theorem 16, which says that composition with $\tilde{\theta}_{\mathbb{C}}$ takes the holomorphic subspace of $L^2(W(K_{\mathbb{C}}), \mu)$ onto the holomorphic subspace of $L^2(W(\mathfrak{f}_{\mathbb{C}}), m)$.

THEOREM 13. *For all $f \in L^2(W(K), \rho)$,*

$$(Rf) \circ \theta_{\mathbb{C}} = RS(f \circ \tilde{\theta}).$$

COROLLARY 14. *The restriction map R is one-to-one on $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$. The coherent states $\{\psi_g\}$ span a dense subspace of $L^2(W(K), \rho)$. The coherent states $\{\chi_g\}$ span a dense subspace of $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$.*

COROLLARY 15. *Suppose that F is holomorphic on $H(K_{\mathbb{C}})$. Then F is of the form $F = R\tilde{F}$ for $\tilde{F} \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ if and only if*

$$\|F \circ \theta_{\mathbb{C}}\|_m < \infty,$$

in which case \tilde{F} is unique and

$$\|\tilde{F}\|_{L^2(W(K_{\mathbb{C}}), \mu)} = \|F \circ \theta_{\mathbb{C}}\|_m.$$

Here $\|\cdot\|_m$ refers to the norm defined in [GM1, Eq. (3.14)], with $t = 1$.

The norm used in [GM1] (and many other papers) is defined as follows. Take a holomorphic function G on $H(\mathfrak{k}_\mathbb{C})$, restrict G to a finite-dimensional subspace V , and then compute the L^2 norm of G with respect to a suitably normalized Gaussian measure on V . The supremum of this quantity over all finite-dimensional subspaces V is defined to be $\|G\|_m$. As it turns out, $\|\cdot\|_m$ is a norm on the space of all holomorphic functions G for which $\|G\|_m$ is finite, and this space is complete. Moreover, the restriction map R is an isometric isomorphism of $\mathcal{H}L^2(W(\mathfrak{k}_\mathbb{C}), m)$ onto the space of all G 's in $\mathcal{H}(H(\mathfrak{k}_\mathbb{C}))$ for which $\|G\|_m < \infty$. These results are known collectively as the Skeleton Theorem, reflecting the fact that a function in the holomorphic subspace of $L^2(W(\mathfrak{k}_\mathbb{C}), m)$ is related in a nice way to its "skeleton"—its restriction to the finite-energy subspace—a fact which is, to quote Gross and Malliavin, "notoriously false in the absence of holomorphy." For more information see [GM1, Rem. 4.5] or [BSZ].

The two corollaries above represent a form of the Skeleton Theorem for the path-group in $K_\mathbb{C}$. However, it would be helpful to have a definition for the norm of a holomorphic function on $H(K_\mathbb{C})$ which does involve the Itô mapping. Further study of this issue is merited.

THEOREM 16. *Let F be in $L^2(W(K_\mathbb{C}), \mu)$, so that $F \circ \tilde{\theta}_\mathbb{C} \in L^2(W(\mathfrak{k}_\mathbb{C}), m)$. Then $F \in \mathcal{H}L^2(W(K_\mathbb{C}), \mu)$ if and only if $F \circ \tilde{\theta}_\mathbb{C} \in \mathcal{H}L^2(W(\mathfrak{k}_\mathbb{C}), m)$.*

THEOREM 17. *The following diagram is well-defined and commutative:*

$$\begin{array}{ccccc}
 L^2(W(\mathfrak{k}), \rho) & \xleftarrow{S} & \mathcal{H}L^2(W(\mathfrak{k}_\mathbb{C}), m) & \xrightarrow{R} & \mathcal{H}(H(\mathfrak{k}_\mathbb{C})) \\
 \uparrow \tilde{\theta} & & \uparrow \tilde{\theta}_\mathbb{C} & & \uparrow \theta_\mathbb{C} \\
 L^2(W(K), \rho) & \xleftarrow{B} & \mathcal{H}L^2(W(K_\mathbb{C}), \mu) & \xrightarrow{R} & \mathcal{H}(H(K_\mathbb{C}))
 \end{array}$$

2.5. The Transform for K , Revisited

The statements of this section are already implicit in earlier sections, but it is worth making them explicit. If ψ is a function on K , then for any $t \in [0, 1]$ we can construct the cylinder function $\psi(x_t)$ on $W(K)$. Applying the transform B to this cylinder function yields a cylinder function $\Psi(g_t)$ on $W(K_\mathbb{C})$, where Ψ is a holomorphic function on $K_\mathbb{C}$. Since the distribution of x_t with respect to the Wiener measure ρ is the heat kernel measure ρ_t on K (Section 3.1), Ψ will be precisely $B_t \psi$, where B_t is the transform of [H1, D]. The L^2 norm of $\psi(x_t)$ with respect to ρ is the same as the L^2 norm of ψ with respect to ρ_t . Similarly, the L^2 norm of $\Psi(g_t)$ with respect to μ is the same as the L^2 norm of Ψ with respect to the heat kernel measure μ_t .

Meanwhile, we may compose $\psi(x_t)$ with the Itô map to get a function $\psi(\tilde{\theta}(X)_t)$ on $W(\mathfrak{f})$. This function is not a cylinder function on $W(\mathfrak{f})$. Similarly we may compose $\Psi(g_t)$ with either $\tilde{\theta}_c$ or θ_c to get functions $\Psi(\tilde{\theta}_c(Z)_t)$ and $\Psi(\theta_c(Z)_t)$ on $W(\mathfrak{f}_c)$ or $H(\mathfrak{f}_c)$. Applying Theorem 17 to functions of this form, and noting that $R(\Psi(g_t)) = \Psi(g_t)$, since $\Psi(g_t)$ is a cylinder function, we obtain the following result.

THEOREM 18. *The following diagram is well-defined and commutative, and all maps are isometric.*

$$\begin{array}{ccccc}
 L^2(W(\mathfrak{f}), p) & \xleftarrow{S} & \mathcal{H}L^2(W(\mathfrak{f}_c), m) & \xrightarrow{R} & \mathcal{H}(H(\mathfrak{f}_c)) \\
 \uparrow & & \uparrow & \nearrow & \\
 L^2(K, \rho_t) & \xleftarrow{B_t} & \mathcal{H}L^2(K_c, \mu_t) & &
 \end{array}$$

Here the vertical maps are, respectively, $\psi \rightarrow \psi(\tilde{\theta}(X)_t)$, $\Psi \rightarrow \Psi(\tilde{\theta}_c(Z)_t)$, and $\Psi \rightarrow \Psi(\theta_c(Z)_t)$, and the norm on $\mathcal{H}(H(\mathfrak{f}_c))$ is $\|\cdot\|_m$. The map B_t is the one of [H1, D].

Recall the definition of $\|\cdot\|_m$ from Section 2.3. If $t = 1$, then Gross and Malliavin have proved in [GM1, Cors. 7.9, 7.12] that the above diagram would commute if the middle vertical arrow were removed. Note that our map RS coincides with the map S_1 in [GM1]. So we have gone beyond [GM1] in two ways: first by allowing any time $t \leq 1$ and second by including the middle vertical arrow. Note that Gross and Malliavin consider only the “smooth” Itô map θ_c on the complex side; the map $\tilde{\theta}_c$ does not occur in [GM1]. The advantage of using θ_c is that it is reasonably easy to show that $\Psi(\theta_c(Z)_t)$ is holomorphic as a function of $Z \in H(\mathfrak{f}_c)$. The disadvantage is that it is not clear why the m -norm of $\Psi(\theta_c(Z)_t)$ should be the same as the L^2 norm of Ψ . This equality of norms is proved in [GM1] using the deep results of [DG]. The advantage of using $\tilde{\theta}_c$ is that since $\tilde{\theta}_c$ is measure preserving, it is immediate that the L^2 norm of $\Psi(\tilde{\theta}_c(Z)_t)$ is the same as the L^2 norm of $\Psi(g_t)$ and hence the same as the L^2 norm of Ψ . The disadvantage is that it is difficult to prove that $\Psi(\tilde{\theta}_c(Z)_t)$ is in the holomorphic subspace of $L^2(W(\mathfrak{f}), m)$. In fact, our proof of this (taken from [Sa2]) makes use of the density theorem of [H1, D]. Since the nominal purpose of [GM1] is to rederive the results of [H1] by means of stochastic analysis, it would have been undesirable to use the density theorem in [GM1]. (See Remark 6.2 of [GM1].)

