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THE SEGAL–BARGMANN TRANSFORM FOR TWO-DIMENSIONAL EUCLIDEAN QUANTUM YANG–MILLS

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The Segal–Bargmann transform of certain Wilson loop variables for quantum Yang–Mills theory on the plane is determined explicitly.

1. Introduction

Quantum mechanics for the configuration space $\mathbf{R}^d$ can be described either by a Hilbert space $L^2(\mathbf{R}^d)$ or by a corresponding Hilbert space of holomorphic square integrable functions over the phase space $\mathbf{C}^d$. The unitary transformation relating these two representations is the Segal–Bargmann transform.[4,15] For a free quantum field, $\mathbf{R}^d$ is replaced by a real separable Hilbert space and $\mathbf{C}^d$ by the complexification of the Hilbert space; the Segal–Bargmann transform again describes the relationship between the two representations. An analogous transform has been discovered in Refs. 8 and 9 for the case when the configuration space is a compact connected Lie group.

In this paper we will examine this transform for the case of the two-dimensional Euclidean quantum Yang–Mills field over the plane $\mathbf{R}^2$. We are concerned with the transform SF of a function of the form $F = f(h(C_1), \ldots, h(C_n))$, where $C_1, \ldots, C_n$ are loops in the plane, based at the origin, $h(C)$ denotes the holonomy around C, and f is a function on K^n, K being the compact structure group. We will determine SF explicitly for certain types of loops $C_1, \ldots, C_n$.

The Segal–Bargmann transform of a function f on $\mathbf{R}^d$ is the function Sf on $\mathbf{C}^d$ specified by the formula

$$(Sf)(z) = \int_{\mathbf{R}^d} f(x)\rho(x-z)\,dx\,,$$

where ρ is the analytic continuation to $\mathbf{C}^d$ of the standard Gaussian density on $\mathbf{R}^d$, i.e. $\rho(w) = (2\pi)^{-d/2}e^{-(w_1^2+\cdots+w_d^2)/2}$ for every $w = (w_1,\ldots,w_d) \in \mathbf{C}^d$. For any $f \in L^2(\mathbf{R}^d;\rho(x)\;dx)$, Sf exists and is a holomorphic function on $\mathbf{C}^d$, square integrable with respect to the measure $\mu(z)\;dz$, where $\mu(z) = \pi^{-d}e^{-|z|^2}$ and dz is Lebesgue measure on $\mathbf{C}^d$; the Segal–Bargmann theorem, in this context, implies that the map

$$L^2(\mathbf{R}^d,\rho(x)\;dx) \to \mathcal{H}L^2(\mathbf{C}^d,\mu(z)\;dz) : f \mapsto Sf$$

is a unitary isomorphism, where $\mathcal{H}L^2(\mathbf{C}^d,\mu(z)\,dz)$ is the subspace of $L^2(\mathbf{C}^d,\mu(z)\,dz)$ consisting of holomorphic functions. As explained in Sec. 2, the unitary transform S is also meaningful when $\mathbf{R}^d$ is replaced by an infinite-dimensional linear space equipped with an appropriate Gaussian measure.

Now let $\mathcal{A}$ be the space of all connections on a principal K-bundle $\pi : P \to \Sigma$, where Σ is a two-dimensional Riemannian manifold and K is a compact connected real Lie group, with an Ad-invariant metric on its Lie algebra $L(K)$. The Euclidean Yang–Mills measure ρ^{YM} is, informally, a probability measure on $\mathcal{A}$ given by the heuristic formula

$$\int_{\mathcal{A}} f\,d\rho^{\mathrm{YM}} = \frac{1}{\int_{\mathcal{A}} e^{-S_{\mathrm{YM}}(A)}\,\mathcal{D}A} \int_{\mathcal{A}} e^{-S_{\mathrm{YM}}(A)} f(A)\,\mathcal{D}A\,,$$

where $\mathcal{D}A$ is an informal "Lebesgue measure" on $\mathcal{A}$, and S_{YM} the Yang–Mills action. One is interested only in functions f on $\mathcal{A}$ which are invariant under the action of the group $\mathcal{G}_o$ of gauge transformations fixing pointwise the fiber over a basepoint o, and thus the "measure" ρ^{YM} is viewed as a measure on the quotient $\mathcal{A}/\mathcal{G}_o$. The Euclidean quantum field theory of the Yang–Mills field is then described by the Hilbert space $L^2(\mathcal{A}/\mathcal{G}_o,\rho^{\mathrm{YM}})$.

In this paper we shall consider the special case in which the underlying manifold Σ is $\mathbf{R}^2$. In this case $\mathcal{A}/\mathcal{G}_o$ can be identified with a linear space $\mathcal{E}'$ by means of a particular gauge-fixing procedure (the radial gauge), and the Yang–Mills measure is a Gaussian measure on $\mathcal{E}'$. We may therefore consider the usual Segal–Bargmann transform for $\mathcal{E}'$ and think of it as a Segal–Bargmann transform for $\mathcal{A}/\mathcal{G}_o$. The range Hilbert space is an L^2-space of holomorphic functions on the complexified space $\mathcal{E}'_{\mathbf{C}}$, where (using again the radial gauge) $\mathcal{E}'_{\mathbf{C}}$ may be thought of as a moduli space of complex connections. We then consider functions on $\mathcal{A}/\mathcal{G}_o$ of the form $F = f(h(C_1),\ldots,h(C_n))$, where $h(C)$ denotes the holonomy of a connection around a curve C (based at the origin), f is a function on K^n, and K is the structure group. Now, suppose that $C_1,\ldots,C_n$ are simple closed curves with disjoint interiors, and suppose further that they are of a special class which are "nice" with respect to the radial gauge. (This means curves having the polar form $(r(t),\theta(t))$, where $\theta(t)$ is monotone.) In that case, we get a simple answer, namely SF is just a function of the $K_{\mathbf{C}}$-valued holonomy of a complex connection around the same curves $C_1, \ldots, C_n$, given by the generalized Segal–Bargmann transform for K^n.

If the structure group K is non-commutative, then this result does not hold for general collections of simple closed curves with disjoint interiors, even if we assume, say, that the curves are real-analytic. While this seems disappointing, it is actually an unavoidable situation. One might be tempted to *define* a transform by defining it to be the generalized Segal–Bargmann transform on arbitrary collections of simple closed curves with disjoint interiors. This seems more intrinsic and geometrically natural than the gauge-fixing approach we are using. However, this "intrinsic" transform is not well-defined. For if K is not commutative, then there is an inconsistency that arises when the same function on $\mathcal{A}/\mathcal{G}_o$ is expressed as a function of holonomies around disjoint simple closed curves in two different ways.

Our difficulty can be understood more clearly when one compares our transform with that of Ashtekar *et al.*[3] At first glance it appears as if our transform might be definable by the formulas used in Ref. 3 (but using different measures on both domain and range). However, Ref. 3 required a "length-functional" on curves. We would need such a length functional which, in the case of a simple closed curve, gives the area enclosed. There is no such positive length functional. For the length functional would have to be something like the integral of $x\, dy$ over the curve (or some other one-form whose differential is $dx \wedge dy$), which indeed gives the enclosed area for simple closed curves, but which is not always positive.

It should be noted, though, that if the structure group K is commutative, then Theorem 2.1 does indeed hold for arbitrary collections of simple closed curves with disjoint interiors, despite the absence of a positive length functional. This is because our transform is defined only on connections modulo gauge transformations, whereas that of Ref. 3 is defined first on connections and then descends to $\mathcal{A}/\mathcal{G}$. Things work well in the case of a commutative gauge group because in that case the curvature is a gauge-invariant quantity. Thus the labeling of gauge-equivalence classes in terms of the curvature (which is used to identify $\mathcal{A}/\mathcal{G}$ with $\mathcal{E}'$) does not rely on a gauge-fixing procedure, as it does in the non-commutative case.

In other settings, the Segal–Bargmann transform is defined as $e^{t\Delta}$ followed by analytic continuation, where Δ is a Laplacian. Indeed our transform can be understood in this way, where Δ is the Laplacian for $\mathcal{E}'$, identified with $\mathcal{A}/\mathcal{G}_o$ by gauge-fixing. Moreover, this Laplacian has the property (informally) that the Euclidean Yang–Mills measure is the fundamental solution of the heat equation for this Laplacian at "the" flat connection. On the other hand, there can be several different Laplacians all of which have the same fundamental solution at the trivial connection, for example one's obtained by using different gauge-fixing procedures, such as axial gauge. Thus although the Yang–Mills measure on $\mathcal{A}/\mathcal{G}_o$ is independent of the gauge-fixing procedure, the Segal–Bargmann transform is not. So although our transform is gauge-invariant in the sense that it is defined on $\mathcal{A}/\mathcal{G}_o$, it nevertheless depends on the gauge-fixing procedure.

Parenthetically, let us note that there is another Laplacian on $\mathcal{A}/\mathcal{G}_o$, defined in an intrinsic way, as for example in Singer.[17] However, this Laplacian seems not to be related to the Euclidean Yang–Mills measure. In particular, the fundamental solution at the flat connection, defined with respect to the Singer Laplacian, is not the Euclidean Yang–Mills measure. Nevertheless, a Segal–Bargmann transform on $\mathcal{A}/\mathcal{G}_o$ defined with respect to the Singer Laplacian might well be of interest for the study of $(2+1)$-dimensional Yang–Mills theory from a canonical quantization point of view. Driver-Hall[5] considered the canonical quantization of $(1+1)$-dimensional Yang–Mills theory using precisely such a Segal–Bargmann transform.

Other works have investigated Segal–Bargmann type transforms for two-dimensional gauge theory. The work[18] is in the Hamiltonian approach, and, in some sense, corresponds in our approach to transforming with the one-dimensional Yang–Mills measure. We have already discussed the work of Ashtekar *et al.*[3] There has been much recent activity in the area of Segal–Bargmann transforms; Refs. 7–10 contain more references.

Part of the motivation of the present work comes from recent investigations[1] of Chern–Simons theory from an infinite-dimensional distribution-theoretic approach. This raised natural questions concerning parallel-transport and S-transforms of holonomy variables in the three-dimensional setting. We hope that an exploration of the simpler two-dimensional Yang–Mills setting will be useful in gaining insights, technical and conceptual, in the three-dimensional Chern–Simons context.

