

The Segal–Bargmann transform and the Gross ergodicity theorem

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This paper is dedicated to my advisor and friend, Leonard Gross.

ABSTRACT. This paper gives a new proof of the Gross ergodicity theorem by making use of the Segal–Bargmann transform for an infinite-dimensional linear space and the generalized Segal–Bargmann transform for a compact Lie group.

1. Introduction

In [G3], Leonard Gross discovered an analog for compact Lie groups of the classical Hermite expansion. Specifically, let K be a simply connected Lie group of **compact type**. Compact type means that the Lie algebra \mathfrak{k} of K admits an Ad- K -invariant inner product. It is known that every simply connected Lie group of compact type is the product of a compact simply connected group and \mathbb{R}^n . Let $\rho(x)$ denote the **heat kernel** on K , at the identity and at time one, and consider also the **heat kernel measure** $\rho(x) dx$, where dx is the Haar measure on K . Then Gross gives a unitary isomorphism of $L^2(K, \rho(x) dx)$ onto a certain completion of the universal enveloping algebra of the Lie algebra \mathfrak{k} . (See also [G4, G5, G6] and the survey paper [H4].)

In the case $K = \mathbb{R}^n$ (so that $\mathfrak{k} = \mathbb{R}^n$ as well), the universal enveloping algebra is just the symmetric algebra over \mathbb{R}^n and the completion of the enveloping algebra is just the usual (bosonic) Fock space over \mathbb{R}^n . In this case the measure ρ on \mathbb{R}^n is just a Gaussian measure, and the isomorphism of $L^2(\mathbb{R}^n, \rho(x) dx)$ with the Fock space is just the expansion of a function $f \in L^2(\mathbb{R}^n, \rho(x) dx)$ into Hermite polynomials.

In the case where K is compact and simply connected (hence necessarily non-commutative) the symmetric algebra is replaced by the **universal enveloping algebra** $U(\mathfrak{k})$ over the Lie algebra \mathfrak{k} . The universal enveloping algebra is the associative algebra with identity generated by the elements of \mathfrak{k} subject to the relation $XY - YX = [X, Y]$, where $[X, Y]$ is the bracket operation in the Lie algebra. The universal enveloping algebra is naturally isomorphic to the algebra of left-invariant differential operators on K . In this case the Hermite polynomials are replaced by

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certain logarithmic-type derivatives of the heat kernel, that is, functions of the form $\alpha\rho/\rho$, where α is a left-invariant differential operator on K . (In the \mathbb{R}^n case such derivatives *are* simply the Hermite polynomials.) The isomorphism between $L^2(K, \rho(x) dx)$ was given originally in Theorem 2.1 of [G3]. A more explicit description of the isomorphism was given by Hijab [Hi1] and new proofs of the unitarity of the isomorphism were given by Driver [D1] and Hijab [Hi1, Hi2]. See also the survey paper [H4] and the papers [M1, M2] which study the generalized Hermite functions.

Now, the main objective of the paper [G3] was not to obtain the generalized Hermite expansion, but rather to obtain a certain result in infinite-dimensional analysis. That result, the **Gross ergodicity theorem**, states that the finite-energy loop group over K acts ergodically on the continuous loop group over K , with respect to the pinned Wiener measure. (This result is explained in more detail below.) The generalized Hermite expansion came out as consequence of Gross's method of proof of the ergodicity theorem.

At the time of the writing of [G3] I was a Ph.D. student of Gross. His discovery of the generalized Hermite expansion for compact groups led him to suggest to me that I look for an analog on compact groups of the Segal–Bargmann transform. This project was successful and became the basis of my Ph.D. thesis, published in [H1].

Even though the origin of my thesis problem was in infinite-dimensional analysis, the proofs in [H1] were purely finite-dimensional. (There are also purely finite-dimensional proofs of the generalized Hermite expansion, given by Driver and by Hijab.) Later, Gross and Malliavin showed in [GM] how to obtain the generalized Segal–Bargmann for K by means of infinite-dimensional analysis and the Gross ergodicity theorem. The purpose of this paper is to return the favor by using the Segal–Bargmann transform for K to give a new proof of the Gross ergodicity theorem. Many of the intermediate results we will need are already in the paper of Gross and Malliavin. We may thus say (oversimplifying slightly) that the present paper consists of showing that the arguments of Gross and Malliavin can be run backwards to prove the ergodicity theorem using the generalized Segal–Bargmann transform.

The Gross ergodicity theorem was re-proved using quasi-sure analysis by G. Sadasue [Sa]. The proof given here is much closer to the original proof of Gross, differing primarily in the use of the Segal–Bargmann transform in place of the multiple Itô integral expansion of [G3]. The point of the present paper is not so much to give a new proof of a result that already has more than one proof, but rather to demonstrate the power of the Segal–Bargmann transform and to further illustrate the connection between the ordinary and generalized Segal–Bargmann transforms. Section 5 shows how the approach used in this paper (using the Segal–Bargmann transform) can be combined with the original approach of Gross (using the expansion into multiple Itô integrals).

It is a great pleasure to thank Leonard Gross for starting me on this path of research and for much advice and encouragement over the years. I also thank Bruce Driver for many helpful discussions and specifically for crucial advice concerning the proof of Proposition 4.6. Finally, I thank the referee for useful comments and corrections.