We should emphasize that although the ergodicity results of [Gr1] (reproved in a more direct way in [Sa1]) play an important role in both [GM1] and [Sa2], we have made no use of ergodicity in the proof of Theorem 18. We have managed to avoid ergodicity because we have the

Segal–Bargmann transform B for the path-space in K , and B manifestly takes functions of the form $\psi(x_t)$ to functions of the form $\Psi(g_t)$. On the other hand, we have had to use the density results of [H1, D] to show that the middle vertical arrow is well-defined.

3. PROOFS

3.1. Preliminaries

For each partition $0 \leq t_1 < t_2 < \dots < t_n \leq 1$ we may define a projection from $W(K)$ into K^n by $x \rightarrow (x_{t_1}, \dots, x_{t_n})$. The push-forward of ρ to K^n under this projection is a probability measure on K^n , which in probabilistic language we call the joint distribution of x_{t_1}, \dots, x_{t_n} . We will denote this measure by ρ_{t_1, \dots, t_n} . The measure ρ_{t_1, \dots, t_n} is absolutely continuous with respect to Haar measure on K^n , and in a slight abuse of notation we will denote its density by ρ_{t_1, \dots, t_n} , the context making it clear whether we are thinking of ρ_{t_1, \dots, t_n} as a function or as a measure. Explicitly, the density is given by

$$\rho_{t_1, \dots, t_n}(x_1, \dots, x_n) = \rho_{t_1}(x_1) \rho_{t_2-t_1}(x_1^{-1}x_2) \cdots \rho_{t_n-t_{n-1}}(x_{n-1}^{-1}x_n), \tag{9}$$

where ρ_t is the heat kernel on K [MM]. Although ρ_t is a class function on K , ρ_{t_1, \dots, t_n} is not a class function on K^n (unless K is commutative). Nevertheless, $\rho_{t_1, \dots, t_n}(x_1^{-1}, \dots, x_n^{-1}) = \rho_{t_1, \dots, t_n}(x_1, \dots, x_n)$, as follows from the identities $\rho_t(xy) = \rho_t(yx)$ and $\rho_t(x^{-1}) = \rho_t(x)$.

Similarly we may define the measure μ_{t_1, \dots, t_n} on $K_{\mathbb{C}}^n$ to be the joint distribution of g_{t_1}, \dots, g_{t_n} with respect to μ . This measure has a density with respect to Haar measure on $K_{\mathbb{C}}^n$ given by

$$\mu_{t_1, \dots, t_n}(g_1, \dots, g_n) = \mu_{t_1}(g_1) \mu_{t_2-t_1}(g_1^{-1}g_2) \cdots \mu_{t_n-t_{n-1}}(g_{n-1}^{-1}g_n). \tag{10}$$

Since the μ_i 's are not in general class functions, this measure is not the same as the one in which $g_{i-1}^{-1}g_i$ is replaced by $g_i g_{i-1}^{-1}$, reflecting the fact that our Brownian motion in $K_{\mathbb{C}}$ is left- but not right-invariant.

Both densities ρ_{t_1, \dots, t_n} and μ_{t_1, \dots, t_n} are strictly positive everywhere.

3.2. The Transform for Paths in K

Proof of Lemma 2. For each $y \in W(K)$, the map $x \rightarrow x^{-1}y$ is a continuous, hence measurable, map of $W(K)$ to itself. Thus $f_1(x^{-1}y)$ and $f_2(x^{-1}y)$ are measurable. By taking a common refinement, we may assume that f_1 and f_2 are cylinder functions based on the same partition t_1, \dots, t_n ,

so $f_1(x) = \psi_1(x_{t_1}, \dots, x_{t_n})$ and $f_2(x) = \psi_2(x_{t_1}, \dots, x_{t_n})$. If $f_1 = f_2$ a.e. with respect to ρ , then $\psi_1 = \psi_2$ a.e. with respect to ρ_{t_1, \dots, t_n} . The strict positivity of the density ρ_{t_1, \dots, t_n} and the invariance of Haar measure on K^n under inversion and left-multiplication show that for each $y \in K^n$, $\psi_1(x^{-1}y) = \psi_2(x^{-1}y)$ for ρ_{t_1, \dots, t_n} -almost every x . Thus for every $y \in W(K)$, $f_1(x^{-1}y) = f_2(x^{-1}y)$ for ρ -almost every x . So it remains only to address the convergence of the integral in the lemma.

If \tilde{K} is the universal cover of K , then \tilde{K} is again of compact type, and has its own Wiener measure $\tilde{\rho}$. Here we identify the Lie algebras of K and \tilde{K} and use the same inner product in the construction of ρ and $\tilde{\rho}$. If we use the canonical projection $\pi: \tilde{K} \rightarrow K$ to push forward $\tilde{\rho}$ from $W(\tilde{K})$ to $W(K)$, it is easy to see that the resulting measure is ρ . Moreover, if f is a cylinder function on $W(K)$, then $f \circ \pi$ is cylinder function on $W(\tilde{K})$. So it suffices to prove convergence on $W(\tilde{K})$.

Now, \tilde{K} is of the form $\tilde{K} = K_1 \times \mathbb{R}^d$, where K_1 is compact and where the Lie algebras of K_1 and \mathbb{R}^d are orthogonal with respect to the invariant inner product on $\mathfrak{k} = \text{Lie}(K) = \text{Lie}(\tilde{K})$. Thus the heat kernel on \tilde{K} factors. Since f is a cylinder function, we need to prove that the integral

$$\int_{\tilde{K}^n} |\psi(x^{-1}y)| \tilde{\rho}_{t_1, \dots, t_n}(x) dx = \int_{\tilde{K}^n} |\psi(x)| \tilde{\rho}_{t_1, \dots, t_n}(yx^{-1}) dx$$

is finite for all $y \in \tilde{K}^n$ and for all $\psi \in L^2(\tilde{K}^n, \tilde{\rho}_{t_1, \dots, t_n})$. For this it is sufficient to show that

$$\frac{\tilde{\rho}_{t_1, \dots, t_n}(yx^{-1})}{\tilde{\rho}_{t_1, \dots, t_n}(x)} \tag{11}$$

is square-integrable in x with respect to $\tilde{\rho}_{t_1, \dots, t_n}$.

But the function in (11) factors, and so it suffices to prove convergence for the two cases K_1 and \mathbb{R}^d . Square-integrability of (11) in the compact case follows from compactness and strict positivity. Square-integrability in the \mathbb{R}^d case is a simple explicit computation—the function in (11) grows only exponentially in x for each y , and so is square-integrable with respect to a Gaussian measure. ■

Proof of Theorem 3. Two holomorphic functions on $K^n_{\mathbb{C}}$ which are equal on K^n must be equal on all of $K^n_{\mathbb{C}}$. (See the proof of [V, Lem. 4.11.13].) It follows that a holomorphic cylinder function on $W(K_{\mathbb{C}})$ is determined by its values on $W(K)$. So if f is an L^2 cylinder function on $W(K)$ there can be at most one holomorphic cylinder function on $W(K_{\mathbb{C}})$ whose values on $W(K)$ are given by the integral in Theorem 3.