2. Statement of the Result

Fix, as before, a compact connected real Lie group K, and an Ad-invariant inner product on its Lie algebra $L(K)$. We work with the plane $\mathbf{R}^2$, equipped with some Riemannian metric.

We review briefly some of the conceptual framework behind the two-dimensional quantum Yang–Mills measure. Let $\mathcal{A}$ be the space of connections on the (necessarily trivial) principal K-bundle over $\pi : P \to \mathbf{R}^2$, and $\mathcal{G}_o$ the group of smooth fiber-preserving automorphisms $\phi : P \to P$ which fix the fiber over the origin $o \in \mathbf{R}^2$; then $\mathcal{G}_o$ acts on $\mathcal{A}$ in the usual way: $(\phi, A) \mapsto \phi^* A$. The quotient space $\mathcal{A}/\mathcal{G}_o$ can be identified with the linear space of smooth $L(K)$-valued functions on $\mathbf{R}^2$ in the following way. Fix, once and for all, a point u in the fiber over o; for $A \in \mathcal{A}$ and any point $p \in \mathbf{R}^2$ let $s_A(p)$ denote the element in $\pi^{-1}(p)$ obtained by parallel-translating u along the radial segment op by the connection A; then $s_A : \mathbf{R}^2 \to P$ is a section of the bundle P. Let F^A be the curvature of A; thus $s_A^* F^A$ is a smooth $L(K)$-valued two-form on $\mathbf{R}^2$ and hence can be expressed as $f^A d\sigma$, where f^A is a smooth $L(K)$-valued function on $\mathbf{R}^2$ and $d\sigma$ is the Riemannian area two-form on $\mathbf{R}^2$. The association $A \mapsto f^A$ identifies $\mathcal{A}/\mathcal{G}_o$ with the space of smooth $L(K)$-valued functions on $\mathbf{R}^2$. The informal Yang–Mills measure goes over formally to the Gaussian measure on the space of $L(K)$-valued functions on $\mathbf{R}^2$. We proceed to a more precise framework for the latter.

We work with the Schwartz space

$$\mathcal{E} = \mathcal{S}(\mathbf{R}^2) \otimes L(K)$$

and let ρ^{YM} be the Gaussian measure on the dual space $\mathcal{E}' = \mathcal{S}'(\mathbf{R}^2) \otimes L(K)$. Thus for any function $f \in L^2_{\text{real}}(\mathbf{R}^2)$, there is a ρ^{YM}-a.e. defined $L(K)$-valued function $\tilde{f}$ on $\mathcal{E}'$ such that for any $u, v \in L(K)$ and $f, g \in L^2_{\text{real}}(\mathbf{R}^2)$, the random variables $\langle u, \tilde{f}(\cdot)\rangle_{L(K)}$ and $\langle \tilde{g}(\cdot), v\rangle_{L(K)}$ are mean 0 Gaussians with covariance $\langle u, v\rangle_{L(K)}\langle f, g\rangle_{L^2(\mathbf{R}^2)}$.

Now consider a curve in the plane which can be expressed in polar coordinates by means of a smooth function $C : [0, T] \to \mathbf{R}^2 : t \mapsto (r_C(t)\cos t, r_C(t)\sin t)$. We shall consider curves only of this type. Let L_t denote the loop formed by the radial segment from the origin o to $C(0)$, followed by the curve $C|[0,t]$, followed by the radial segment back to o. Denote by $h_t(A) \in K$ the holonomy of a connection A around the loop L_t (i.e. $uh_t(A)$ is the result of parallel-transport by A of the fixed initial point u around the loop A). If f^A is the function described earlier corresponding to a smooth connection A, then it is a fact that

$$dh_t = (-dM_t)h_t\,,$$

where M_t is the integral of f^A over the region enclosed by the loop L_t.

In passing to the case where f^A is no longer a smooth function but an element of $\mathcal{E}'$, one replaces the above smooth differential equation by a stochastic differential equation

$$dh_t = (-dM_t) \circ h_t\,,$$

where on the right we have a Stratonovich composite. Let us spell this out more clearly. Let E_t be the region enclosed by the loop L_t. Recall the $L(K)$-valued Gaussian random variable $\tilde{1}_{E_t}$ defined almost everywhere on $\mathcal{E}'$. It is clear that for $0 \le s \le t \le 2\pi$, the random variables $\tilde{1}_{E_t} - \tilde{1}_{E_s}$ and $\tilde{1}_{E_s}$ are independent, and the variance of any component of $\tilde{1}_{E_t} - \tilde{1}_{E_s}$ is bounded by constant times $|t-s|$. Then by Kolmogorov's lemma, there is a map $[0,T] \times \mathcal{E}' \to L(K) : (t, f) \mapsto M_t(f)$ such that: (i) for every $t_1, \ldots, t_n$, the random variables $(M_{t_1}, \ldots, M_{t_n})$ and $(\tilde{1}_{E_{t_1}}, \ldots, \tilde{1}_{E_{t_n}})$ are equal a.e., and (ii) $t \mapsto M_t(f)$ is continuous for every $f \in \mathcal{E}'$. Then there is a continuous K-valued process $t \mapsto h_t$, beginning at the identity e in K, satisfying the Itô stochastic differential equation

$$dh_t = -dM_t h_t + \frac{1}{2}(dM_t)^2 h_t\,.$$

We are now working with matrix-valued processes.

The stochastic holonomy around the loop L_t is, by definition, h_t.

Holonomies, both stochastic and deterministic, with values in $K_{\mathbf{C}}$ around loops are defined exactly analogously to the procedure above.

We now recall rapidly the basics of the Segal–Bargmann transform for an infinite-dimensional Gaussian measure space (details are in Sec. 3.2). Recall that ρ^{YM} is the variance 1 Gaussian measure on $\mathcal{E}'$. Let μ^{YM} be the variance 1/2 Gaussian measure on $\mathcal{E}'_{\mathbf{C}}$. Suppose that F is a cylinder function in $L^2(\rho^{\mathrm{YM}})$. Then we define SF to be the function on $\mathcal{E}'_{\mathbf{C}}$ which is the analytic continuation of the function

$$\mathcal{E}' \to \mathbf{C} : x \mapsto \int F(\phi + x)\, d\rho^{\mathrm{YM}}(\phi)\,.$$

Since F is a cylinder function, the integrand is meaningful and the integral is really finite-dimensional. Then S, so defined, maps cylinder functions in $L^2(\rho^{\mathrm{YM}})$ to holomorphic cylinder functions in $L^2(\mu^{\mathrm{YM}})$. This map is isometric and thus extends by continuity to an isometric map of $L^2(\rho^{\mathrm{YM}})$ into $L^2(\mu^{\mathrm{YM}})$. The image of this map is the L^2 closure of the L^2 holomorphic cylinder functions. We will call this closure the "holomorphic" subspace of $L^2(\mu^{\mathrm{YM}})$, denoted $\mathcal{H}L^2(\mu^{\mathrm{YM}})$ (Lemma 3.1.2).

Now, the elements of $\mathcal{H}L^2(\mu^{\mathrm{YM}})$ might not be actually holomorphic or even continuous on $\mathcal{E}'_{\mathbf{C}}$. However, if F is a holomorphic cylinder function on $\mathcal{E}'_{\mathbf{C}}$ (so everywhere-defined) then $|F(z)| \leq e^{||z||^2/2}||F||_{L^2(\mu^{\mathrm{YM}})}$. Thus, pointwise evaluation at an element of $\mathcal{E}_{\mathbf{C}}$ is a continuous linear functional on the space of L^2 holomorphic cylinder functions, and therefore extends continuously to a linear functional on $\mathcal{H}L^2(\mu^{\mathrm{YM}})$. We now define a "restriction map" R from $\mathcal{H}L^2(\mu^{\mathrm{YM}})$ to the space of functions on $\mathcal{E}_{\mathbf{C}}$ by defining R to be honest restriction for holomorphic cylinder functions and then extending by continuity at each point. This restriction map is one-to-one on $\mathcal{H}L^2(\mu^{\mathrm{YM}})$. The details, following a strategy slightly different from that just described, are in Sec. 3.1.

Consider loops $C_1, \ldots, C_n$, all based at the origin, each consisting of a radial segment, followed by a cross-radial segment, followed by a radial segment back to the origin. Here "cross-radial" means the form $(r(t)\cos t, r(t)\sin t)$. For any function f on K^n, we have the function $F = f(h(C_1), \ldots, h(C_n))$ on $\mathcal{E}'$; suppose that F is in $L^2(\rho^{\mathrm{YM}})$. Then the Segal–Bargmann transform $SF \in \mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu^{\mathrm{YM}})$ is meaningful. The main result of the present paper provides an explicit formula for SF for certain types of systems of loops $C_1, \ldots, C_n$:

Theorem 2.1. *Let $C_1, \ldots, C_n$, be loops, all based at the origin, of the type described above. Assume that the loops enclose disjoint regions $\Delta_1, \ldots, \Delta_n$, of the plane. Let f be a measurable function on K^n such that the function $F = f(h(C_1), \ldots, h(C_n))$ on $\mathcal{E}'$ is in $L^2(\rho^{\mathrm{YM}})$. Then the restriction $R(SF)$ of the Segal–Bargmann transform SF, as a function on $\mathcal{E}_{\mathbf{C}}$, is given explicitly by*

$$R(SF)(A) = \int_{K^n} f(x_1, \ldots, x_n)\rho_{|\Delta_1|}(x_1^{-1}z_1(A)) \cdots \rho_{|\Delta_n|}(x_n^{-1}z_n(A))\, dx_1 \cdots dx_n\,, \tag{2.1}$$

where $\rho_t(x)$ is the usual heat kernel, relative to Haar measure dx on K, analytically continued to $K_{\mathbf{C}}$, $|\Delta_j|$ is the area of Δ_j, and $z_j(A) \in K_{\mathbf{C}}$ is the complex holonomy of $A \in \mathcal{E}_{\mathbf{C}}$ around the loop C_j.