2. The Gross ergodicity theorem

2.1. Statement of the theorem. Let K be a connected Lie group whose Lie algebra \mathfrak{k} admits an Ad- K -invariant inner product. (In the way I am formulating things here, it is not necessary to assume that K is simply connected.) Such a group is necessarily of the form $K = K_1 \times \mathbb{R}^n$, where K_1 is compact. (See [H6, Sect. 7].) Fix once and for all an Ad- K -invariant inner product on \mathfrak{k} . This inner product determines a bi-invariant Riemannian metric on K .

Now let $W(K)$ be the **continuous pathgroup**, that is, the set of continuous maps $x : [0, 1] \rightarrow K$ with $x_0 = e$. Let $W^0(K)$ be the (based) **continuous loop group**, that is, the set of x in $W(K)$ for which $x_1 = e$. Let p be the **Wiener measure** on $W(K)$, that is, the probability distribution of Brownian motion in K , starting from the identity. Let p^0 be the **pinned Wiener measure**, that is, the probability distribution of Brownian motion in K conditioned to return to the identity at time one. Let $H(K)$ be the **finite energy pathgroup**, that is, the subset of $W(K)$ consisting of paths having one derivative in L^2 . Let $\mathcal{L}(K)$ be the (based) **finite energy loop group**, that is, $\mathcal{L}(K) = H(K) \cap W^0(K)$. It is known that the Wiener measure p on $W(K)$ is quasi-invariant under the left and right action of $H(K)$ and that the pinned Wiener measure p^0 is quasi-invariant under the left and right actions of $\mathcal{L}(K)$. (See [MM, G03].)

THEOREM 2.1 (Gross). *Suppose f in $L^2(W^0(K), p^0)$ is such that for all l in the finite-energy loop group $\mathcal{L}(K)$ we have*

$$f(xl) = f(x)$$

for p^0 -almost every $x \in W^0(K)$. Then there is a constant c such that

$$f(x) = c$$

for p^0 -almost every $x \in W^0(K)$.

This is Theorem 2.8 of [G3]. The result is equivalent to saying that the action of $\mathcal{L}(K)$ on $W^0(K)$ is ergodic, i.e. that every $\mathcal{L}(K)$ -invariant measurable set has either measure zero or measure one. Note that the statement of the theorem makes sense only in light of the quasi-invariance of the pinned Wiener measure under the finite-energy loop group (since f is actually an equivalence class of functions equal almost everywhere).

Gross shows that the ergodicity theorem is equivalent to the following.

THEOREM 2.2 (Gross). *Suppose f is in $L^2(W(K), p)$ is such that for all l in the finite-energy loop group $\mathcal{L}(K)$ we have*

$$f(xl) = f(x)$$

for p -almost every $x \in W(K)$. Then there exists a measurable function ϕ on K such that

$$f(x) = \phi(x_1)$$

for p -almost every $x \in W(K)$.

From now on I will consider only this form of the ergodicity theorem.

The difficulty in proving Theorem 2.2 is the disparity in the smoothness classes of the *continuous* path group $W(K)$ and the *finite-energy* loop group $\mathcal{L}(K)$. If f were an everywhere-defined function on $W(K)$ and f were invariant under the *continuous* loop group, then we would have $f(x) = f(y)$ whenever $x_1 = y_1$, since if we set $l_\tau = y_\tau^{-1}x_\tau$ then l is a loop and $x = yl$. Such a function f would then have

to be a function of x_1 . In reality, though, f is defined only almost everywhere and f is invariant only under the finite-energy loop group, so this simple argument does not work.

Note also that we are not at liberty to replace the continuous path group with the finite-energy path group, because the finite-energy paths are a set of Wiener measure zero. Nor are we at liberty to replace the finite-energy loop group with the continuous loop group, because the Wiener measure is quasi-invariant only under finite-energy loops. (Without quasi-invariance the expression $f(xl)$ does not make sense when f is an equivalence class of almost-everywhere-equal functions.)

2.2. Outline of Gross's proof. Gross (and many others working on related problems) linearizes the problem by using the **Itô map** to move from the path-group in K to the pathspace in the Lie algebra \mathfrak{k} . So let $W(\mathfrak{k})$ denote the space of continuous paths $B : [0, 1] \rightarrow \mathfrak{k}$ with $B_0 = 0$ and let P denote the Wiener measure on $W(\mathfrak{k})$. We consider the following Stratonovich differential equation

$$dx_\tau = x_\tau \circ dB_\tau, \quad x_0 = e,$$

which can also be written in Itô form with an extra dt term. The Itô map $\tilde{\theta} : W(\mathfrak{k}) \rightarrow W(K)$ is the map $\tilde{\theta}(B) = x$. This map is defined for P -almost every B in $W(\mathfrak{k})$. The Itô map can be inverted by the formula $B_\tau = \int_0^\tau x_\sigma^{-1} \circ dx_\sigma$. It is known that x is a Brownian motion in K , which is to say that $\tilde{\theta}$ is a measure-preserving map of $(W(\mathfrak{k}), P)$ to $(W(K), p)$. This means that the map

$$f \rightarrow f \circ \tilde{\theta}$$

is a unitary map of $L^2(W(K), p)$ onto $L^2(W(\mathfrak{k}), P)$.