To show that there is at least one such holomorphic cylinder function, let $f(x) = \psi(x_{t_1}, \dots, x_{t_n})$ be an L^2 cylinder function on $W(K)$. Then the function

$$F(y) = \int_{W(K)} f(x^{-1}y) d\rho(x)$$

on the right side of (7) is also a cylinder function $F(y) = \Psi(y_{t_1}, \dots, y_{t_n})$, where

$$\Psi(y_1, \dots, y_n) = \int_{K^n} \psi(x_1^{-1}y_1, \dots, x_n^{-1}y_n) \rho_{t_1, \dots, t_n}(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

For the rest of this proof, let x stand for a variable $x = (x_1, \dots, x_n)$ in K^n . So we may write $\Psi(y) = \int_{K^n} \psi(x^{-1}y) \rho_{t_1, \dots, t_n}(x) dx$. Making the change of variable $x \rightarrow yx^{-1}$ and noting that by (9) and [H1, Prop. 1] and [D, Cor. 4.6] ρ_{t_1, \dots, t_n} has an analytic continuation to $K_{\mathbb{C}}^n$, we see that Ψ has an analytic continuation to $K_{\mathbb{C}}^n$ given by

$$\Psi(g) = \int_{K^n} \rho_{t_1, \dots, t_n}(gx^{-1}) \psi(x) dx \quad g \in K_{\mathbb{C}}^n.$$

The convergence of the integral is proved as in the proof of Lemma 2, and the analyticity of Ψ is proved by Morera's Theorem.

We now need to establish isometricity, which amounts to saying that the norm of ψ in $L^2(K^n, \rho_{t_1, \dots, t_n})$ is the same as the norm of Ψ in $L^2(K_{\mathbb{C}}^n, \mu_{t_1, \dots, t_n})$. To this end, let us consider the map $L: K_{\mathbb{C}}^n \rightarrow K_{\mathbb{C}}^n$ given by

$$L(g_1, g_2, \dots, g_n) = (g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n). \tag{12}$$

This map is a biholomorphism of $K_{\mathbb{C}}^n$ whose inverse is given by

$$L^{-1}(a_1, a_2, \dots, a_n) = (a_1, a_1a_2, \dots, a_1a_2 \cdots a_n).$$

We will call the g_i 's the direct coordinates on $K_{\mathbb{C}}^n$ and the a_i 's the incremental coordinates. The map L preserves the Haar measure on $K_{\mathbb{C}}^n$, as can be seen by making successive changes of variable. Furthermore, the restriction of L to K^n is a diffeomorphism of K^n which preserves the Haar measure on K^n . Note that L converts the function ρ_{t_1, \dots, t_n} into a product function:

$$\rho_{t_1, \dots, t_n}(x) = \rho_{t_1}(u_1) \rho_{t_2-t_1}(u_2) \cdots \rho_{t_n-t_{n-1}}(u_n) \quad u = L(x). \tag{13}$$

The map L is not a group homomorphism, unless K (and therefore $K_{\mathbb{C}}$) is commutative. However, L is "close" to being a homomorphism. Let

$\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$ be the product function on the right in (13). Then we have the following.

LEMMA 19. For all $x, y \in K^n$

$$(\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}})(L(xy^{-1})) = (\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}})(L(x)L(y)^{-1}).$$

The same formula holds on $K_{\mathbb{C}}^n$ for the analytic continuation of $\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$.

Proof. Writing things out, the desired identity is

$$\begin{aligned} &\rho_{t_1}(x_1 y_1^{-1}) \rho_{t_2 - t_1}(y_1 x_1^{-1} x_2 y_2^{-1}) \cdots \rho_{t_n - t_{n-1}}(y_{n-1} x_{n-1}^{-1} x_n y_n^{-1}) \\ &= \rho_{t_1}(x_1 y_1^{-1}) \rho_{t_2 - t_1}(x_1^{-1} x_2 y_2^{-1} y_1) \cdots \rho_{t_n - t_{n-1}}(x_{n-1}^{-1} x_n y_n^{-1} y_{n-1}), \end{aligned}$$

which follows from the fact that each ρ_s is a class function. By uniqueness of analytic continuation, this identity continues to hold on $K_{\mathbb{C}}^n$. ■

Our strategy to now simple. We first write down the integral that defines the norm of ψ , and convert to incremental coordinates. We then apply the isometric transform of [H1] (as generalized to groups of compact type in [D]) in each variable, and then convert back to direct coordinates. The transform in incremental coordinates is just convolution with $\rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}$, followed by analytic continuation. Our lemma shows that upon conversion back to direct coordinates, the transform is convolution with ρ_{t_1, \dots, t_n} , followed by analytic continuation.

So

$$\begin{aligned} &\int_{K^n} |\psi(x)|^2 \rho_{t_1, \dots, t_n}(x) dx \\ &= \int_{K^n} |\psi \circ L^{-1}(u)|^2 \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}(u) du \\ &= \int_{K_{\mathbb{C}}^n} \left| \int_{K^n} \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}(au^{-1}) \psi \circ L^{-1}(u) du \right|^2 \\ &\quad \times \mu_{t_1} \otimes \cdots \otimes \mu_{t_n - t_{n-1}}(a) da \\ &= \int_{K_{\mathbb{C}}^n} \left| \int_{K^n} \rho_{t_1} \otimes \cdots \otimes \rho_{t_n - t_{n-1}}(L(g)L(x)^{-1}) \psi(x) dx \right|^2 \\ &\quad \times \mu_{t_1, \dots, t_n}(g) dg. \end{aligned}$$

Between the second and third lines we use [H1, D]. But applying the lemma in the last line gives

$$\begin{aligned} \int_{K^n} |\psi(x)|^2 \rho_{t_1, \dots, t_n}(x) dx &= \int_{K_{\mathbb{C}}^n} \left| \int_{K^n} \rho_{t_1, \dots, t_n}(gx^{-1}) \psi(x) dx \right|^2 \mu_{t_1, \dots, t_n}(g) dg \\ &= \int_{K_{\mathbb{C}}^n} |\Psi(g)|^2 \mu_{t_1, \dots, t_n}(g) dg. \end{aligned}$$

This establishes the desired isometricity.

Meanwhile, suppose that Ψ is any holomorphic function on $K_{\mathbb{C}}^n$ which is square-integrable with respect to μ_{t_1, \dots, t_n} . Then $\Psi \circ L^{-1}$ is a holomorphic function on $K_{\mathbb{C}}^n$ which is square-integrable with respect to $\mu_{t_1} \otimes \dots \otimes \mu_{t_n - t_{n-1}}$. By the surjectivity of the transform of [H1, D], there exists ϕ in $L^2(K^n, \rho_{t_1} \otimes \dots \otimes \rho_{t_n - t_{n-1}})$ such that $\Psi \circ L^{-1}$ is the analytic continuation of the convolution of ϕ with $\rho_{t_1} \otimes \dots \otimes \rho_{t_n - t_{n-1}}$. Then $\psi = \phi \circ L$ will be in $L^2(K^n, \rho_{t_1, \dots, t_n})$. By the above argument, our transform takes ψ to Ψ . Thus every L^2 holomorphic cylinder function is the image under B of an L^2 cylinder function, which is necessarily unique, since B is isometric.

Finally, we recall that cylinder functions are dense in $L^2(W(K), \rho)$ and that by definition holomorphic cylinder functions are dense in $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$. So B extends to an isometric isomorphism. ■

3.3. The Coherent States and the Restriction Map

The main issue is the estimate in Theorem 4 for the norm of the pointwise evaluation functional, of an L^2 holomorphic cylinder function F at a finite-energy path g . We obtain this estimate by applying the pointwise bounds of Driver [D].