The full Segal–Bargmann transform SF is obtained by "analytic continuation" (from the *measure zero subspace* $\mathcal{E}$ to $\mathcal{E}'_{\mathbf{C}}$):

Theorem 2.2. *With hypotheses and notation as in the preceding result, the Segal–Bargmann transform SF, as a function in $\mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}})$, is given by*

$$(SF)(A) = \int_{K^n} f(x_1, \ldots, x_n)\rho_{|\Delta_1|}(x_1^{-1}z_1(A)) \cdots \rho_{|\Delta_n|}(x_n^{-1}z_n(A))\, dx_1 \cdots dx_n\,, \tag{2.2}$$

where now $z_j(A) \in K_{\mathbf{C}}$ is the complex stochastic holonomy of $A \in \mathcal{E}'_{\mathbf{C}}$ around the loop C_j.

For more complicated configurations of loops, the expression for SF does not appear to be as simple.

3. The Segal–Bargmann Transform and the Restriction Map

We will set up the technical background in this section. Many of the results of this section have appeared before in the literature in related but different frameworks.

Unless otherwise stated, $\mathcal{E}$ will be a real vector space with an inner product $\langle \cdot, \cdot \rangle_0$ such that the induced topology is separable. The completion of $\mathcal{E}$ will be denoted by $\mathcal{E}_0$. The complexification of a real vector space will denoted by a subscript $\mathbf{C}$. There is a real vector space $\mathcal{E}'$, a measure ρ on (some σ-algebra of subsets of) $\mathcal{E}'$, and there is a linear isometry $\mathcal{E}_0 \to L^2(\mathcal{E}', \rho) : x \mapsto \hat{x}$ such that

$$\int_{\mathcal{E}'} e^{i\hat{x}(\phi)}\, d\rho(\phi) = e^{-\frac{1}{2}|x|_0^2}\,, \qquad \text{for every } x \in \mathcal{E}_0\,.$$

We take ρ to be defined on the completed σ-algebra generated by random variables $\hat{x}$, as x runs over $\mathcal{E}$. Each $\hat{x}$ is a Gaussian random variable on $(\mathcal{E}', \rho)$ with mean 0 and variance $|x|_0^2$.

It may be assumed without loss of generality that $\mathcal{E}$ is a subspace of $\mathcal{E}'$. The significance of this lies only in the fact that for each $x \in \mathcal{E}$ there is a translation map $\tau_x : \mathcal{E}' \to \mathcal{E}' : \phi \mapsto \phi + x$, having natural algebraic properties and $d(\tau_x)_*\rho/d\rho = e^{-\hat{x}-\frac{1}{2}|x|_0^2}$.

The inner product $\langle \cdot, \cdot \rangle_0$ on the real vector space $\mathcal{E}$ gives rise to a hermitian inner product, also denoted $\langle \cdot, \cdot \rangle_0$ on $\mathcal{E}_{\mathbf{C}}$, whose real part $\mathrm{Re}\langle \cdot, \cdot \rangle_0$ is the natural real inner product on the real vector space $\mathcal{E} \oplus \mathcal{E} = \mathcal{E}_{\mathbf{C}}$. In contrast to these inner products, there is also induced a natural *complex bilinear* pairing $(\cdot, \cdot)_0$ on $\mathcal{E}_{\mathbf{C}}$.

We also work with the Gaussian measure μ on (a complex vector space) $\mathcal{E}'_{\mathbf{C}}$ specified by

$$\int_{\mathcal{E}'_{\mathbf{C}}} e^{i\hat{z}(\psi)}\, d\mu(\psi) = e^{-\frac{1}{4}|z|_0^2}\,, \qquad \text{for all } z \in (\mathcal{E}_0)_{\mathbf{C}}\,.$$

For each $z \in (\mathcal{E}_0)_{\mathbf{C}}$ there is a *complex* linear map $\mathcal{E}_{0\mathbf{C}} \to L^2(\mathcal{E}'_{\mathbf{C}}, \mu) : z \mapsto \tilde{z}$ given as follows:

$$\tilde{z} = \hat{\bar{z}} + i(\widehat{i\,\bar{z}}),$$

with $\bar{z}$ always denoting the complex conjugate of $z \in \mathcal{E}_{0\mathbf{C}}$.

Thus $\tilde{z}$ is a *complex* Gaussian random variable on $(\mathcal{E}'_{\mathbf{C}}, \mu)$, in contrast to the real Gaussian $\hat{z}$. For example, if $x \in \mathcal{E}_0$, then $\tilde{x}$ is a *complex* Gaussian on $(\mathcal{E}'_{\mathbf{C}}, \mu)$ whose real and imaginary parts are independent Gaussians each with mean 0 and variance $\frac{1}{2}|x|_0^2$.

To keep the notation close to the conceptual meaning, we shall often write $\psi \cdot z$ to mean $\tilde{z}(\psi)$, and $\bar{\psi} \cdot w$ to mean $\overline{\tilde{\bar{w}}(\psi)}$:

$$\psi \cdot z \stackrel{\text{def}}{=} \tilde{z}(\psi) \qquad \text{and} \qquad \bar{\psi} \cdot w \stackrel{\text{def}}{=} \overline{\tilde{\bar{w}}(\psi)}\,.$$

In our application, $\mathcal{E}$ will be $\mathcal{S}(\mathbf{R}^2) \otimes L(K)$, where $L(K)$ is the Lie algebra of a matrix group K, and $\langle \cdot, \cdot \rangle_0$ obtained from the $L^2(\mathbf{R}^2)$-inner product on $\mathcal{S}(\mathbf{R}^2)$ coupled with an Ad-invariant inner product on $L(K)$.

3.1. *The restriction map R*

For $z \in \mathcal{E}_{0\mathbf{C}}$, define the μ-a.e. defined random variable $\exp_z = e^{\tilde{z}}$:

$$\exp_z : \mathcal{E}'_{\mathbf{C}} \to \mathbf{C} : \psi \mapsto e^{\psi \cdot z}\,.$$

Then a straightforward calculation gives

$$\langle\langle \exp_z, \exp_w \rangle\rangle_{L^2(\mathcal{E}'_{\mathbf{C}}, \mu)} = e^{\langle z, w \rangle_0}\,.$$

So $\|\exp_z - \exp_w\|^2_{L^2(\mu)} = e^{|z|_0^2} - 2\mathrm{Re}(e^{\langle z, w\rangle_0}) + e^{|w|_0^2} \to 0$ as $w \to z$ in $\mathcal{E}_{0\mathbf{C}}$. Thus $\mathcal{E}_{0\mathbf{C}} \to L^2(\mathcal{E}'_{\mathbf{C}}, \mu) : z \mapsto \exp_z$ is continuous.

For $f \in L^2(\mathcal{E}', \rho)$, we define the function Rf on $\mathcal{E}_{\mathbf{C}}$ by

$$Rf(z) = \int_{\mathcal{E}'_{\mathbf{C}}} f(\psi) e^{\bar{\psi} \cdot z}\, d\mu(\psi) = \langle f, \exp_{\bar{z}} \rangle_{L^2(\mu)}\,. \tag{3.1a}$$

Note that $Rf(z)$ is meaningful for $z \in \mathcal{E}_{0\mathbf{C}}$.

We also work with the space

$$\mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu) = \text{closed linear span of } \{\exp_x : x \in \mathcal{E}\} \text{ in } L^2(\mu)\,. \tag{3.1b}$$

(We will see later, by Lemma 3.2, that $\mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$ also contains $\exp_z$, for all $z \in \mathcal{E}_{0\mathbf{C}}$.)

A *holomorphic cylinder function* on $\mathcal{E}'_{\mathbf{C}}$ is a function of the form $F = H(\tilde{f}_1, \ldots, \tilde{f}_n)$ for some holomorphic function H on $\mathbf{C}^n$ and some $f_1, \ldots, f_n \in \mathcal{E}_{\mathbf{C}}$. There is a natural *restriction* of a holomorphic cylinder function F on $\mathcal{E}'_{\mathbf{C}}$ to a function $R'F$ on $\mathcal{E}_{\mathbf{C}}$: for F as given before, we can define the evaluation of $R'F$ on $(v_1, \ldots, v_n) \in \mathcal{E}_{\mathbf{C}}$ to be $H((f_1, v_1)_0, \ldots, (f_n, v_n)_0)$, where $(\cdot, \cdot)_0$ is the complex bilinear pairing on $\mathcal{E}_{\mathbf{C}}$ induced by the inner product on $\mathcal{E}$. Part (i) of the following result says that $R'F = RF$, and so, in particular, the restriction $R'F$ is specified uniquely by the random variable F and does not depend on the choice of H or $f_1, \dots, f_n$.

Lemma 3.1. *The restriction map R has the following properties:*

(i) *if $f \in L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$ is a holomorphic cylinder function, then Rf is the restriction of f to $\mathcal{E}_{\mathbf{C}}$,*
(ii) $|Rf(z)| \le |f|_{L^2(\mu)} e^{|z|_0^2/2}$,
(iii) *Rf is continuous on $\mathcal{E}_{\mathbf{C}}$,*
(iv) *for each $f \in \mathcal{H}L^2(\mu)$, Rf is holomorphic on every finite-dimensional subspace of $\mathcal{E}_{\mathbf{C}}$,*
(v) *R is one-to-one on $\mathcal{H}L^2(\mu)$,*
(vi) *for any $f, h \in \mathcal{H}L^2(\mu)$, if Rf and Rh agree on $\mathcal{E}$, then $f = h$,*
(vii) *for any $f \in \mathcal{H}L^2(\mu)$ and $z \in \mathcal{E}_{\mathbf{C}}$,*

$$Rf(z) = \int_{\mathcal{E}'_{\mathbf{C}}} f(\psi + z)\, d\mu(\psi)\,. \tag{3.1c}$$

(*In* (3.1c) *we have used the identification of $\mathcal{E}_{\mathbf{C}}$ with a subspace of $\mathcal{E}'_{\mathbf{C}}$.*)

Proof. (i) Let f be a holomorphic cylinder function in $L^2(\mu)$. Then, for any $z \in \mathcal{E}_{\mathbf{C}}$, changing variables $\psi \mapsto \psi - z$ and using Cauchy–Schwarz, the function $\psi \mapsto f(\psi + z)e^{-\psi\cdot\bar{z}}$ is in $L^1(\mu)$. The integral $\int f(\psi + z)e^{-\psi\cdot\bar{z}}\, d\mu(\psi)$, being a radially symmetric average of a holomorphic function $R'f$ on a finite-dimensional space, is equal to the value of the integrand at 0, i.e. to $R'f(z)$. On the other hand, changing variables $\psi \mapsto \psi - z$ transforms the integral $\int f(\psi + z)e^{-\psi\cdot\bar{z}}\, d\mu(\psi)$ to $Rf(z)$.