The action of the loop group $\mathcal{L}(K)$ on $W(K)$ can be transferred to $W(\mathfrak{k})$ by means of the Itô map. The resulting action of $\mathcal{L}(K)$ on $W(\mathfrak{k})$ is given by

$$(2.1) \quad (l \cdot B)_\tau = \int_0^\tau l_\sigma^{-1} (dB_\sigma) l_\sigma + \int_0^\tau l_\sigma^{-1} \frac{dl_\sigma}{d\sigma} d\sigma.$$

The first term can be interpreted as a stochastic integral, but in fact a simple integration by parts argument shows that for all finite-energy loops l , the first integral is defined for every continuous path B .

So the ergodicity theorem can now be reformulated as follows.

THEOREM 2.3. *Suppose $f \in L^2(W(\mathfrak{k}), P)$ has the property that for all $l \in \mathcal{L}(K)$ we have*

$$f(l \cdot B) = f(B)$$

for P -almost every $B \in W(\mathfrak{k})$. Then there exists a measurable function ϕ on K such that

$$f(B) = \phi(\tilde{\theta}(B)_1).$$

Since we will use repeatedly the value of the Itô map at time one, we will introduce a notation for it. I call it the holonomy since in my paper [DH] with Driver it has the interpretation of the holonomy of a connection.

DEFINITION 2.4. The **holonomy** is the (P -almost-everywhere-defined) map $h : W(\mathfrak{k}) \rightarrow K$ given by

$$h(B) = \tilde{\theta}(B)_1.$$

The holonomy can be computed as

$$h(B) = \lim_{n \rightarrow \infty} \exp(B_{1/n} - B_0) \exp(B_{2/n} - B_{1/n}) \cdots \exp(B_1 - B_{(n-1)/n}),$$

where it can be shown that the limit exists for P -almost every path B in $W(\mathfrak{k})$.

By itself this linearization of the problem does not resolve the fundamental difficulty of having continuous paths but finite-energy loops. We would *like to* argue that if B and C are paths in \mathfrak{k} with $h(B) = h(C)$ then there exists a loop l in $\mathcal{L}(K)$ with $l \cdot B = C$. Indeed this is true if B and C are finite-energy paths. But if B and C are only continuous then the relevant loop l will not be of finite energy, that is, not an element of $\mathcal{L}(K)$. As in the pathgroup setting we are not allowed to use finite-energy paths (since $H(\mathfrak{k})$ is a set of P -measure zero in $W(\mathfrak{k})$) or to use continuous loops (since P is quasi-invariant only under the action of finite-energy loops).

To resolve this difficulty Gross uses the isomorphism between $L^2(W(\mathfrak{k}), P)$ and the Fock space. In this setting the isomorphism with the Fock space takes the form of the chaos expansion, that is, the expansion of a function $f \in L^2(W(\mathfrak{k}), P)$ into a sum of iterated Itô integrals. The action of the loop group can be transferred from $W(\mathfrak{k})$ to the Fock space. In the Fock space Gross was able to identify precisely those elements that are invariant under the loop group action [G3, Sect. 5]. From this he was able to work back to the loop-invariant elements in $L^2(W(\mathfrak{k}), P)$.

In this paper I will give a different proof of the ergodicity result, using the Segal–Bargmann transform for $L^2(W(\mathfrak{k}), P)$ instead of the chaos expansion. The advantage of the Segal–Bargmann transform is that in the Segal–Bargmann space we can actually use the argument that we wanted to use in $L^2(W(\mathfrak{k}), P)$.

3. Segal–Bargmann transform

3.1. The transform for finite-dimensional linear spaces. Let P denote the standard Gaussian measure on \mathbb{R}^n , namely,

$$dP(x) = (2\pi)^{-n/2} e^{-x^2/2} dx.$$

Then we define a map $S : L^2(\mathbb{R}^n, P) \rightarrow \mathcal{H}(\mathbb{C}^n)$ by

$$(3.1) \quad Sf(z) = \int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-(z-x)^2/2} f(x) dx$$

$$(3.2) \quad = e^{-z^2/2} \int_{\mathbb{R}^n} e^{z \cdot x} f(x) dP(x), \quad z \in \mathbb{C}^n.$$

Here $\mathcal{H}(\mathbb{C}^n)$ denotes the space of entire holomorphic functions on \mathbb{C}^n and $z^2 = z_1^2 + \cdots + z_n^2$. For $z \in \mathbb{R}^n$ we can make the change of variable $y = x - z$ in (3.1) (and then re-name the integration variable to x). This gives

$$(3.3) \quad Sf(z) = \int_{\mathbb{R}^n} f(x+z) dP(x), \quad z \in \mathbb{R}^n.$$

Since P is just the heat kernel at time one we see from (3.1) that for $z \in \mathbb{R}^n$, Sf is just the solution of the heat equation at time one, with initial condition f . That is,

$$(3.4) \quad Sf = \text{analytic continuation of } e^{\Delta/2} f,$$

where the analytic continuation is from \mathbb{R}^n to \mathbb{C}^n . Here $\exp(\Delta/2)$ is the (forward) heat operator at time one.