Proof of Theorem 4. To begin with, let us assume that there is a partition t_1, \dots, t_n such that on each interval $[t_i, t_{i+1}]$, g_t is a constant-speed length-minimizing geodesic, with respect to our left-invariant Riemannian metric on $K_{\mathbb{C}}$. Now fix an L^2 holomorphic cylinder function F . By passing to a common refinement, we may assume that g is piecewise geodesic on the same partition on which F is based. If L is the change of variable (12) in the previous section, then $\Psi \circ L^{-1} \in \mathcal{H}L^2(K_{\mathbb{C}}^n, \mu_{t_1} \otimes \dots \otimes \mu_{t_n - t_{n-1}})$, and $\|\Psi \circ L^{-1}\| = \|\Psi\|$. Thus by Cor. 5.5 (together with Thm. 5.7 and Cor. 5.9) of [D],

$$|\Psi \circ L^{-1}(a_1, \dots, a_n)|^2 \leq \|\Psi\|^2 \exp \left(\sum_{i=1}^n \frac{|a_i|^2}{t_i - t_{i-1}} \right).$$

Here we set $t_0 = 0$, and in the notation of $[D]$ $|a|$ denotes the distance from e to a with respect to our left-invariant metric on K_C . (A more general version of Driver's estimate is found in $[DG, \text{Cor. 3.10}]$.) Thus, recalling the definition of L and expressing things in terms of F , we have

$$|F(g)|^2 \leq \|F\|^2 \exp \left(\sum_{i=1}^n \frac{|g_{t_{i-1}}^{-1} g_{t_i}|^2}{t_i - t_{i-1}} \right), \tag{14}$$

where $g_{t_0} = g_0 = e$. Note that since the metric on K_C is left-invariant,

$$|g_{t_{i-1}}^{-1} g_{t_i}| = d(g_{t_{i-1}}^{-1} g_{t_i}, e) = d(g_{t_i}, g_{t_{i-1}}) = d(g_{t_{i-1}}, g_{t_i}).$$

But since g is constant-speed and length-minimizing on $[t_{i-1}, t_i]$,

$$d(g_{t_{i-1}}, g_{t_i})^2 = (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} \left| g_t^{-1} \frac{dg}{dt} \right|^2 dt.$$

Thus (14) becomes precisely Theorem 4.

Now suppose that g is an arbitrary finite-energy path. Then take a sequence g^n of curves such that g^n coincides with g at times k/n , $k = 0, 1, \dots, n$, and such that g^n is a constant-speed length-minimizing geodesic on each interval $[k/n, (k+1)/n]$. Then

$$\int_0^1 \left| (g^n)^{-1} \frac{dg^n}{dt} \right|^2 dt \leq \int_0^1 \left| g^{-1} \frac{dg}{dt} \right|^2 dt$$

for each n . This is seen as follows. On each subinterval, the L^2 norm of the derivative of g^n is just a constant times the L^1 norm of its derivative (since g^n is constant speed), which is at most the L^1 norm of the derivative of g (since g^n is length-minimizing), which is at most a constant times the L^2 norm of the derivative of g (by Jensen). Keeping track of the constants and summing over the subintervals gives the desired inequality.

It is easy to see that $g_t^n \rightarrow g_t$ for each $t \in [0, 1]$. So for each fixed holomorphic cylinder function F ,

$$\begin{aligned} |F(g)|^2 &= \lim_{n \rightarrow \infty} |F(g^n)|^2 \leq \|F\|^2 \lim_{n \rightarrow \infty} \exp \int_0^1 \left| (g_t^n)^{-1} \frac{dg_t^n}{dt} \right|^2 dt \\ &\leq \|F\|^2 \exp \int_0^1 \left| g_t^{-1} \frac{dg}{dt} \right|^2 dt. \end{aligned}$$

Thus Theorem 4 holds for all $g \in H(K_C)$. ■

Proof of Theorem 7. If F is a holomorphic cylinder function, then for each $g \in H(K_C)$, $F(ge^Z)$ will be holomorphic as a function of $Z \in H(\mathfrak{k}_C)$,

and jointly continuous in g, Z . This implies that $F(ge^Z)$ is Fréchet differentiable at $Z=0$ with complex-linear derivative, and (using simple results of [HP, Chap. III]) that this derivative is continuous with respect to g . So $RF=F$ is holomorphic in the sense of [GM1].

For general $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$,

$$RF(g) = \langle \chi_g, F \rangle = \lim_{n \rightarrow \infty} \langle \chi_g, F_n \rangle = \lim_{n \rightarrow \infty} F_n(g),$$

where the F_n 's are holomorphic cylinder functions and where by the estimates of Theorem 4, the limit is locally uniform. It follows then by [HP, Thm. 3.18.1] that $RF(ge^Z)$ is holomorphic as a function of Z and jointly continuous in g, Z so that RF is holomorphic in the sense of [GM1]. ■

Proof of Theorem 8. The desired result follows from [GM1, Thm. 5.7] together with the corresponding result for the Lie algebra, which is elementary. (See below.) Theorem 5.7 of [GM1] states that a function F on $H(K_{\mathbb{C}})$ is holomorphic if and only if $F \circ \theta_{\mathbb{C}}$ is holomorphic, where $\theta_{\mathbb{C}} : H(\mathfrak{k}_{\mathbb{C}}) \rightarrow H(K_{\mathbb{C}})$ is the Itô map (Section 2.4). We will here sketch another proof of Theorem 8 which does not rely on the Itô map. Our proof is easily adapted to the setting of loop groups, using the fact that the fundamental group of $K_{\mathbb{C}}$ is the same as that of K .

We first prove the theorem in the Lie algebra: a holomorphic function F on $H(\mathfrak{k}_{\mathbb{C}})$ which is zero on $H(\mathfrak{k})$ is identically zero. To see this, note that for $X, Y \in H(\mathfrak{k})$, $F(X + aY)$ is a holomorphic function of a which is zero for $a \in \mathbb{R}$, hence identically zero. Taking $a = i$ we see that $F(X + iY) = 0$ for all $X, Y \in H(\mathfrak{k})$, so $F \equiv 0$.

Now suppose that F is holomorphic on $H(K_{\mathbb{C}})$. According to Gross and Malliavin's definition, this implies that for each $g \in H(K_{\mathbb{C}})$, $F(ge^Z)$ is Fréchet differentiable with respect to $Z \in H(\mathfrak{k}_{\mathbb{C}})$ at $Z=0$, and that Fréchet derivative is complex-linear. We assert that this implies that $F(ge^Z)$ is holomorphic as a function of $Z \in H(\mathfrak{k}_{\mathbb{C}})$. To see this, consider $F(ge^{Z+h}) = F(ge^Z e^{\tilde{h}})$, where $\tilde{h}_t = \log(e^{-Z_t} e^{Z_t + h_t})$. Note that \tilde{h} is defined for h in a neighborhood of the identity in $H(\mathfrak{k}_{\mathbb{C}})$. Using [HP, Thm. 3.17.1] and [HP, Thm. 3.10.1] and Morera's Theorem, it is not hard to verify that \tilde{h} is a holomorphic function of h , that is, Fréchet differentiable with complex-linear derivative. Thus by the chain rule $F(ge^{Z+h}) = F(ge^Z e^{\tilde{h}})$ is Fréchet differentiable with respect to h at $h=0$ with complex-linear derivative. That is, $F(ge^Z)$ is holomorphic as a function of Z , as claimed. It is then straightforward to verify that $F(ge^Z h)$ is holomorphic as a function of Z for each fixed $g, h \in H(K_{\mathbb{C}})$. Thus $F(e^{Z^1} \dots e^{Z^n})$ is holomorphic as a function of each Z^i with the other Z^j 's fixed.