Part (ii) follows by the Cauchy–Schwarz inequality.

(iii) As noted before, $\mathcal{E}_{\mathbf{C}} \to L^2(\mathcal{E}'_{\mathbf{C}}, \mu) : z \mapsto \exp_z$ is continuous with respect to the $|\cdot|_0$-norm on $\mathcal{E}_{\mathbf{C}}$. Hence so is $z \mapsto \exp_{\bar{z}}$ and therefore also $z \mapsto Rf(z) = \langle f, \exp_{\bar{z}}\rangle_{L^2(\mu)}$.

(iv) By Hartog's lemma, it will suffice to prove that the function $H(\lambda) = RF(a+\lambda b)$ is holomorphic in $\lambda \in \mathbf{C}$, where a, b are arbitrary elements of $\mathcal{E}_{\mathbf{C}}$. Now, as λ runs over a closed loop C in the complex plane, we have $\sup_{\lambda\in C} |f(\psi)e^{\bar{\psi}\cdot(a+\lambda b)}| \le |f(\psi)|e^{|\psi\cdot\bar{a}|+K|\psi\cdot\bar{b}|}$, for some finite constant K, and the latter function is in $L^1(\mu)$. Thus, by dominated convergence, H is continuous, and, by Fubini's theorem, $\int_C H(\lambda)\, d\lambda = 0$. Then, by Morera's theorem, H is holomorphic.

(v) Suppose $Rf = 0$. Then, by definition of Rf, f is orthogonal to $\exp_z$ for all $z \in \mathcal{E}_{\mathbf{C}}$. Since $\mathcal{H}L^2(\mu)$ is the closed linear span of $\{\exp_x : x \in \mathcal{E}\}$, we conclude that $f \perp \mathcal{H}L^2(\mu)$. Thus R is one-to-one on $\mathcal{H}L^2(\mu)$.

(vi) Suppose $Rf|\mathcal{E} = 0$, where $f \in \mathcal{H}L^2(\mu)$. Then, for any $a, b \in \mathcal{E}$, the holomorphic function $\lambda \mapsto Rf(a + \lambda b)$ is 0 on the real line and so is 0 everywhere; in particular, $Rf(a + ib) = 0$. Thus $Rf = 0$ on $\mathcal{E}_{\mathbf{C}}$ and so, by (v), $f = 0$.

(vii) The integral is well-defined by the Cameron–Martin theorem. For $f \in \mathcal{H}L^2(\mu)$, let $R^z f = Rf(z)$ and $R_1^z f = \int_{\mathcal{E}'_{\mathbf{C}}} f(\psi + z)\, d\mu(\psi)$. Changing variables, we have $R_1^z f = \int_{\mathcal{E}'_{\mathbf{C}}} f(\psi) e^{\psi\cdot\bar{z}+\bar{\psi}\cdot z-|z|_0^2}\, d\mu(\psi)$. Then by Cauchy–Schwarz, R_1^z is a bounded linear functional on $L^2(\mu)$, as is R^z (by (ii)). If $f = \exp_w$ for some $w \in \mathcal{E}_{\mathbf{C}}$, then $R_1^z f = \int_{\mathcal{E}'_{\mathbf{C}}} f(\psi+z)\, d\mu(\psi)$, being a radially symmetric average of a holomorphic function of essentially one variable, equals the value of the integrand at 0, i.e. to $f(z)$, which, by (i), is also equal to $R^z f$. Thus R^z and R_1^z agree on $\{\exp_x : z \in \mathcal{E}\}$ and hence on the closure $\mathcal{H}L^2(\mu)$. □

Lemma 3.2. *$\mathcal{H}L^2(\mu)$ is the closed linear span in $L^2(\mu)$ of all holomorphic cylinder functions on $\mathcal{E}'_{\mathbf{C}}$ which belong to $L^2(\mu)$.*

Proof. Since each $\exp_x$ is a holomorphic cylinder function, it will suffice to show that every holomorphic cylinder function in $L^2(\mu)$ is in $\mathcal{H}L^2(\mu)$.

First assume that $\mathcal{E}$ has finite dimension n. Let H be a holomorphic function on $\mathbf{C}^n$ which is in $L^2(\mu)$. Denote by $H_\perp$ the component of H which is $L^2(\mu)$-orthogonal to the closed subspace spanned by the "real" exponentials $\exp_x$, with x running over $\mathcal{E}$. Note that each $\exp_x$ is holomorphic. Then, by definition of the map R, we have $RH_\perp = 0$. Since holomorphic functions on $\mathbf{C}^n$ form a closed subspace of the $L^2(\mathbf{C}^n)$-space corresponding to radially symmetric Gaussian measure (distinct monomials on $\mathbf{C}^n$ are orthogonal with respect to the measure), it follows that $H_\perp$ is also holomorphic. So by Lemma 3.1(i), $RH_\perp = H_\perp$. So $H_\perp = 0$, i.e. $H \in \mathcal{H}L^2(\mu)$. Now consider the general case, i.e. $\mathcal{E}$ may be infinite-dimensional. Let h be a holomorphic cylinder function on $\mathcal{E}'_{\mathbf{C}}$ which is in $L^2(\mu)$. Then $h = H(\tilde{z}_1, \ldots, \tilde{z}_n)$ for some $z_1, \ldots, z_n \in \mathcal{E}_{\mathbf{C}}$ and some holomorphic function H on $\mathbf{C}^n$. Let $e_1, \ldots, e_n$ be an $\langle\cdot,\cdot\rangle_0$-orthonormal basis of a complex subspace of $\mathcal{E}_{\mathbf{C}}$ containing $z_1, \ldots, z_n$. Then $h = H'(\tilde{e}_1, \ldots, \tilde{e}_n)$, for some holomorphic function H' on $\mathbf{C}^n$. Now the Borel measure $\mu_n = \mu \circ (\tilde{e}_1, \ldots, \tilde{e}_n)^{-1}$ on $\mathbf{C}^n$ is the Gaussian $\pi^{-d} e^{-|z|^2} dz$, and so, by the finite-dimensional case we have already proven, H' is the $L^2(\mu_n)$-limit of a sequence of functions H'_k each of which is a linear combination of "real" exponentials on $\mathbf{C}^n$ (i.e. functions of the form $\exp_x$ for $x \in \mathbf{R}^n$). Let $h_k = H'_k(\tilde{e}_1, \ldots, \tilde{e}_n)$. Then $h'_k \to H'(\tilde{e}_1, \ldots, \tilde{e}_n) = h$ in $L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$, and each h_k is in $\mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$. □

Lemma 3.3. *Let H be a holomorphic function on $\mathbf{C}^n$, and $x = (x_1, \ldots, x_n) \in \mathcal{E}_0$. Assume that the random variable $H_x = H \circ (\tilde{x}_1, \ldots, \tilde{x}_n)$ is in $L^2(\mu)$. Then $H_x \in \mathcal{H}L^2(\mu)$.*

Proof. Since distinct monomials on $\mathbf{C}^n$ are orthogonal under radially symmetric Gaussian measure, it will suffice to prove the result in the case when H is a monomial. So assume that H is a monomial. For $w \in (\mathcal{E}_0)_{\mathbf{C}}$, we have the random variable $\tilde{w} : \mathcal{E}'_{\mathbf{C}} \to \mathbf{C} : \psi \mapsto \psi \cdot w$. Fix $x \in \mathcal{E}_0^n$. Then for $y \in \mathcal{E}_0^n$ close to x, the difference $H_x - H_y$ can be expressed as a polynomial in the $\tilde{x}_j - \tilde{y}_j$ with coefficients which are products in $\tilde{y}_i$ and $\tilde{x}_i$. Using Cauchy–Schwarz repeatedly and the fact that $|\tilde{w}|_p$ (for $p \geq 2$) is bounded by constant times a power of $|w|_0$, it follows that $y \mapsto H_y$ is continuous at x. □

3.2. *The Segal–Bargmann transform*

The Segal–Bargmann transform is a unitary isomorphism $S : L^2(\rho) \to \mathcal{H}L^2(\mu)$. We summarize the standard definition within our framework in the following.

Proposition 3.1. *There is a unique unitary isomorphism*

$$S : L^2(\mathcal{E}', \rho) \to \mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$$

such that for any $F \in L^2(\rho)$, the restriction of SF to $\mathcal{E}$ is given by

$$R(SF)(x) = \int_{\mathcal{E}'} F(\phi + x)\, d\rho(\phi) \qquad \textit{for all } x \in \mathcal{E}\,. \tag{3.2a}$$

Proof. For each $z \in \mathcal{E}_{\mathbf{C}}$, define χ_z on $\mathcal{E}'$ and $\exp_z$ on $\mathcal{E}'_{\mathbf{C}}$ by

$$\chi_z(\phi) = e^{\phi\cdot z - \frac{1}{2}(z,z)_0}, \qquad \text{and} \quad \exp_z(\psi) = e^{\psi\cdot z} \qquad \text{for all } \phi \in \mathcal{E}' \text{ and } \psi \in \mathcal{E}'_{\mathbf{C}}\,, \tag{3.2b}$$

i.e. $\chi_z = e^{\tilde{z}|\mathcal{E}' - \frac{1}{2}(z,z)_0}$ (with $\tilde{z}$ defined in the obvious way on $\mathcal{E}'$) and $\exp_z = e^{\tilde{z}}$. Then, for any $x \in \mathcal{E}$, the integral $\int_{\mathcal{E}'} \chi_z(\phi + x)\, d\rho(\phi)$ reduces to a one-dimensional Gaussian integral which works out to be $e^{x\cdot z}$, which is $(R\exp_z)(x)$. Thus, for (3.2a) to hold for $f = \chi_z$, we must have

$$S\chi_z = \exp_z \tag{3.2c}$$

for all $z \in \mathcal{E}_{\mathbf{C}}$.