Now let μ be the Gaussian measure on \mathbb{C}^n given by

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dz$$

where dz is $2n$ -dimensional Lebesgue measure.

THEOREM 3.1 (Segal–Bargmann). *The map S is a unitary map of $L^2(\mathbb{R}^n, P)$ onto $\mathcal{HL}^2(\mathbb{C}^n, \mu)$, where \mathcal{HL}^2 denotes the space of square-integrable holomorphic functions.*

See [H3, Sect. 6.4] and [G7] for some of the history of this theorem. It is easily shown that $\mathcal{HL}^2(\mathbb{C}^n, \mu)$ is a closed subspace of $L^2(\mathbb{C}^n, \mu)$, hence a Hilbert space [H3]. Functions in the Segal–Bargmann space $\mathcal{HL}^2(\mathbb{C}^n, \mu)$ satisfy the pointwise bounds [B, H3]

$$(3.5) \quad |F(z)|^2 \leq \|F\|_{L^2(\mathbb{C}^n, \mu)}^2 e^{|z|^2}.$$

3.2. The transform for infinite-dimensional linear spaces. We now consider the infinite-dimensional limit of the transform described in the previous section. This should properly be done in the setting of abstract Wiener spaces, but in the interests of notational consistency I will consider only the concrete Wiener space of interest in this problem. (Only the notation need be changed to do this in general.) There are actually two substantially different versions of the transform in the infinite-dimensional case (see [H3, HS]). The one presented here is essentially the one in [S3, BSZ].

We want to consider the Segal–Bargmann space on \mathbb{C}^n and let n tend to infinity. This means that \mathbb{C}^n should be replaced by an infinite-dimensional separable complex Hilbert space $H_{\mathbb{C}}$. Unfortunately, the Gaussian measures $d\mu^{(n)}(z) = \pi^{-n} \exp(-|z|^2) dz$ do not have a limit as a measure on $H_{\mathbb{C}}$. To see this, let z_1, z_2, \dots be the coordinates with respect to an orthonormal basis for $H_{\mathbb{C}}$. If there were a limiting measure on $H_{\mathbb{C}}$, the z_k 's would be independent and identically distributed with respect to this measure (as z_1, \dots, z_n are with respect to $\mu^{(n)}$). But if all the z_k 's have the same distribution, then with probability one $\sum |z_k|^2 = \infty$. This shows that the limiting measure cannot exist on $H_{\mathbb{C}}$, since for each $Z \in H_{\mathbb{C}}$, $\|Z\|^2 = \sum |z_k|^2 < \infty$.

Nevertheless, there *is* a limiting Gaussian measure M ; it simply is not a measure on $H_{\mathbb{C}}$. Rather M must “live” on some Banach space B containing $H_{\mathbb{C}}$, where B is enough bigger to capture the measure. (See [G1, K].) We want to take $H_{\mathbb{C}}$ to be $H(\mathfrak{k}_{\mathbb{C}})$, in which case we may take $B = W(\mathfrak{k}_{\mathbb{C}})$ and M is the Wiener measure on $W(\mathfrak{k}_{\mathbb{C}})$.

It would then seem natural to define the Segal–Bargmann space to be the space of M -square-integrable functions on $W(\mathfrak{k}_{\mathbb{C}})$ that are holomorphic in the standard sense (Definition 6 below). Unfortunately, this space is not a closed subspace of $L^2(W(\mathfrak{k}_{\mathbb{C}}), M)$. The reason for this is that the $n \rightarrow \infty$ limit of the pointwise bounds (3.5) only allow us to control the values of holomorphic functions on $H(\mathfrak{k}_{\mathbb{C}})$, but not the values on $W(\mathfrak{k}_{\mathbb{C}})$.

We have, then, a dilemma. We want L^2 holomorphic functions, but (roughly) the functions are holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$ but square-integrable on $W(\mathfrak{k}_{\mathbb{C}})$. So we must adjust either the notion of square-integrability or the notion of holomorphicity to get our functions defined on a single space. The approach we will use here is to consider holomorphic functions on $H(\mathfrak{k}_{\mathbb{C}})$ whose “ L^2 ” norm is defined by a limit over finite-dimensional subspaces.

First we introduce the standard definition of a holomorphic function on a Hilbert (or Banach) space. (E.g., [HP].)

DEFINITION 3.2. A function $F : H(\mathfrak{k}_{\mathbb{C}}) \rightarrow \mathbb{C}$ is called **holomorphic** if F is locally bounded and F is holomorphic on each finite-dimensional subspace of $H(\mathfrak{k}_{\mathbb{C}})$. The space of holomorphic functions on $H(\mathfrak{k}_{\mathbb{C}})$ will be denoted $\mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$.

We then introduce the Segal–Bargmann space.

DEFINITION 3.3. If F is a holomorphic function on $H(\mathfrak{k}_{\mathbb{C}})$ (in the sense of Definition 3.2) define the **Segal–Bargmann norm** $\|F\|_{SB}$ by

$$\|F\|_{SB}^2 = \sup_V \int_V |F(Z)|^2 \frac{e^{-\|Z\|^2}}{\pi^d} dZ,$$

where the supremum is over all finite-dimensional subspaces V of $H(\mathfrak{k}_{\mathbb{C}})$ and where $d = \dim V$. Here $\|Z\|$ is the Cameron–Martin norm given by $\|Z\|^2 = \int_0^1 |dZ/dt|^2 dt$. Define the **Segal–Bargmann space** to be the set of $F \in \mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$ for which $\|F\| < \infty$.