Now assume that F is holomorphic on $H(K_{\mathbb{C}})$ and zero on $H(K)$. Then $F(e^{Z^1} \dots e^{Z^n})$ is zero whenever Z^1, \dots, Z^n are in $H(K)$. But then by

the theorem for the Lie algebra, $F(e^{Z^1} \cdots e^{Z^n})$ is zero for $Z^1 \in H(\mathfrak{k}_{\mathbb{C}})$, with $Z^2, \dots, Z^n \in H(K)$. Extending one variable at a time, we see that $F(e^{Z^1} \cdots e^{Z^n})$ is zero for all $Z^1, \dots, Z^n \in H(\mathfrak{k}_{\mathbb{C}})$. If g is any element of $H(K_{\mathbb{C}})$, then there exists a homotopy g^s of g to the trivial path with the property that for each s , $g^s \in H(K_{\mathbb{C}})$. Taking $h^i = (g^{i/n})^{-1} g^{(i+1)/n}$ we have $g = h^1 \cdots h^n$. For n large enough, each h^i_t will be close to the identity for all t , so we may define $Z^i_t = \log h^i_t$, giving $g = e^{Z^1} \cdots e^{Z^n}$. Thus $F(g) = 0$ for all g .

If F_1 and F_2 are two holomorphic functions on $H(K_{\mathbb{C}})$ which are equal on $H(K)$, then $F_1 - F_2 = 0$ on $H(K)$, hence on all of $H(K_{\mathbb{C}})$. ■

LEMMA 20. *The Wiener measure ρ on $W(K)$ is invariant under the map $x \rightarrow x^{-1}$.*

Proof. If f is a cylinder function, then we may integrate over K^n with respect to the measure ρ_{t_1, \dots, t_n} . But then the desired result follows from the identities $\rho_t(x) = \rho_t(x^{-1})$ and $\rho_t(xy) = \rho_t(yx)$ [H1, Lem. 1]. So if $\hat{\rho}$ denotes ρ composed with the map $x \rightarrow x^{-1}$, then ρ and $\hat{\rho}$ agree on cylinder sets, that is, sets of the form $f^{-1}(E)$, where f is a measurable cylinder function and E is a Borel set in \mathbb{R} . It then follows from the Monotone Class Lemma that ρ and $\hat{\rho}$ agree on the σ -algebra generated by cylinder sets, which is the σ -algebra of all measurable sets in $W(K)$. ■

Proof of Theorem 10. The quasi-invariance is well-known, as is the square-integrability of the Radon–Nikodym derivative [MM]. So now assume that $y \in H(K)$ and that f is a square-integrable cylinder function. Then by the isometricity of B

$$\langle \psi_y, f \rangle = \langle \chi_y, Bf \rangle = RBf(y) = Bf(y).$$

So recalling how B is defined on cylinder functions and using Lemma 20

$$\begin{aligned} \langle \psi_y, f \rangle &= \int_{W(K)} f(x^{-1}y) \, d\rho(x) \\ &= \int_{W(K)} f(xy) \, d\rho(x) \\ &= \int_{W(K)} f(x) \, d\rho(xy^{-1}) \\ &= \int_{W(K)} f(x) \frac{d\rho(xy^{-1})}{d\rho(x)} \, d\rho(x). \end{aligned}$$

Since the Radon–Nikodym derivative is real, the last integral is the inner product of the Radon–Nikodym derivative with f .

So for each y , $d\rho(xy^{-1})/d\rho(x)$ is an L^2 function of x whose inner product with every L^2 cylinder function f agrees with $\langle \psi_y, f \rangle$. Since cylinder functions are dense, this implies that $d\rho(xy^{-1})/d\rho(x) = \psi_y$. ■

Proof of Corollary 11. The formula for $RBf(y)$ follows from the definition of R , the definition of ψ_y , and Theorem 10. The uniqueness is a consequence of Theorem 8. ■

Proof of Theorem 12. Theorem 3.2 of [Sh1] shows that the Wiener measure μ on $W(K_{\mathbb{C}})$ is quasi-invariant under the right action of $H(K)$. The restriction to $H(K)$ as opposed to $H(K_{\mathbb{C}})$ is necessary because the inner product on $\mathfrak{k}_{\mathbb{C}}$ is $\text{Ad-}K$ -invariant but not $\text{Ad-}K_{\mathbb{C}}$ -invariant. Furthermore, the Radon–Nikodym derivative [Sh1, Eq. (3.7)] is readily seen to be square-integrable, since it is essentially the same as the Radon–Nikodym derivative in the classical Cameron–Martin theorem. (See [Sh1, p. 420].) Thus the integral in the theorem is well-defined and continuous in F for fixed y . So it suffices to prove the result for L^2 holomorphic cylinder functions.

If F is a cylinder function then both sides of the theorem will be cylinder functions of y . Thus it suffices to prove that for Ψ holomorphic and square-integrable on $K_{\mathbb{C}}^n$,

$$\begin{aligned} & \Psi(y_1, \dots, y_n) \\ &= \int_{K_{\mathbb{C}}^n} \Psi(g_1 y_1, \dots, g_n y_n) \mu_{t_1}(g_1) \cdots \mu_{t_n - t_{n-1}}(g_{n-1}^{-1} g_n) dg_1 \cdots dg_n \quad (15) \end{aligned}$$

for all $y \in K^n$. Using density results [H1, D] we may assume that Ψ and its derivatives grow only exponentially at infinity. In this case there is no problem (see [H2, Sec. 4]) with differentiating under the integral sign and integrating by parts on the right to give

$$\begin{aligned} & 4 \frac{d}{dt_n} \int_{K_{\mathbb{C}}^n} \Psi(g_1 y_1, \dots, g_n y_n) \mu_{t_1}(g_1) \cdots \mu_{t_n - t_{n-1}}(g_{n-1}^{-1} g_n) dg_1 \cdots dg_n \\ &= \int_{K_{\mathbb{C}}^n} \Delta_{g_n} \Psi(g_1 y_1, \dots, g_n y_n) \mu_{t_1}(g_1) \cdots \mu_{t_n - t_{n-1}}(g_{n-1}^{-1} g_n) dg_1 \cdots dg_n. \end{aligned}$$

Here we use the fact that the Laplacian on $K_{\mathbb{C}}$ is left-invariant.

Since $\Psi(g_1 y_1, \dots, g_n y_n)$ is holomorphic in g_n with g_1, \dots, g_{n-1} and y_1, \dots, y_n fixed, the Laplacian is zero. Thus the integral in (15) is independent of t_n , and we may let t_n tend to t_{n-1} . Since μ_t is the fundamental solution

at the identity of the heat equation, then in the limit the g_n integral has the effect of setting g_n equal to g_{n-1} . So we get

$$\int_{K_{\mathbb{C}}^n} \Psi(g_1 y_1, \dots, g_{n-1} y_{n-1}, g_{n-1} y_n) \mu_{t_1}(g_1) \cdots \mu_{t_{n-1}-t_{n-2}}(g_{n-2}^{-1} g_{n-1}) dg_1 \cdots dg_{n-1}.$$

Now the integrand is a holomorphic function of g_1, \dots, g_{n-1} which does not grow too rapidly at infinity. Repeating the argument we eventually set $g_n = g_{n-1} = \cdots = g_1 = e$, giving the desired result. ■

3.4. The Itô Mapping

Proof of Theorem 13. If $y \in H(K)$ and $X \in W(\mathfrak{f})$, define $y \cdot X \in W(\mathfrak{f})$ by the formula

$$(y \cdot X)_t = \int_0^t Ad(y_s^{-1}) dX_s \tag{16}$$

(stochastic integral). As it stands, (16) is defined for almost every X , but since y_s is “nice” it can actually defined for all X by means of integration-by-parts. (See [GM1, Notation 7.1].) For fixed $y \in H(K)$, the map $X \rightarrow y \cdot X$ is a bounded linear transformation of $W(\mathfrak{f}_{\mathbb{C}})$. Furthermore, $y^{-1} \cdot (y \cdot X) = X$.