A simple calculation involving one- and two-dimensional Gaussian integrals shows that

$$\langle \chi_z, \chi_w \rangle_{L^2(\rho)} = e^{\langle z,w\rangle_0} = \langle \exp_z, \exp_w \rangle_{L^2(\mu)}\,, \tag{3.2d}$$

for all $z, w \in \mathcal{E}_{\mathbf{C}}$. Thus there is indeed a unique unitary isomorphism S from the closed linear span of $\{\chi_z : z \in \mathcal{E}_{\mathbf{C}}\}$, i.e. from $L^2(\rho)$, onto the closed linear span of $\{\exp_z : z \in \mathcal{E}_{\mathbf{C}}\}$, i.e. $\mathcal{H}L^2(\mu)$, satisfying (3.2c).

Fix $x \in \mathcal{E}$. If $F \in L^2(\mathcal{E}', \rho)$, we have, by change of variables, $\int_{\mathcal{E}'} F(\phi + x)\, d\rho(\phi) = \int_{\mathcal{E}'} F(\phi)\chi_x(\phi)\, d\rho(\phi)$. Since $\chi_x \in L^2(\rho)$, the mapping $S_1^x : F \mapsto \int_{\mathcal{E}'} F(\phi + x)\, d\rho(\phi)$ is continuous on $L^2(\rho)$. By continuity of S and of $R^x : F \mapsto (RF)(x)$ (Lemma 3.1(ii)), the linear map $S_2^x : F \mapsto R(SF)(x)$ is continuous. Thus S_1^x and S_2^x, being continuous linear functionals on $L^2(\rho)$ which agree on the dense subspace spanned by $\{\chi_z : z \in \mathcal{E}_{\mathbf{C}}\}$, are equal. This proves that (3.2a) holds for all $F \in L^2(\rho)$. □

3.3. *Independence and products for S and R*

Recall that for each $x \in \mathcal{E}_0$ we have a ρ-a.e. defined Gaussian random variable $\hat{x}$ on $(\mathcal{E}', \rho)$ with mean 0 and variance $|x|_0^2$. For V a subspace of $\mathcal{E}_0$, let σ_V be the completed σ-algebra generated by the (ρ-a.e. defined) functions $\hat{x} : \mathcal{E}' \to \mathbf{R}$, as x runs over V. The algebra of functions spanned by $\{e^{\hat{x}} : x \in V\}$ is dense in $L^2(\rho|\sigma_V)$. If V and W are orthogonal subspaces of $\mathcal{E}_0$, then σ_V and σ_W are independent relative to the measure μ (on $\sigma_{\mathcal{E}_\mathbf{C}}$), because for any $x_1, \ldots, x_m \in V$ and $y_1, \ldots, y_n \in W$, the (Gaussian) random variables $(\hat{x}_1, \ldots, \hat{x}_m)$ and $(\hat{y}_1, \ldots, \hat{y}_n)$ are independent. So the map j_{VW} specified as

$$j_{VW} : L^2(\mathcal{E}', \rho|\sigma_V) \otimes L^2(\mathcal{E}', \rho|\sigma_W) \to L^2(\mathcal{E}', \rho) : f \otimes g \mapsto fg$$

is a unitary isomorphism onto $L^2(\mathcal{E}', \rho|\sigma_{V+W})$. We have similarly a map $j_{V_1 \cdots V_n}$ associated to orthogonal subspaces $V_1, \ldots, V_n$.

Let $\sigma_{\tilde{V}}$ be the completed σ-algebra of subsets of $\mathcal{E}'_\mathbf{C}$ generated by the functions $\tilde{x}$, where x runs over a subspace V of $\mathcal{E}_0$. Then for V and W orthogonal subspaces of $\mathcal{E}_0$, $\sigma_{\tilde{V}}$ and $\sigma_{\tilde{W}}$ are independent and so the map $\tilde{j}_{VW}$ specified by

$$\tilde{j}_{VW} : L^2(\mathcal{E}'_\mathbf{C}, \mu|\sigma_{\tilde{V}}) \otimes L^2(\mathcal{E}'_\mathbf{C}, \mu|\sigma_{\tilde{W}}) \to L^2(\mathcal{E}'_\mathbf{C}, \mu) : f \otimes g \mapsto fg$$

is a linear isometry. We have similarly a map $\tilde{j}_{V_1 \cdots V_n}$ associated to orthogonal subspaces $V_1, \cdots, V_n$.

Proposition 3.2. (i) *If $f \in L^2(\mathcal{E}', \rho|\sigma_V)$, then $Sf \in L^2(\mathcal{E}'_\mathbf{C}, \mu|\sigma_{\tilde{V}})$, for any subspace V of $\mathcal{E}_0$.*

(ii) *If $V_1, \ldots, V_n$ are orthogonal subspaces of $\mathcal{E}_0$ then for any $f_j \in L^2(\rho|\sigma_{V_j})$,*

$$S(f_1 \cdots f_j) = Sf_1 \cdots Sf_n\,.$$

In other words, $S \circ j_{V_1 \cdots V_n} = \tilde{j}_{V_1 \cdots V_n} \circ (S_{V_1} \otimes \cdots \otimes S_{V_n})$, where S_X is $S|L^2(\mathcal{E}', \rho|\sigma_X)$.

Proof. (i) By (3.2c), $S\chi_x = \exp_x$ for every $x \in V$. Since $\{\chi_x : x \in V\}$ spans a dense subspace of $L^2(\rho|\sigma_V)$ and $\exp_x \in L^2(\mu|\sigma_{\tilde{V}})$ for every $x \in V$, it follows by continuity of S that S maps $L^2(\rho|\sigma_V)$ into $L^2(\mu|\sigma_{\tilde{V}})$.

(ii) Since $\{\chi_x : x \in X\}$ spans a dense subspace of $L^2(\rho|\sigma_X)$, and since S is continuous, we may assume that $f_j = \chi_{v_j}$ for some $v_j \in V_j$. In this case, the equality $S(f_1 \cdots f_j) = Sf_1 \cdots Sf_n$ follows directly by using $S\chi_x = \exp_x$ and the equality $\chi_{v_1 + \cdots + v_n} = \chi_{v_1} \cdots \chi_{v_n}$, which follows by orthogonality of the subspaces V_j. □

For any complex number λ, we have $\lambda = \mathrm{Re}(\lambda) - i\,\mathrm{Re}(i\lambda)$. Thus

$$\langle a, b \rangle_0 = \mathrm{Re}\,\langle a, b \rangle_0 - i\,\mathrm{Re}\langle ia, b \rangle_0\,.$$

It follows that if V and W are complex subspaces of $(\mathcal{E}_0)_\mathbf{C}$, then they are orthogonal with respect to the hermitian inner product $\langle \cdot, \cdot \rangle_0$ if and only if they are orthogonal as real subspaces with respect to the real inner product $\langle \cdot, \cdot \rangle_0$.

Recall that for $w \in (\mathcal{E}_0)_\mathbf{C}$, there is a real Gaussian $\hat{w}$ on $(\mathcal{E}'_\mathbf{C}, \mu)$ and a complex Gaussian $\tilde{w} = \hat{\bar{w}} + i(\widehat{i\,\bar{w}})$.

If S is a subset of $(\mathcal{E}_0)_\mathbf{C}$, we have two corresponding completed σ-algebras of subsets of $\mathcal{E}'_\mathbf{C}$:

$$\sigma_S = \sigma\{\hat{w} : w \in S\} \qquad \text{and} \qquad \sigma_{\tilde{S}} = \sigma\{\tilde{w} : w \in S\}\,.$$

If V is a complex subspace of $(\mathcal{E}_0)_{\mathbf{C}}$, then $\sigma_V = \sigma_{\tilde{V}}$. If V is a real subspace of $\mathcal{E}_0 \subset \mathcal{E}_{0\mathbf{C}}$, then $\sigma_{\tilde{V}} = \sigma_{V+iV}$.

If V is a complex subspace of $(\mathcal{E}_0)_{\mathbf{C}}$ then the real-orthogonal complement $V^{\perp}$ coincides with the hermitian-orthogonal complement, and the σ-algebras σ_V and $\sigma_{V^{\perp}}$ are independent.

If V and W are orthogonal subspaces of $\mathcal{E}_0$, then $V + iV$ and $W + iW$ are orthogonal complex subspaces of $(\mathcal{E}_0)_{\mathbf{C}}$, and so $\sigma_{\tilde{V}}$ and $\sigma_{\tilde{W}}$ are independent.

Proposition 3.3. (i) *Let V be a complex subspace of $(\mathcal{E}_0)_{\mathbf{C}}$. If $f \in L^2(\mathcal{E}'_{\mathbf{C}}, \mu|\sigma_{\tilde{V}})$ and $z \in \mathcal{E}_{0\mathbf{C}}$, then $(Rf)(z) = (Rf)(z_V)$, where z_V is the orthogonal projection of z onto $\bar{V}$.*

(ii) *If $V_1, \ldots, V_n$ are orthogonal subspaces of $\mathcal{E}_0$ and $f_j \in L^2(\mu|\sigma_{\tilde{V}_j})$, then*

$$R(f_1 \cdots f_n) = Rf_1 \cdots Rf_n .$$

Proof. (i) Decomposing z into orthogonal pieces $z = z_V + z_{V^{\perp}}$, with $z_V \in V$ and $z_{V^{\perp}} \in V^{\perp}$, we have $\exp_z = \exp_{z_V} \exp_{z_{V^{\perp}}}$. Since the pair $(f, \exp_z)$ is σ_V-measurable and $\overline{\exp_{z_{V^{\perp}}}}$ is $\sigma_{V^{\perp}}$-measurable, they are independent, and so the definition of R implies:

$$\begin{aligned}(Rf)(z) &= \int_{\mathcal{E}'_{\mathbf{C}}} f\, \overline{\exp_z}\, d\mu \\ &= \int_{\mathcal{E}'_{\mathbf{C}}} f\, \overline{\exp_{z_V}}\, d\mu \int_{\mathcal{E}'_{\mathbf{C}}} \overline{\exp_{z_{V^{\perp}}}}\, d\mu \\ &= (Rf)(z_V)(R1)(z_{V^{\perp}}) \\ &= (Rf)(z_V)\end{aligned}$$