It is not immediately obvious that the Segal–Bargmann “norm” is indeed a norm, or that the Segal–Bargmann space is complete with respect to this norm. Nevertheless, both assertions are true. The key to proving these assertions is the **monotonicity lemma**, which states that if F is holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$ and V, W are finite-dimensional subspaces of $H(\mathfrak{k}_{\mathbb{C}})$ with $V \subset W$, then the integral of $|F|^2$ over V (with respect to the standard Gaussian measure in Definition 3.3) is less than or equal to the corresponding integral of $|F|^2$ over W . Nothing like this is true for general non-holomorphic functions.

In the proof of the ergodicity theorem it is essential that we work with functions defined on the *finite-energy* path space $H(\mathfrak{k}_{\mathbb{C}})$. This is the advantage of transferring the problem to the Segal–Bargmann space—that we can work with finite-energy rather than continuous paths. This can be done only in the holomorphic setting—the norm given in Definition 3.3 is not actually a norm when applied to arbitrary (not necessarily holomorphic) functions.

THEOREM 3.4. *The quantity $\|\cdot\|_{SB}$ is a norm on the Segal–Bargmann space, and the Segal–Bargmann space is complete in this norm. The Segal–Bargmann space becomes a Hilbert space with inner product defined by polarizing this norm.*

Suppose V_n is any sequence of finite-dimensional subspaces such that $V_n \subset V_{n+1}$ and such that $\cup V_n$ is dense in $H(\mathfrak{k}_{\mathbb{C}})$. Then for any holomorphic function on $H(\mathfrak{k}_{\mathbb{C}})$ (in the Segal–Bargmann space or not) we have

$$(3.6) \quad \|F\|_{SB}^2 = \lim_{n \rightarrow \infty} \int_{V_n} |F(Z)|^2 \frac{e^{-\|Z\|^2}}{\pi^{d_n}} dZ,$$

where $d_n = \dim V_n$. Furthermore, the integral on the right in (3.6) is a non-decreasing function of n for fixed F .

See [S3, BSZ, GM] for more information.

One can define another “Segal–Bargmann space” as the closure in $L^2(W(\mathfrak{k}_{\mathbb{C}}), M)$ of the holomorphic cylinder functions. However, the elements of this space are not necessarily holomorphic on $W(\mathfrak{k}_{\mathbb{C}})$ in the sense of Definition 3.2, indeed, not necessarily continuous or locally bounded on $W(\mathfrak{k}_{\mathbb{C}})$. We will not make use of this form of the Segal–Bargmann space in this paper. (See [Su, Sh, HS, Go1, Go2] for

more information, including the “skeleton map” or “restriction map” that relates the two forms of the Segal–Bargmann space.)

We then introduce the Segal–Bargmann transform.

DEFINITION 3.5. The Segal–Bargmann transform is the map

$$S : \bigcup_{p>1} L^p(W(\mathfrak{k}), P) \rightarrow \mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$$

given by

$$(3.7) \quad Sf(Z) = e^{-Z^2/2} \int_{W(\mathfrak{k})} e^{Z \cdot B} f(B) dP(B), \quad Z \in H(\mathfrak{k}_{\mathbb{C}}).$$

Here Z^2 is the complex-valued quadratic (not Hermitian) form given by

$$Z^2 = \int_0^1 \left(\frac{dZ}{d\tau}, \frac{dZ}{d\tau} \right) d\tau,$$

(\cdot, \cdot) is the complex-bilinear extension to $\mathfrak{k}_{\mathbb{C}}$ of the inner product on \mathfrak{k} . Here also $Z \cdot B = \int_0^1 (dZ/d\tau, dB)$ (stochastic integral), well-defined for P -almost every B in $W(\mathfrak{k})$.

If we restrict attention to $Z \in H(\mathfrak{k})$ then we may recognize $e^{-Z^2/2} e^{Z \cdot B}$ as the Cameron–Martin density. This allows us to give an alternative expression for the transform. (Compare (3.3).)

PROPOSITION 3.6. *For any $f \in L^p(W(\mathfrak{k}), P)$, Sf is the unique holomorphic function on $H(\mathfrak{k}_{\mathbb{C}})$ whose restriction to $H(\mathfrak{k})$ is given by*

$$(3.8) \quad Sf(X) = \int_{W(\mathfrak{k})} f(B + X) dP(B), \quad X \in H(\mathfrak{k}).$$

Note that the integral makes sense because of quasi-invariance of the Wiener measure under translations in $H(\mathfrak{k})$ and because the Cameron–Martin density is in L^q for all $q < \infty$. Note that the restriction of Sf to $H(\mathfrak{k})$ is roughly the convolution of f with P . Since P is just the infinite-dimensional limit of the heat kernel at time one on Euclidean space we have the following informal description of the Segal–Bargmann transform:

$$(3.9) \quad Sf = \text{analytic continuation of } e^{\Delta/2} f.$$

Here the analytic continuation is from $H(\mathfrak{k})$ to $H(\mathfrak{k}_{\mathbb{C}})$ and the Laplacian is (roughly) the sum of squares of derivatives in the directions of an orthonormal basis of $H(\mathfrak{k})$.