LEMMA 21. *For each $y \in H(K)$, the map $X \rightarrow y \cdot X$ is a measure preserving map of $(W(\mathfrak{f}), p)$ to itself.*

Proof. The map $X \rightarrow y \cdot X$ is an invertible linear transformation of $W(\mathfrak{f}_{\mathbb{C}})$ to itself which restricts to an invertible, isometric transformation of $H(\mathfrak{f}_{\mathbb{C}})$ to itself. This is sufficient to show that the map is measure-preserving. (See the proof of [GM1, Cor. 7.12].) ■

LEMMA 22. *For each $Y \in H(\mathfrak{f})$,*

$$\tilde{\theta}(X) \theta(Y) = \tilde{\theta}(\theta(Y) \cdot X + Y)$$

for p -almost every $X \in W(\mathfrak{f})$.

Proof. This can be seen in the proof of Theorem 3.2 on p. 420 of [Sh1], with $x_t = \tilde{\theta}(X)$ and $z_t = \tilde{\theta}(X) \theta(Y)$. See also [GM1, Lem. 7.3]. ■

Now, according to Theorem 11,

$$RBf(y) = \int_{W(K)} f(x^{-1}y) d\rho(x).$$

Since (Lemma 20) the Wiener measure ρ is invariant under $x \rightarrow x^{-1}$, we may write this as

$$RBf(y) = \int_{W(K)} f(xy) d\rho(x).$$

Taking $y = \theta(Y) = \theta_{\mathbb{C}}(Y)$ with $Y \in H(\mathfrak{f})$ we get

$$RBf \circ \theta_{\mathbb{C}}(Y) = \int_{W(K)} f(x\theta(Y)) d\rho(x).$$

Using our two lemmas and the fact that $\tilde{\theta}$ is measure-preserving

$$\begin{aligned} RBf \circ \theta_{\mathbb{C}}(Y) &= \int_{W(\mathfrak{t})} f(\tilde{\theta}(X) \theta(Y)) d\rho(X) \\ &= \int_{W(\mathfrak{t})} f \circ \tilde{\theta}(\theta(Y) \cdot X + Y) d\rho(X) \\ &= \int_{W(\mathfrak{t})} f \circ \tilde{\theta}(X + Y) d\rho(X). \end{aligned}$$

Since the Wiener measure ρ is invariant under $X \rightarrow -X$, we may recognize the last expression as $RS(f \circ \tilde{\theta})(Y)$. Thus $RBf \circ \theta_{\mathbb{C}} = RS(f \circ \tilde{\theta})$ on $H(\mathfrak{f})$. But $RS(f \circ \tilde{\theta})$ is holomorphic on $H(\mathfrak{f}_{\mathbb{C}})$, and RBf is holomorphic on $H(K_{\mathbb{C}})$, from which it follows [GM1, Thm. 5.7] that $RBf \circ \theta_{\mathbb{C}}$ is holomorphic on $H(\mathfrak{f}_{\mathbb{C}})$. (Gross and Malliavin consider only the case in which K is compact, but the proof applies just as well if K is of compact type.) A holomorphic function on $H(\mathfrak{f}_{\mathbb{C}})$ is determined by its values on $H(\mathfrak{f})$, so $RBf \circ \theta_{\mathbb{C}} = S(f \circ \tilde{\theta})$. ■

Proof of Corollary 14. Suppose $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ and $RF \equiv 0$. Then by Theorem 3 there is a unique $f \in L^2(W(K), \rho)$ such that $Bf = F$. So $RBf \equiv 0$. But then by Theorem 13 $RS(f \circ \tilde{\theta}) = (RBf) \circ \theta_{\mathbb{C}} \equiv 0$. Then the isometricity of the classical transform RS [GM1, Thm. 4.8] implies that $f \circ \tilde{\theta} = 0$ a.e. and hence that $f = 0$ a.e. and hence that $F = Bf = 0$.

If the closed span of χ_g 's were not all of $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, then there would exist a non-zero $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$ such that $\langle \chi_g, F \rangle = 0$ for all g . But then by the definition of R , we would have $RF \equiv 0$, which we have just seen implies that $F \equiv 0$, thus giving a contradiction. So the χ_g 's span a

dense subspace of $\mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, and so by the unitarity of B the ψ_g 's span a dense subspace of $L^2(W(K), \rho)$. ■

Proof of Corollary 15. Suppose that $F = R\tilde{F}$ for some $\tilde{F} \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$. Then there is a unique $f \in L^2(W(K), \rho)$ such that $\tilde{F} = Bf$. So $F = RBf$ and $F \circ \theta_{\mathbb{C}} = (RBf) \circ \theta_{\mathbb{C}} = RS(f \circ \tilde{\theta})$. But $f \circ \tilde{\theta} \in L^2(W(\mathfrak{k}), p)$, so by [GM1, Thm. 4.8]

$$\|F \circ \theta_{\mathbb{C}}\|_m = \|RS(f \circ \tilde{\theta})\|_m < \infty.$$

Conversely, suppose $\|F \circ \theta_{\mathbb{C}}\|_m < \infty$. Then by [GM1, Thm. 4.8] there is $g \in L^2(W(\mathfrak{k}), p)$ such that $RSg = F \circ \theta_{\mathbb{C}}$. Let $f = g \circ \tilde{\theta}^{-1}$. Then $(RBf) \circ \theta_{\mathbb{C}} = RS(f \circ \tilde{\theta}) = RSg = F \circ \theta_{\mathbb{C}}$. So $F = R\tilde{F}$, where $\tilde{F} = Bf$.

If $F = R\tilde{F}$, then again exists f such that $R\tilde{F} = RBf$. So $F \circ \theta_{\mathbb{C}} = (RBf) \circ \theta_{\mathbb{C}} = RS(f \circ \tilde{\theta})$. Thus

$$\begin{aligned} \|F \circ \theta_{\mathbb{C}}\|_m &= \|RS(f \circ \tilde{\theta})\|_m = \|f \circ \tilde{\theta}\|_{L^2(W(\mathfrak{k}), p)} \\ &= \|f\|_{L^2(W(K), \rho)} = \|\tilde{F}\|_{L^2(W(K_{\mathbb{C}}), \mu)}. \end{aligned}$$

Finally, if $R\tilde{F}_1 = R\tilde{F}_2$, then $R(\tilde{F}_1 - \tilde{F}_2) = 0$ so $\|\tilde{F}_1 - \tilde{F}_2\| = \|R(\tilde{F}_1 - \tilde{F}_2)\|_m = 0$, which implies that $\tilde{F}_1 = \tilde{F}_2$ a.e. ■

Proof of Theorem 16. We will prove fairly directly that if $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, then $F \circ \tilde{\theta}_{\mathbb{C}} \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$, using a “multiplication lemma,” the density results of [H1, D], and a result of Sadasue [Sa2, Lem. 5.1]. To go in the other direction, we will have to prove that parts of the diagram in Theorem 17 commute, and use the isometricity of the maps involved. Thus by the time Theorem 16 is proved, we will essentially have proved Theorem 17 as well.

LEMMA 23 [Multiplication Lemma]. *Suppose F_1 and F_2 are elements of $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$. Suppose further that $F_1 \in L^p(W(\mathfrak{k}_{\mathbb{C}}), m)$ for some $p > 2$, and that $F_1 F_2 \in L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$. Then $F_1 F_2 \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$.*

Proof. The result is reasonable since we expect the product of holomorphic functions to be holomorphic, and since we assume that $F_1 F_2$ is square-integrable. The result is clear if F_1 and F_2 are polynomials, and we will prove the general result by approximation.