(ii) This follows as in (i) by using independence. □

3.4. *Brownian motion*

Basic facts of stochastic differentials may be found in Ref. 11. If A and B are matrix-valued random variables, then we denote by $\langle\langle A, B\rangle\rangle_{L^2}$ the inner product $E(\mathrm{tr}(AB^*))$, where tr is normalized trace. Since the norm $\|A\|_{\mathrm{tr}} = \sqrt{\mathrm{tr}(A^*A)}$ on the (finite-dimensional) space of matrices A is equivalent to the usual sup-norm $\|A\|_{\mathrm{sup}}$, we have

$$\mathrm{tr}(ACB) \le (\mathrm{constant} \cdot \|C\|_{\mathrm{tr}})\|A\|_{\mathrm{tr}}\|B\|_{\mathrm{tr}} . \tag{3.4a}$$

Lemma 3.4. *Let $t \mapsto M_t$, $0 \le t \le 1$, be a matrix-valued continuous martingale, and denote by $t \mapsto h_M(t)$ the solution of the stochastic differential equation*

$$dh_M(t) = \left[-dM(t) + \frac{1}{2}dM(t)^2\right] h_M(t) \tag{3.4b}$$

with $h_M(0) = I$, *the identity. Then, with* $dN = -dM + \frac{1}{2}dM^2$,

$$h_M(t) = 1 + \int_0^t dN(t_1) + \int_0^t dN(t_2)\int_0^{t_2} dN(t_1) + \cdots\,, \tag{3.4c}$$

the series converging almost-surely and in L^2.

Proof. The series on the right in (3.4c), obtained by the usual iteration method, may be expressed as $\sum_{k=0}^{\infty}\eta_k$, where $\eta_0(t) = 1$ and, for $k \geq 1$, $\eta_k(t) = \int_0^t dN(s)\eta_{k-1}(s)$. We will show that $\sum_{k=0}^{\infty}\|\eta_k(t)\|_{L^2} < \infty$. Let $D_k(t) = \|\eta_k(t)\|_{L^2}^2$. Then, for $k \geq 1$,

$$\begin{aligned} D_k(t) &= E\left[\operatorname{tr}\int_0^t d(\eta_k(s)^*\eta_k(s))\right] \\ &= E\left[\operatorname{tr}\int_0^t \eta_{k-1}(s)^*[dN(s)^* + dN(s) + dN(s)^*dN(s)]\eta_{k-1}(s)\right]. \end{aligned}$$

The term $[\cdots]$ in the integrand is of the form d(martingale) + $d\Phi(s)$, where Φ is a process of bounded variation. So, using the trace inequality (3.4a), $D_k(t) \leq C\int_0^t D_{k-1}(s)\,d\phi(s)$, for some finite constant C, with ϕ being an increasing function. So $D_k(t) \leq (C\phi(t))^k/k!$. Thus $\sum_{k=0}^{\infty}\|\eta_k(t)\|_{L^2} < \infty$, i.e. the series in (3.4c) converges in L^2.

The above iterative scheme is the traditional way to solve a stochastic differential equation[12,13] and the right side of (3.4c) converges almost everywhere to the left-hand side. □

Note. If in the above result we work with complex matrices, in particular if (3.4b) is in the complex matrix sense, then the standard Itô formula is valid only for functions which are differentiable in the complex sense, i.e. for holomorphic functions.

The following is valid for real and complex matrix groups.

Lemma 3.5. *Let* π *be a continuous representation of a connected matrix group, and* $t \mapsto H_t$, $0 \leq t \leq 1$, *a Brownian motion with values in the Lie algebra of the group. Let* $t \mapsto h(t)$ *be the solution of* $dh = [-dH + \frac{1}{2}dH^2]h$, *with* $h(0) = I$, *the identity matrix. Then* $z(t) = \pi(h(t))$ *is the solution of* $dz = [-d\pi_* H + \frac{1}{2}(d\pi_* H)^2]z$ *starting at the identity. In particular, the expansion* (3.4c) *holds for* $\pi(h(t))$ *with* dM *replaced by* $d\pi_* H$.

Proof. Let $h_n(0) = I$ and, for $t > 0$, $h_n(t) = \exp[H(t) - H(k2^{-n})]h_n(k2^{-n})$, where $k = [2^n t]$. Then by p. 120 of Ref. 13, $h_n(t) \to h(t)$ with probability 1. So $\pi(h_n(t)) \to \pi(h(t))$ with probability 1. Now $\pi(h_n(t)) = \exp[\pi_* H(t) - \pi_* H(k2^{-n})]\pi(h_n(k2^{-n}))$. So, again by the result from McKean[13] just quoted, $\pi(h_n(t))$ converges to $z(t)$ with probability 1. Thus $\pi(h(t)) = z(t)$, with probability 1. □

Each iterated integral appearing in (3.4c) is an iterated integral over a simplex and, approximating each such simplex by a union of small "rectangles", it follows that each iterated integral appearing in (3.4b) is the L^2-limit of matrices whose entries are polynomials in the entries of the matrices $M(u) - M(v)$, for $u, v \in [0, t]$. Combining this with Lemma 3.5, we have:

Proposition 3.4. (i) *Let π be a continuous representation of a matrix group H onto another matrix group, and M an $L(H)$-valued Brownian motion on $(\mathcal{E}', \rho)$. Suppose that V is a subspace of $\mathcal{E}_0$ such that, for each $t \geq 0$, each entry of the matrix $M(t)$ is of the form $\hat{v}$ for some $v \in V$. Then, for any $t \geq 0$, $\pi(h_M(t)) \in L^2(\rho|\sigma_V)$.*

(ii) *Let π be a holomorphic representation of a complex matrix group G onto a complex matrix group, Z a Brownian motion on $(\mathcal{E}'_{\mathbf{C}}, \mu)$ with values in the complex matrix Lie algebra $L(G)$. Suppose that V is a subspace of $\mathcal{E}_0$ such that, for each $t \geq 0$, each entry of the matrix $Z(t)$ is of the form $\tilde{v}$ for some $v \in V$. Then, for any $t \geq 0$, $\pi(h_Z(t)) \in \mathcal{H}L^2(\mu|\sigma_{\tilde{V}})$.*

We will say that random variables X and X' have the same distribution conditional on a σ-algebra $\mathcal{A}$ (or a collection $\{X_i\}$ of random variables) if for any $\mathcal{A}$-measurable random variable f, the random variables (f, X) and (f, X') have the same distribution. The following observation, in which we take h_M to be meaningful for semimartingale M, will be useful:

Lemma 3.6. *If A and E are continuous independent semimartingales and dE has Ad-invariant distribution given the past, then h_{A+E} and $h_A h_E$ have the same law. Moreover, for any process B such that (B, A) is independent of E,*

$$\boxed{\text{the joint law of } (B, h_{A+E}) \text{ is the same as that of } (B, h_A h_E).}$$

Proof. By Itô calculus, we have

$$\begin{aligned} d(h_A h_E)(h_A h_E)^{-1} &= [(dh_A)h_E + h_A\, dh_E + (dh_A)(dh_E)](h_A h_E)^{-1} \\ &= (dh_A)h_A^{-1} + \mathrm{Ad}(h_A)(dh_E)h_E^{-1} + 0\,. \end{aligned}$$

Since $(dh_E)h_E^{-1} = -dE + \frac{1}{2}(dE)^2$, and since the hypotheses imply that $\mathrm{Ad}(h_A)\, dE$ has the same distribution as dE, given (A, B), it follows that $(d(h_A h_E))(h_A h_E)^{-1}$ has the same distribution, on any interval, as $-d(A + E) + \frac{1}{2}[d(A + E)]^2$, i.e. the same distribution as $(dh_{A+E})h_{A+E}^{-1}$, given B. □

In our application, E will be deterministic.

Note. The effect of deterministic translations on Wiener measure and paths on groups[2,6] has been used in Refs. 7 and 10, and by Sadasue in Ref. 14, to relate the Segal–Bargmann transform for the path spaces on $L(K)$ to the corresponding transform for K. The general case, with non-deterministic E, has been used in the context of two-dimensional gauge theory in Lemma 4.10 of Ref. 16.

3.5. *R and S for functions of* $h_M(t)$

We work with a compact connected matrix group K, with Lie algebra $L(K)$ equipped with an Ad-invariant metric. Denote by ρ_t the heat kernel on K at time t, and by μ_t the heat kernel on $K_{\mathbf{C}}$ (relative to a fixed bi-invariant Haar measure) at time $t/2$. As shown in Ref. 8, ρ_t has an analytic continuation to a holomorphic function on $K_{\mathbf{C}}$, which we will denote also by ρ_t.

The Lie algebra $L(K)_{\mathbf{C}}$ and the group $K_{\mathbf{C}}$ are taken to consist of complex matrices, with $L(K)$ and K consisting of real matrices of the same dimension. For the Ad-invariant metric on $L(K)$ and the corresponding metric on $L(K)_{\mathbf{C}}$, there is the standard Brownian motion on $L(K)$ and $L(K)_{\mathbf{C}}$.

Lemma 3.7. *Let* $Z : [0,T] \to L(K)_{\mathbf{C}}$ *be Brownian over the probability space* $(\mathcal{E}'_{\mathbf{C}}, \mu)$. *Suppose there is a subspace* V *of* $\mathcal{E}_0$ *such that each entry of the matrix* $Z(t)$ *is of the form* $\tilde{v}$ *for some* $v \in V$. *Then for any function* f *on* K *which is a linear combination of matrix elements of representations of* K, *the function* $\hat{f}$ *given* μ*-a.e. on* $\mathcal{E}'_{\mathbf{C}}$ *by*

$$\hat{f}(\omega) = \int_K f(x)\rho_T(x^{-1}h_Z(T;\omega))\,dx \tag{3.5a}$$

is in $\mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu|\sigma_{\tilde{V}})$.

Proof. Under the hypothesis on f, the integral $\int_K f(x)\rho_t(x^{-1}y)\,dx$ is a finite linear combination of matrix entries $\pi_{ij}(y)$, where π runs over representations of K. Using analytic continuation and putting $h_Z(T;\omega)$ in place of y, it follows by Proposition 3.4(ii) that $\hat{f} \in \mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu|\sigma_{\tilde{V}})$. □

As we shall see later, the above result holds for all $f \in L^2(K, \rho_T)$.