We now have the following infinite-dimensional version of the Segal–Bargmann theorem.

THEOREM 3.7. *The Segal–Bargmann transform S is a unitary map of the space $L^2(W(\mathfrak{k}), P)$ onto the Segal–Bargmann space.*

Note that the Segal–Bargmann transform is defined on all L^p spaces with $p > 1$, even though the unitarity theorem applies only to L^2 .

3.3. The transform for Lie groups of compact type. In [H1] (motivated by the results of Gross concerning the generalized Hermite expansion), I introduced a version of the Segal–Bargmann transform for compact-type Lie groups. My idea was to replace the Gaussian measures appearing in the conventional Segal–Bargmann transform with their natural geometric analogs, namely, heat kernel measures. This means that I defined things by analogy to the ordinary transform.

The subsequent paper of Gross and Malliavin [GM] goes beyond an analogy and actually obtains the *generalized* transform for a compact group as a special case of the *ordinary* transform for an infinite-dimensional linear space. See also [H4, Sect. 4], [HS, AHS], and Theorem 4.1 below.

Let K be a connected Lie group of compact type. Let ρ be the heat kernel on K , at the identity and at time one. This means that ρ is the unique function on K such that

$$\left(e^{\Delta_K/2} f\right)(e) = \int_K f(x)\rho(x) dx$$

for all bounded measurable functions f on K . Here dx is the Haar measure on K . We will consider the Hilbert space $L^2(K, \rho(x)dx)$, which we abbreviate as $L^2(K, \rho)$.

Now let $K_{\mathbb{C}}$ be the complexification of K as described in [H1]. (For example, if $K = \mathbb{R}^n$ then $K_{\mathbb{C}} = \mathbb{C}^n$ and if $K = SU(n)$ then $K_{\mathbb{C}} = SL(n; \mathbb{C})$.) It is shown in [H1] that the function ρ admits a unique analytic continuation to $K_{\mathbb{C}}$, also denoted ρ . Define the generalized Segal–Bargmann transform $S_K : L^2(K, \rho) \rightarrow \mathcal{H}(K_{\mathbb{C}})$ by

$$(3.10) \quad S_K f(g) = \int_K \rho(gx^{-1})f(x) dx, \quad g \in K_{\mathbb{C}}$$

where $\mathcal{H}(K_{\mathbb{C}})$ denotes the space of holomorphic functions on $K_{\mathbb{C}}$. For $g \in K$ we may make the change of variable $y = xg^{-1}$, use the invariance of ρ under $y \rightarrow y^{-1}$, and then re-name the integration variable to x . This gives

$$(3.11) \quad S_K f(g) = \int_K f(xg)\rho(x) dx, \quad g \in K.$$

(Compare (3.3).) From (3.10) we may also write

$$(3.12) \quad S_K f = \text{analytic continuation of } e^{\Delta_K/2} f$$

where Δ_K is the (negative) Laplacian for K and the analytic continuation is from K to $K_{\mathbb{C}}$.

Finally, we need the heat kernel μ (at the identity, at time one) on $K_{\mathbb{C}}$. See [H1] for details. If $\mu(g) dg$ is the associated heat kernel measure on $K_{\mathbb{C}}$ (where dg is Haar measure on $K_{\mathbb{C}}$) then we have the following result.

THEOREM 3.8. *For any connected Lie group K of compact type the map S_K given by (3.10) is a unitary map of $L^2(K, \rho)$ onto $\mathcal{HL}^2(K_{\mathbb{C}}, \mu)$, where \mathcal{HL}^2 denotes the space of square-integrable holomorphic functions on $K_{\mathbb{C}}$.*

This is Theorem 1' of [H1]. More precisely [H1] proves the compact case; the \mathbb{R}^n case is the ordinary Segal–Bargmann transform.

Once one knows the “right” definition of the transform, proving the isometricity of the transform is relatively easy. Proving the surjectivity of the transform is the hard part. Surjectivity follows as soon as one knows that the matrix entries of finite-dimensional holomorphic representations span a dense subspace of $\mathcal{HL}^2(K_{\mathbb{C}}, \mu)$ —something that seems obvious but requires some thought to prove. In [H1] the density of the holomorphic matrix entries is proven by means of the “averaging lemma” [H1, Lem. 11]. It is the surjectivity of the generalized Segal–Bargmann transform that will ultimately show that every $\mathcal{L}(K)$ -invariant function in $L^2(W(\mathfrak{k}), P)$ is of the form we want.