Let \mathcal{P} denote the space of functions f on $W(\mathfrak{k}_{\mathbb{C}})$ which can be expressed as $f(Z) = p(Z_{i_1}, \dots, Z_{i_n})$, where p is a not-necessarily-holomorphic polynomial on $\mathfrak{k}_{\mathbb{C}}^n$. Let \mathcal{P}_1 denote the subspace of \mathcal{P} consisting of those f 's for which p is holomorphic, and let \mathcal{P}_2 consist of those elements of \mathcal{P} which are orthogonal to \mathcal{P}_1 . It is not hard to see that \mathcal{P}_1 is dense in $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$ and that \mathcal{P}_2 is dense in the orthogonal complement of $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$.

Clearly both \mathcal{P} and \mathcal{P}_1 are closed under pointwise multiplication. Thus if p_1, p_2 are in \mathcal{P}_1 and q is in \mathcal{P}_2 , then $p_1 p_2$ is in \mathcal{P}_1 , so

$$0 = \langle p_1 p_2, q \rangle = \langle p_1, \overline{p_2} q \rangle.$$

Thus \mathcal{P}_2 is closed under multiplication by anti-holomorphic polynomials.

Since \mathcal{P}_2 is dense in the orthogonal complement of $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$, $F_1 F_2$ will be in $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$ provided that $\langle F_1 F_2, q \rangle = 0$ for all $q \in \mathcal{P}_2$. Since \mathcal{P}_1 is dense in $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$, we may take a sequence $\{p_n\}$ of holomorphic polynomials such that $p_n \rightarrow F_2$ in the L^2 sense. For any n ,

$$\langle F_1 p_n, q \rangle = \langle F_1, \overline{p_n} q \rangle = 0.$$

Since F_1 is assumed to be in L^p for some $p > 2$, $F_1 p_n$ will converge to $F_1 F_2$ in some L^r with $r > 1$. Since q is polynomial, it will be in $L^{r'}$ (r' the conjugate exponent to r), and so

$$\langle F_1 F_2, q \rangle = \lim_{n \rightarrow \infty} \langle F_1 p_n, q \rangle = 0.$$

Thus $F_1 F_2 \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$. ■

LEMMA 24. *Suppose that $\pi : K_{\mathbb{C}} \rightarrow \text{GL}(n; \mathbb{C})$ is a finite-dimensional holomorphic representation of $K_{\mathbb{C}}$. Then for each $t \in [0, 1]$ and each $1 \leq l, m \leq n$, the function*

$$Z \rightarrow \pi^{lm}(\tilde{\theta}_{\mathbb{C}}(Z)_t)$$

is in $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$.

This is Lemma 5.1 of [Sa2]. Since this is an important technical result, we will reproduce Sadasue’s proof in the appendix, and will also give another proof due to Gross and Malliavin [GM2].

LEMMA 25. *If $F \in \mathcal{H}L^2(W(K_{\mathbb{C}}), \mu)$, then $F \circ \tilde{\theta}_{\mathbb{C}} \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$.*

Proof. Since $\tilde{\theta}_{\mathbb{C}}$ is measure-preserving, $F \circ \tilde{\theta}_{\mathbb{C}} \in L^2(W(\mathfrak{k}_{\mathbb{C}}), m)$. We must show that $F \circ \tilde{\theta}_{\mathbb{C}}$ is in the holomorphic subspace. If F is of the form $F(g) = \pi^{lm}(g_t)$, then this is the content of Lemma 24. Furthermore, the rapid decay of the heat kernel measure μ_t on $K_{\mathbb{C}}$ ensures that such an F is in L^p for all $p < \infty$, and so $F \circ \tilde{\theta}_{\mathbb{C}} \in L^p$ for all $p < \infty$. Finally, the same results hold if F is of the form $\pi^{lm}(g_t^{-1})$, since $\pi^{lm}(g^{-1}) = \sigma^{ml}(g)$, where σ is the dual representation to π .

Now fix a partition t_1, \dots, t_n of $[0, 1]$, and consider a function F of the form

$$F(g) = \text{trace}(\pi_1(g_{t_1}) A_1) \text{trace}(\pi_2(g_{t_1}^{-1} g_{t_2}) A_2) \cdots \text{trace}(\pi_n(g_{t_{n-1}}^{-1} g_{t_n}) A_n). \tag{17}$$

Here the π_i 's are finite-dimensional holomorphic representations acting on spaces V_i , and the A_i 's are operators on the V_i 's. Expanding everything out in terms of bases for the V_i 's, and applying Lemmas 23 and 24 we see that $F \circ \tilde{\theta}_C$ is in $\mathcal{H}L^2(W(\mathfrak{k}_C), m)$. On the other hand, the density results of [H1, D], together with the change of variables L from Section 3.2, show that every L^2 holomorphic cylinder function is an L^2 limit of functions of the form (17). Since L^2 holomorphic cylinder functions are dense in $\mathcal{H}L^2(W(K_C), \mu)$ and composition with $\tilde{\theta}_C$ is an isometry, we obtain the desired result for all F . ■

LEMMA 26. *If $F \in \mathcal{H}L^2(W(K_C), \mu)$, then $(RF) \circ \theta_C = R(F \circ \tilde{\theta}_C)$.*

Proof. By [GM1, Thm. 5.7] (extended to groups of compact type), $(RF) \circ \theta_C$ is holomorphic on $H(\mathfrak{k}_C)$. Moreover, $R(F \circ \tilde{\theta}_C)$ is also holomorphic. We have already remarked in the proof of Theorem 13 that a holomorphic function on $H(\mathfrak{k}_C)$ is determined by its values on $H(\mathfrak{k})$. Thus it suffices to show that the two functions in the lemma are equal on $H(\mathfrak{k})$. We will compute both of the restriction maps using Theorem 12. In the case of \mathfrak{k}_C we think of \mathfrak{k}_C as a commutative Lie group, which is therefore of compact type, or else we use [Su1, Su2].

Using Theorem 12, the proof is almost the same as the proof of Theorem 13. In particular, Lemmas 21 and 22 continue to hold with $(W(\mathfrak{k}), p)$ replaced with $(W(\mathfrak{k}_C), m)$, provided that the finite-energy paths y and Y are still required to lie in K and \mathfrak{k} , respectively. The reason for this restriction is that the inner product on \mathfrak{k}_C is $\text{Ad-}K$ -invariant but not $\text{Ad-}K_C$ -invariant. With this restriction, the proof of Lemma 21 goes through unchanged, and the hypotheses of [Sh1, Thm. 3.2] are satisfied to give Lemma 22. The rest of the proof is the same as the proof of Theorem 13. ■

LEMMA 27. *If $G \in \mathcal{H}L^2(W(\mathfrak{k}_C), m)$, then $G \circ \tilde{\theta}_C^{-1} \in \mathcal{H}L^2(W(K_C), \mu)$.*

Proof. The proof is “diagram chasing.” Let $g = S^{-1}(G)$, let $f = g \circ \theta$, and let $F = B(f)$. We will show that $G = F \circ \tilde{\theta}_C$. Once this is established we are done, since then $G \circ \tilde{\theta}_C^{-1} = F = B(f)$ is in $\mathcal{H}L^2(W(K_C), \mu)$.