Let V be a subspace of $\mathcal{E}_0$, and $E_1, \ldots, E_d$ a basis of $L(K)$. Consider a path $t \mapsto (v_1(t), \ldots, v_d(t)) \in V^d$, and let

$$M(t) = \hat{v}_1(t)E_1 + \cdots + \hat{v}_d(t)E_d\,, \tag{3.5b}$$

as an $L(K)$-valued random variable over $(\mathcal{E}', \rho)$. There is also the $L(K)_{\mathbf{C}}$-valued random variable over $(\mathcal{E}'_{\mathbf{C}}, \mu)$ given by

$$Z_t = \tilde{M}_t = \tilde{v}_1(t)E_1 + \cdots + \tilde{v}_d(t)E_d\,. \tag{3.5c}$$

If M is standard Brownian motion on $L(K)$, then Z is Brownian motion on $L(K)_{\mathbf{C}}$ with time scaled by half (because of the way μ is defined).

Recall that for $v \in \mathcal{E}_0$ the function $\tilde{v} = \hat{v} + i\widehat{(iv)}$ on $\mathcal{E}'_{\mathbf{C}}$ is, directly by definition, a holomorphic cylinder function on $\mathcal{E}'_{\mathbf{C}}$. By Lemma 3.1(i), the restriction $R\tilde{v}$ to $\mathcal{E}_{\mathbf{C}}$ is given by

$$(R\tilde{v})(\omega) = \tilde{v} \cdot i_{\mathbf{C}}(\omega) = (v, w)_0\,,$$

where $(\cdot,\cdot)_0$ is the complex-bilinear extension to $\mathcal{E}_{0\mathbf{C}}$ of the inner product $\langle\cdot,\cdot\rangle_0$ on $\mathcal{E}$.

For each time t, the matrix function $Z_t = \tilde{M}_t$ is, entrywise, an L^2-holomorphic cylinder function on $\mathcal{E}'_{\mathbf{C}}$ whose restriction to $\mathcal{E}_{\mathbf{C}}$ is given by

$$RZ_t(\omega) = \sum_{j=1}^{d} (v_j(t), \omega)_0 E_j \,. \tag{3.5d}$$

Since each $v_j(t)$ has been assumed to be in $\mathcal{E}_0$, it follows that for each $\omega \in \mathcal{E}$, the inner product $(v_j(t), \omega)_0$ is real and so $RZ_T(\omega)$ is in $L(K)$.

Thus, if $A \in \mathcal{E}$, then $RZ(A) : t \mapsto RZ_t(A)$ is an $L(K)$-valued (deterministic) path.

Lemma 3.8. *With notation and hypotheses as above,*

$$R\hat{f}(A) = \int_K f(x)\rho_T(x^{-1}h_{RZ(A)}(T))\,dx \qquad \textit{for all } A \in \mathcal{E} \tag{3.5e}$$

for any function f on K which is a linear combination of matrix entries of representations of K.

Proof. Since, by Lemma 3.7, $\hat{f} \in \mathcal{H}L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$, we have

$$\begin{aligned} R\hat{f}(A) &= \int_{\mathcal{E}'_{\mathbf{C}}} \hat{f}(\omega + A)\,d\mu(\omega) \\ &= \int_{\mathcal{E}'_{\mathbf{C}}} \int_K f(x)\rho_T(x^{-1}h_Z(T;\omega + A))\,dx\,d\mu(\omega)\,. \end{aligned} \tag{3.5f}$$

As noted in the proof of Lemma 3.5, $h_Z(T;\omega)$ is the pointwise μ-a.e. limit of time-ordered products of exponentials $\exp(Z(t_j;\omega) - Z(t_{j-1};\omega))$. Since $A \in \mathcal{E}$, the Cameron–Martin theorem says that $\omega \mapsto \omega + A$ carries sets of measure zero into sets of measure zero, and so $h_Z(T;\omega + A)$ is the pointwise μ-a.e. limit of time-ordered products of exponentials $\exp(Z(t_j;\omega + A) - Z(t_{j-1};\omega + A))$.

Now

$$Z(t;\omega + A) = Z(t;\omega) + RZ_t(A)$$

So

$$h_Z(T;\omega + A) = h_{Z+RZ(A)}(T;\omega) \qquad \text{for all } \omega \in \mathcal{E}'_{\mathbf{C}}\,. \tag{3.5g}$$

Here $RZ(A) : t \mapsto (RZ_t)(A)$ is a deterministic path in $L(K)$. By Lemma 3.6, the $K_{\mathbf{C}}$-valued random variables $h_{Z+RZ(A)}(T)$ and $h_{RZ(A)}(T)h_Z(T)$ have the same distribution. Then, by (3.5f),

$$R\hat{f}(A) = \int_{\mathcal{E}'_{\mathbf{C}}} \int_K f(x)\rho_T(x^{-1}h_{RZ(A)}(T;\omega)h_Z(T))\,dx\,d\mu(\omega)\,. \tag{3.5h}$$

Let $f = \pi_{ij}$, the ij-th matrix entry of a representation π of K. Then, using (3.5h),

$$\int_K f(x)\rho_T(x^{-1}y)\,dx = c_\pi f(y)$$

for some constant c_π. This continues to be valid for $y \in K_{\mathbf{C}}$, if we take f via the analytic continuation of π to $K_{\mathbf{C}}$. Then

$$\begin{aligned} R(\hat{f})(A) &= \int_{\mathcal{E}'_{\mathbf{C}}} c_\pi f\left(h_{RZ(A)}(T)h_Z(T;\omega)\right) d\mu(\omega) \\ &= \int_{K_{\mathbf{C}}} c_\pi f\left(h_{RZ(A)}(T)g\right) d\mu_t(g) \\ &= c_\pi f\left(h_{RZ(A)}(T)\right) \qquad \text{(because } f \text{ is holomorphic on } K_{\mathbf{C}}\text{)} \\ &= \int_K f(x)\rho_T(x^{-1}h_{RZ(A)}(T))\, dx \end{aligned}$$

Thus $R(\hat{f})(A)$ is as claimed. □

Let M and Z be as before in (3.5b) and (3.5c). Then, for $\phi \in \mathcal{E}'$ and $A \in \mathcal{E}$, we have, on using the expression for $RZ_t(A)$ given in (3.5d),

$$M_t(\phi + A) = M_t(\phi) + \sum_{j=1}^{d} \langle v_j(t), A\rangle_0 E_j = M_t(\phi) + RZ_t(A)\,. \tag{3.5i}$$

So, by Lemma 3.6,

$$h_M(t; \cdot + A) \text{ has the same distribution as } h_{RZ(A)}(t)h_M(t;\cdot)\,. \tag{3.5j}$$

We turn next to the Segal–Bargmann transform of functions of h_Z.

For f a function on K, denote by F_f the function on $\mathcal{E}'$ given by

$$F_f(\omega) = f(h_M(T;\omega))\,. \tag{3.5k}$$

Then $|f|_{L^2(\rho_T)} = |F_f|_{L^2(\rho)}$.

Proposition 3.5. *With notation and hypotheses as above,*

$$SF_f(\omega) = \int_K f(x)\rho_T(x^{-1}h_Z(T;\omega))\, dx$$

for μ-almost every $\omega \in \mathcal{E}'_{\mathbf{C}}$.

Proof. Let f be a matrix entry of a representation of K. Then $F_f \in L^2(K, \rho_T)$ and, for $A \in \mathcal{E}$,

$$\begin{aligned} R(SF_f)(A) &= \int_{\mathcal{E}'} F_f(\phi + A)\, d\rho(\phi) \\ &= \int_{\mathcal{E}'} f(h_M(T;\phi + A))\, d\rho(\phi) \\ &= \int_{\mathcal{E}'} f(h_{RZ(A)}(T)h_M(\phi))\, d\rho(\phi) \end{aligned}$$

$$= \int_K f(h_{RZ(A)}(T)x)\rho_T(x)\,dx$$

$$= \int_K f(x)\rho_T(x^{-1}h_{RZ(A)}(T))\,dx \qquad (\text{using } \rho_T(x^{-1}) = \rho_T(x))\,. \tag{3.51}$$

Let

$$\hat{f}(\omega) = \int_K f(x)\rho_T(x^{-1}h_Z(T;\omega))\,dx, \qquad \text{for } \omega \in \mathcal{E}'_{\mathbf{C}}\,.$$

By Lemma 3.7, $\hat{f} \in \mathcal{H}L^2(\mu)$. Using Lemma 3.8 and (3.51), we have $R\hat{f} = R(SF_f)$ on $\mathcal{E}$. By Lemma 3.1(v) we then conclude that $\hat{f} = SF_f$.

Now consider a general $f \in L^2(K, \rho_T)$, and choose f_n, linear combinations of matrix entries of representations of K, converging to f in $L^2(\rho_T)$. Then $F_{f_n} \to F_f$ in $L^2(\rho)$ and so $SF_{f_n} \to SF_f$ in $L^2(\mu)$. On the other hand, $f_n \to f$ in $L^2(K, \rho_T)$ implies $\int_K f_n(x)\rho_T(x^{-1}z)\,dx \to \int_K f(x)\rho_T(x^{-1}z)\,dx$ for all $z \in K_{\mathbf{C}}$. So $\hat{f}_n \to \hat{f}$ pointwise. But $\hat{f}_n = SF_{f_n} \to SF_f$ in L^2. So $SF_f = \hat{f}$ in $L^2(\mu)$. □

Note. (i) In view of the above result, and unitarity of S and of $L^2(\rho_T) \to L^2(\rho) : f \mapsto F_f$, it follows that there is a well-defined isometry $L^2(K, \rho_t) \to L^2(\mathcal{E}'_{\mathbf{C}}, \mu)$ given by $f \mapsto \hat{f}$, where $\hat{f}(\omega) = \int_K f(x)\rho_T(x^{-1}h_Z(\omega))\,dx$, with Z being $L(K)_{\mathbf{C}}$-valued time-halved Brownian motion. It also shows therefore that the map $S^H : L^2(K, \rho_T) \to L^2(K_{\mathbf{C}}, \mu_T) : f \mapsto S^H f$, where

$$S^H f(z) = \int_K f(x)\rho_T(x^{-1}z)\,dx\,,$$

is an *isometry*. This result was first proven in Ref. 8 (the perspective just described is due to Ref. 7).