4. A new proof of the Gross ergodicity theorem

4.1. Strategy of proof. Our strategy is to use the Segal–Bargmann transform, both the ordinary Segal–Bargmann transform for the linear space $W(\mathfrak{k})$ and the generalized Segal–Bargmann transform for the group K . First we consider the Segal–Bargmann transform of the functions we really are interested in, functions $f \in L^2(W(\mathfrak{k}), P)$ of the form $f(B) = \phi(h(B))$ where ϕ is a function on K and h is the “holonomy” defined in Definition 2.4. Such functions are certainly $\mathcal{L}(K)$ -invariant and we want to show that every $\mathcal{L}(K)$ -invariant function is of this form. The first main result is that if $f(B) = \phi(h(B))$ then $Sf(Z) = \Phi(h_{\mathbb{C}}(Z))$ where Φ is the holomorphic function on $K_{\mathbb{C}}$ given by

$$\Phi = \text{analytic continuation of } e^{\Delta/2}\phi.$$

Here $h_{\mathbb{C}} : H(\mathfrak{k}_{\mathbb{C}}) \rightarrow K_{\mathbb{C}}$ is the complex holonomy map, defined similarly to the real holonomy function h . This result shows that on functions of the holonomy the infinite-dimensional *ordinary* Segal–Bargmann transform reduces to the *generalized* Segal–Bargmann transform for K . This result was first proved by Gross and Malliavin. (See also [HS, AHS].)

We now want to consider an arbitrary $\mathcal{L}(K)$ -invariant function f in $L^2(W(\mathfrak{k}), P)$ and show that it must be of the form $f(B) = \phi(h(B))$. We consider the Segal–Bargmann transform $F = Sf$, which is a holomorphic function on $H(\mathfrak{k}_{\mathbb{C}})$. It is easily shown that the Segal–Bargmann transform commutes with the action of the loop group, so F is also $\mathcal{L}(K)$ -invariant. Since F is holomorphic, a simple analytic continuation argument shows that F is actually invariant under the action of the complex loop group $\mathcal{L}(K_{\mathbb{C}})$.

Now, the key point is that F is defined on *finite-energy* paths $Z \in H(\mathfrak{k}_{\mathbb{C}})$, as compared to f which is defined on *continuous* paths $B \in W(\mathfrak{k})$. Thus the argument that we would like to make in $L^2(W(\mathfrak{k}), P)$ actually works in the Segal–Bargmann space. That is, two paths in $H(\mathfrak{k}_{\mathbb{C}})$ that have the same holonomy (endpoint of the Itô map) must be equivalent under the (complex) loop group. It follows that the $\mathcal{L}(K_{\mathbb{C}})$ -invariant function F must be of the form $F(Z) = \Phi(h_{\mathbb{C}}(Z))$, where Φ is a function on $K_{\mathbb{C}}$. The function Φ must be holomorphic on $K_{\mathbb{C}}$ in order for F to be holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$.

We next compute the norm of $F = \Phi \circ h_{\mathbb{C}}$ in the Segal–Bargmann space by taking a limit over a sequence of finite-dimensional subspaces. This calculation shows that the norm of F in the Segal–Bargmann space is equal to the norm of Φ in $L^2(K_{\mathbb{C}}, \mu)$, which is therefore finite. We may then apply the surjectivity of the Segal–Bargmann transform for K to conclude that there is a unique $\phi \in L^2(K, \rho)$ such that Φ is the analytic continuation of $e^{\Delta_{\kappa}/2}\phi$. The result in the first paragraph of this subsection then shows that $S(\phi \circ h) = \Phi \circ h_{\mathbb{C}} = F$. But since S is injective (and $F = Sf$) we conclude that $f = \phi \circ h$. This is the desired result.

4.2. Details of the proof. I will state all the intermediate results we need and then give the proofs.

We begin by considering functions in $L^2(W(\mathfrak{k}), P)$ of the form $f(B) = \phi(h(B))$. It is known that h is a measurable (almost-everywhere-defined) map of $W(\mathfrak{k})$ to K , and that the distribution of $h(B)$, if B is distributed according to the Wiener measure P , is the heat kernel measure $\rho(x)dx$ on K . This means that for f of the form $f(B) = \phi(h(B))$ the norm of f in $L^2(W(\mathfrak{k}), P)$ equals the norm of ϕ in $L^2(K, \rho)$.

We next consider the Itô map $\theta_{\mathbb{C}}$ for finite-energy paths in the complex Lie algebra. This is defined in the same way as the Itô map for paths in the real Lie algebra, except that finite-energy paths the equation is an ordinary differential equation rather than a stochastic differential equation. So $\theta_{\mathbb{C}}$ is an everywhere-defined map from $H(\mathfrak{k}_{\mathbb{C}})$ to $H(K_{\mathbb{C}})$. The complex holonomy map $h_{\mathbb{C}} : H(\mathfrak{k}_{\mathbb{C}}) \rightarrow K_{\mathbb{C}}$ is defined as $h_{\mathbb{C}}(Z) = \theta_{\mathbb{C}}(Z)_1$.