Now, by Lemma 25, $F \circ \tilde{\theta}_\mathbb{C} \in \mathcal{H}L^2(W(\mathfrak{k}_\mathbb{C}), m)$. Thus by Lemma 26 and Theorem 13 we have

$$R(F \circ \tilde{\theta}_\mathbb{C}) = (RF) \circ \theta_\mathbb{C} = RBf \circ \theta_\mathbb{C} = RS(f \circ \tilde{\theta}).$$

But $f \circ \tilde{\theta} = g$, so $S(f \circ \tilde{\theta}) = G$, and so we have

$$R(F \circ \tilde{\theta}_\mathbb{C}) = RG.$$

But the restriction map on $\mathcal{H}L^2(W(\mathfrak{k}_\mathbb{C}), m)$ is one-to-one, so $F \circ \tilde{\theta}_\mathbb{C} = G$. ■

We have now completed the proof of Theorem 16. ■

Proof of Theorem 17. We have now shown that the middle vertical arrow in Theorem 17 is well-defined, and we have shown that the right-hand square commutes. So it remains only to show that the left-hand square commutes, that is, that $(Bf) \circ \tilde{\theta}_\mathbb{C} = S(f \circ \tilde{\theta})$. But by Lemma 26 and Theorem 13

$$R((Bf) \circ \tilde{\theta}_\mathbb{C}) = (RBf) \circ \theta_\mathbb{C} = RS(f \circ \tilde{\theta}).$$

Since R is injective, we obtain the desired result. This completes the proof of Theorem 17. ■

APPENDIX

In this appendix, we will reproduce Sadasue’s proof ([Sa2]) of Lemma 24. We will also give another proof of this result, due to Gross and Malliavin ([GM2]).

First Proof of Lemma 24 [Sadasue]. Since μ_t decays rapidly at infinity on $K_\mathbb{C}$, $\pi^{lm}(g_t)$ is μ -square-integrable as a function of g . Thus $\pi^{lm}(\tilde{\theta}_\mathbb{C}(Z)_t)$ is m -square-integrable as a function of Z . We must show that this function is in the holomorphic subspace. Sadasue’s strategy is to use the stochastic differential equation $dg_t = g_t \circ dZ_t$ which defines $\tilde{\theta}_\mathbb{C}$ to obtain a stochastic integral equation for $\pi^{lm}(g_t)$, where $g = \tilde{\theta}_\mathbb{C}(Z)$. Iteration of this equation then yields a convergent expansion for $\pi^{lm}(g_t)$, each term of which is in the holomorphic subspace.

Let X_1, \dots, X_d be an orthonormal basis for \mathfrak{k} , so that $X_1, \dots, X_d, JX_1, \dots, JX_d$ is an orthonormal basis for $\mathfrak{k}_\mathbb{C}$. Viewing the X_j ’s and JX_j ’s as left-invariant differential operators on $K_\mathbb{C}$, we have $X_j \pi(g) = \pi(g) \pi_*(X_j)$ and $JX_j \pi(g) = \pi(g) \pi_*(JX_j) = i\pi(g) \pi_*(X_j)$. Here π_* is the induced Lie algebra representation

$$\pi_*(Z) = \frac{d}{ds} \Big|_{s=0} \pi(e^{sZ}), \tag{18}$$

which is complex-linear since π is holomorphic. Furthermore, since π is holomorphic, each entry $\pi^{lm}(g)$ is a holomorphic and hence harmonic function on $K_{\mathbb{C}}$. Thus the correction term between the Itô and Stratonovich integral formulas for $\pi^{lm}(g_t)$ is zero.

Let $B_t^{1,j}$ and $B_t^{2,j}$ be the X_j and JX_j components of Z , which are independent Brownian motions. Then we have the matrix-valued Itô integral equation

$$\pi(g_t) = \pi(e) + \sum_{j=1}^d \int_0^t \pi(g_s) dB_s^{1,j} \pi_*(X_j) + i \sum_{j=1}^d \int_0^t \pi(g_s) dB_s^{2,j} \pi_*(X_j).$$

If we let $Z_t^j = B_t^{1,j} + iB_t^{2,j}$ (a complex-valued Brownian motion) then we have

$$\pi(g_t) = \pi(e) + \sum_{j=1}^d \int_0^t \pi(g_s) dZ_s^j \pi_*(X_j).$$

Iterating gives

$$\begin{aligned} \pi(g_t) = & \pi(e) + \sum_{k=1}^{n-1} \sum_{j_1, \dots, j_k} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} dZ_{t_k}^{j_k} \dots dZ_{t_1}^{j_1} \pi_*(X_{j_k}) \dots \pi_*(X_{j_1}) \\ & + \sum_{j_1, \dots, j_n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \pi(g_{t_n}) dZ_{t_n}^{j_n} \dots dZ_{t_1}^{j_1} \pi_*(X_{j_n}) \dots \pi_*(X_{j_1}). \end{aligned} \tag{19}$$

But since $\pi^{lm}(g_t)$ is square-integrable, we have

$$E \left\{ \left| \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \pi^{lm}(g_{t_n}) dZ_{t_n}^{j_n} \dots dZ_{t_1}^{j_1} \right|^2 \right\} = \frac{const.}{n!}.$$

Thus the last term in (19) tends to zero in L^2 as $n \rightarrow \infty$, and we obtain an L^2 -convergent series. Using the definition of the Itô stochastic integral, each term in the resulting series is seen to be an L^2 limit of holomorphic polynomials, and is thus in the holomorphic subspace. It follows that $\pi^{lm}(g_t)$ is in the holomorphic subspace. (See also [Sh2].)

Note that this series is just the stochastic version of the series in Lemma 6.4 of [GM1], in the special case which $u = \pi^{lm}$. By the above and the density theorem, the stochastic version of Lemma 6.4 holds for all $u \in \mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$. ■

Second Proof of Lemma 24 [Gross-Malliavin]. Gross and Malliavin's strategy is to use a piecewise-linear approximation Z_t^n to the Brownian paths Z_t in $\mathfrak{k}_{\mathbb{C}}$, and then to apply the smooth Itô map $\theta_{\mathbb{C}}$ to the piecewise-linear paths. Then $\pi^{lm}(\theta_{\mathbb{C}}(Z^n)_t)$ is an L^2 holomorphic cylinder function,

and by a result from [IW], $\pi^{lm}(\theta_{\mathbb{C}}(Z^n)_t)$ converges to $\pi^{lm}(\tilde{\theta}_{\mathbb{C}}(Z)_t)$ in L^2 as $n \rightarrow \infty$.

The function $X_t = \pi(g_t)$ satisfies the matrix-valued Stratonovich stochastic differential equation

$$dX_t = X_t \circ d\pi_*(Z_t),$$

where as in (18) π_* is the induced Lie algebra representation. Consider a partition of $[0, 1]$ into intervals of length $1/n$. For any continuous path $Z \in W(\mathfrak{k}_{\mathbb{C}})$, let Z^n be the piecewise-linear path whose values at times k/n are the same as those of Z . We will regard $\theta_{\mathbb{C}}(Z^n)$ as an approximation to $\tilde{\theta}_{\mathbb{C}}(Z)$. It is easy to see that $\theta_{\mathbb{C}}(Z^n)$ is the piecewise-exponential path satisfying

$$\begin{aligned} \theta_{\mathbb{C}}(Z^n)_t &= \exp(Z_{1/n}) \exp(Z_{2/n} - Z_{1/n}) \cdots \exp(Z_{k/n} - Z_{(k-1)/n}) \\ &\quad \times \exp((t - k/n)(Z_{(k+1)/n} - Z_{k/n})) \end{aligned}$$

for $t \in [k/n, (k+1)/n]$. Thus we see explicitly that $\pi^{lm}(\theta_{\mathbb{C}}(Z^n)_t)$ is a holomorphic cylinder function.

We now apply Theorem VI.7.2 and Example VI.7.1 of [IW], which imply that $\pi^{lm}(\theta_{\mathbb{C}}(Z^n)_t)$ converges to $\pi^{lm}(\tilde{\theta}_{\mathbb{C}}(Z)_t)$ in L^2 as $n \rightarrow \infty$. Thus we have expressed $\pi^{lm}(\tilde{\theta}_{\mathbb{C}}(Z)_t)$ as a limit of L^2 holomorphic cylinder functions. ■

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