(ii) Proposition 3.5 implies that Lemmas 3.7 and 3.8 are valid for all $f \in L^2(K, \rho_T)$.

Using the results on the behavior of the Segal–Bargmann transform and restrictions for independent σ-algebras we have:

Proposition 3.6. *Let $V_1, \ldots, V_n$ be orthogonal subspaces of $\mathcal{E}_0$. Consider processes $[a_j, b_j] \times \mathcal{E}' \to L(K) : (t, \omega) \mapsto M_j(t, \omega)$, for $j = 1, \ldots, n$, which are $L(K)$-valued standard Brownian motions over $(\mathcal{E}', \rho)$, such that each matrix entry of $M_j(t)$ is of the form $\hat{v}$ for some $v \in V_j$. For $f \in L^2(K^n, \rho_{b_1-a_1} \otimes \cdots \otimes \rho_{b_n-a_n})$, and denote by F_f the function on $\mathcal{E}'$ given by*

$$F_f(\omega) = f(h_{M_1}(b_1;\omega), \ldots, h_{M_n}(b_n;\omega))\,.$$

Then

$$SF_f(\omega) = \int_{K^n} f(x) \prod_{j=1}^{n} \rho_{b_j-a_j}(x_j^{-1} h_{\tilde{M}_j}(b_j - a_j;\omega))\,dx$$

for almost every $\omega \in \mathcal{E}'_{\mathbf{C}}$.

Proof. Using the results in Proposition 3.2 on independence and the S-transform, and Proposition 3.5, the result holds for every function of the form $f = f_1 \otimes \cdots \otimes f_n$, where each f_j is a matrix entry of a representation of K. Taking limits, in L^2 and pointwise, we obtain the general case. □

Applying the restriction map R, and using Proposition 3.3(ii) as well as the expression for RSF_f given in (3.5l), we have:

Proposition 3.7. *With notation and hypotheses as in Proposition* 3.6,

$$R(SF_f)(A) = \int_{K^n} f(x) \prod_{j=1}^{n} \rho_{b_j - a_j}(x_j^{-1} h_{R\tilde{M}_j(A)}(b_j - a_j))\, dx$$

for almost every $A \in \mathcal{E}$.

The process $t \mapsto R\tilde{M}_j(A)_t$ is obtained explicitly by (3.5d).

Note. As an example, let $\mathcal{E}$ be the vector space of smooth paths $[0,1] \to L(K)$, with inner product $\langle \cdot, \cdot \rangle_0$ given by $\langle x, y \rangle_0 = \int_0^1 \langle x(t), y(t) \rangle_{L(K)}\, dt$. The space $\mathcal{E}'$ can be taken to be the set of continuous paths on $L(K)$, starting at 0, with evaluation of $\phi \in \mathcal{E}'$ on $x \in \mathcal{E}$ being given by $\phi(x) = \langle x(t), \phi(t) \rangle_{L(K)} - \int_0^1 \langle \phi(t), x'(t) \rangle_{L(K)}\, dt$. The Itô map can be used to identify $(\mathcal{E}', \rho)$ with the Wiener space of Brownian paths on K, and the space $(\mathcal{E}'_{\mathbf{C}}, \mu)$ with a corresponding space of paths on $K_{\mathbf{C}}$. Consider the case where the intervals $[a_j, b_j]$ form an ordered partitioning of $[0, T]$, with $0 = a_1 < b_1 = a_2 < b_2 = a_3 < \cdots < b_n = T$. Take V_j to be the space of functions supported on $[a_j, b_j]$. Then Proposition 3.6, specialized to this setting, is a variant of one of the results of Ref. 10. Another related application may be made in the context of Proposition 5.6 of Ref. 5.

4. Evaluation of the Transform on Loop Variables

The machinery developed in Sec. 3 can now be applied rapidly to the setting of the two-dimensional Yang–Mills measure.

In this section, we take $\mathcal{E} = \mathcal{S}(\mathbf{R}^2) \otimes L(K)$, and $\mathcal{E}' = \mathcal{S}'(\mathbf{R}^2) \otimes L(K)$, where $\mathcal{S}(\mathbf{R}^2)$ is the Schwarz space of rapidly decreasing *real-valued* functions, and $\mathcal{S}'(\mathbf{R}^2)$ the corresponding dual space of distributions. The pairing of $\mathcal{E}$ and $\mathcal{E}'$ is in the obvious way, using the inner product on $L(K)$. The inner product on $\mathcal{E}$ is the one given by the $L^2(\mathbf{R}^2)$-inner product on $\mathcal{S}(\mathbf{R}^2)$ and the inner product on $L(K)$. We also work with the complexification $\mathcal{E}_{\mathbf{C}} = \mathcal{S}(\mathbf{R}^2)_{\mathbf{C}} \otimes L(K)_{\mathbf{C}}$ and the complexification $\mathcal{E}'_{\mathbf{C}}$ of the dual $\mathcal{E}'$ (note that elements of $\mathcal{E}'_{\mathbf{C}}$ are the continuous *real*-linear maps on $\mathcal{E}_{\mathbf{C}}$).

Consider a continuous path $C : [a, T] \to \mathbf{R}^2 : t \mapsto (r_C(t) \cos t, r_C(t) \sin t)$, where $0 \le a \le T \le 2\pi$. Let E_t be the region enclosed by the curve $C|[a, t]$ and the two radii from the origin to the points $C(a)$ and $C(t)$. Then $1_{E_t} \in L^2(\mathbf{R}^2)$ induces an $L(K)$-valued Gaussian random variable $\overline{1}_{E_t}$ on $(\mathcal{E}', \rho^{\mathrm{YM}})$. There is a corresponding *continuous* $L(K)$-valued martingale M_t such that $(M_{t_1}, \ldots, M_{t_n})$ is almost surely equal to $(\bar{1}_{E_{t_1}}, \ldots, \bar{1}_{E_{t_n}})$ for any $t_1, \ldots, t_n \in [0, T]$. We also have, correspondingly, an $L(K)_{\mathbf{C}}$-valued continuous martingale Z on $\mathcal{E}'_{\mathbf{C}}$, with $Z = A + iB$, where A and B are independent identically distributed $L(K)$-valued martingales. In fact, M and Z are Brownian motions on $L(K)$ and on $L(K)_{\mathbf{C}}$, respectively, with time clocked by $|E_t|$ and $\frac{1}{2}|E_t|$, respectively, instead of by t.

As explained in Sec. 2, the *stochastic holonomy* around the loop L_C formed by C and the two terminal radii is the K-valued random variable $\omega \mapsto h(L_C; \omega)$ on $\mathcal{E}'$ which is $h_M(T)$, where $t \mapsto h_M(t)$ is the solution, initiating at the identity, of the stochastic differential equation

$$dh_M = \left[-dM + \frac{1}{2} dM^2\right] h_M \,.$$

Thus, the K-valued random variable $\omega \mapsto h(L_C; \omega)$ has density $\rho_{|L_C|}$ with respect to Haar measure on K, where $\rho_t(x)$ is the heat kernel on K (normalized with respect to the Haar measure), and $|L_C|$ the area enclosed by the loop L_C.

Analogously, there is also the *complex* stochastic holonomy, $\omega \mapsto \xi(L_C; \omega)$, a $K_{\mathbf{C}}$-valued random variable on $(\mathcal{E}'_{\mathbf{C}}, \mu^{\mathrm{YM}})$ which is defined to be $h_Z(T)$, where $t \mapsto h_Z(t)$ starts at the identity and solves the stochastic differential equation

$$dh_Z = \left[-dZ + \frac{1}{2} dZ^2\right] h_Z \,.$$

The density of $\xi(L_C)$ with respect to (the bi-invariant) Haar measure on $K_{\mathbf{C}}$ is $\mu_{|L_C|}$, the corresponding time-scaled heat kernel on $K_{\mathbf{C}}$ with respect to the Laplacian obtained from the metric on K.

Applying Proposition 3.5 to the present situation, and bearing in mind the time-scaling necessary here, we have:

Proposition 4.1. *Let $f \in L^2(K, \rho_{|L_C|})$, and let $F_f(\omega) = f(h(L_C; \omega))$. Then*

$$SF_f(\omega) = \int_K f(x) \rho_{|L_C|}(x^{-1}\xi(L_C; \omega))\, dx \,.$$

Thus we have proven that the Segal–Bargmann transform $S[f(h(L_C))]$ is given by

$$S[f(h(C))](Z) = S^H f(h(L_C; Z))$$

for μ^{YM}-almost every Z; the right-hand side is simply the transform $S^H : f \mapsto f * \rho_{|L_C|}$ of f discovered in Ref. 8, applied to $h(L_C; Z)$.

Now consider curves $C_1, \ldots, C_n$ of the type described above, the corresponding loops L_{C_j} enclosing disjoint regions Δ_j. In this case, we have some non-overlapping intervals $[T_j, T'_j]$, for $j = 1, \ldots, n$, and C_j is described by a path $[T_j, T'_j] \to \mathbf{R}^2 : t \mapsto r_j(t)e^{it}$.

Let $f \in L^2(K^n, \rho_{|L_{C_1}|} \otimes \cdots \otimes \rho_{|L_{C_n}|})$, and let F_f be the function in $L^2(\mathcal{E}', \rho^{\mathrm{YM}})$ given by $F_f(\omega) = f(h(L_{C_1}; \omega), \ldots, h(L_{C_n}; \omega))$. Specializing Proposition 3.6 to this situation we obtain Theorem 2.2:

$$SF_f(\omega) = \int_K f(x_1, \ldots, x_n) \rho_{|L_{C_1}|}(x_1^{-1}\xi(C_1; \omega)) \cdots \rho_{|L_{C_n}|}(x_n^{-1}\xi(C_n; \omega))\, dx_1 \cdots dx_n\,.$$

The restriction by R, as given in Proposition 3.7, yields Theorem 2.1. In this context, $R\tilde{M}_j(A)_t$ works out to be $\int_{E_t} A$.

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