THEOREM 4.1. *Suppose ϕ is in $L^2(K, \rho)$ and consider $f \in L^2(W(\mathfrak{k}), P)$ given by*

$$f(B) = \phi(h(B))$$

where $h(B) = \tilde{\theta}(B)_1$. Then the Segal–Bargmann transform of f satisfies

$$Sf(Z) = \Phi(h_{\mathbb{C}}(Z)),$$

where $h_{\mathbb{C}}(Z) = \theta_{\mathbb{C}}(Z)_1$ and where Φ is the holomorphic function on $K_{\mathbb{C}}$ given by

$$\Phi = \text{analytic continuation of } e^{\Delta_{\kappa}/2}\phi.$$

This result was first obtained in Corollary 7.12 of [GM]. In Section 7 of [GM] Gross and Malliavin are assuming the ergodicity theorem, which we may not do here. Nevertheless, an examination of the proof of Corollary 7.12 shows that ergodicity is not used there. Similar methods were used in Sections 2.4 and 2.5 of [HS] and in [AHS]. The proof of Gross and Malliavin, which I repeat below, is based on probabilistic methods. See Section 8 of [DH] and Section 7 of [H5] for a more geometric perspective on why Theorem 4.1 ought to be true.

We now use the Segal–Bargmann transform to characterize the loop-group-invariant functions on $W(\mathfrak{k})$. The first step is to show that the Segal–Bargmann transform commutes with the action of the loop group.

PROPOSITION 4.2. *For any $l \in \mathcal{L}(K)$ and any $f \in \bigcup_{p>1} L^p(W(\mathfrak{k}), P)$ let f_l denote the function given by $f_l(B) = f(l \cdot B)$. Then*

$$Sf_l(Z) = Sf(l \cdot Z)$$

for all $l \in \mathcal{L}(K)$ and all $Z \in H(\mathfrak{k}_{\mathbb{C}})$.

The second step is to show that for holomorphic functions on $H(\mathfrak{k}_{\mathbb{C}})$, invariance under the real loop group implies invariance under the complex loop group.

PROPOSITION 4.3. *If $F \in \mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$ and $F(l \cdot Z) = F(Z)$ for all l in the real loop group $\mathcal{L}(K)$ and all $Z \in H(\mathfrak{k}_{\mathbb{C}})$ then also $F(l \cdot Z) = F(Z)$ for all l in the complex loop group $\mathcal{L}(K_{\mathbb{C}})$ and all $Z \in H(\mathfrak{k}_{\mathbb{C}})$.*

The next step is to show that if two finite-energy paths in $\mathfrak{k}_{\mathbb{C}}$ have the same endpoint of the Itô map then they are related by an element of $\mathcal{L}(K_{\mathbb{C}})$. This is a critical step in the argument, in that it allows us to identify the loop-invariant elements of the Segal–Bargmann space. It is essential here that we are dealing with finite-energy paths in $\mathfrak{k}_{\mathbb{C}}$ rather than with general continuous paths.

PROPOSITION 4.4. *Suppose that Z and W in $H(\mathfrak{k}_{\mathbb{C}})$ are such that*

$$h_{\mathbb{C}}(Z) = h_{\mathbb{C}}(W).$$

Then there exists $l \in \mathcal{L}(K_{\mathbb{C}})$ such that $l \cdot Z = W$.

This shows that if a function F on $H(\mathfrak{k}_{\mathbb{C}})$ is invariant under the complex loop group, then it must be a function of the complex holonomy, that is, $F(Z) = \Phi(h_{\mathbb{C}}(Z))$. The next step is to show that F is holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$ if and only if Φ is holomorphic on $K_{\mathbb{C}}$. This says essentially that the Itô map is holomorphic.

PROPOSITION 4.5. *Suppose that Φ is a function on $K_{\mathbb{C}}$. Then the function $\Phi \circ h_{\mathbb{C}}$ is holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$ if and only if Φ is holomorphic on $K_{\mathbb{C}}$.*

So now we have a function in the Segal–Bargmann space of the form $F(Z) = \Phi(h_{\mathbb{C}}(Z))$, where Φ is holomorphic on $K_{\mathbb{C}}$. We need to express the norm of F in the Segal–Bargmann space in terms of Φ .

PROPOSITION 4.6. *Suppose Φ is a holomorphic function on $K_{\mathbb{C}}$ with the property that the function $F(Z) = \Phi(h_{\mathbb{C}}(Z))$ is in the Segal–Bargmann space over $H(\mathfrak{k}_{\mathbb{C}})$. Then $\Phi \in L^2(K_{\mathbb{C}}, \mu)$ and*

$$\|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)} = \|F\|_{SB}.$$

Once all these propositions are established we can put them together to get the ergodicity result.

THEOREM 4.7. *Suppose $f \in L^2(W(\mathfrak{k}), P)$ is such that for all $l \in \mathcal{L}(K)$ we have $f(l \cdot B) = f(B)$ for P -almost every B . Then there exists $\phi \in L^2(K, \rho)$ such that $f(B) = \phi(\tilde{\theta}(B)_1)$.*

Proof. We let $F = Sf$. By Propositions 4.2 and 4.3, F is invariant under the complex loop group $\mathcal{L}(K_{\mathbb{C}})$. Thus by Proposition 4.4 there is a function Φ on $K_{\mathbb{C}}$ such that $F(Z) = \Phi(h_{\mathbb{C}}(Z))$. By Proposition 4.5, Φ must be holomorphic on $K_{\mathbb{C}}$ and by Proposition 4.6, Φ is in $L^2(K_{\mathbb{C}}, \mu)$. By the surjectivity of the Segal–Bargmann transform for K , there exists $\phi \in L^2(K, \rho)$ such that Φ is the analytic continuation to $K_{\mathbb{C}}$ of $e^{\Delta\kappa/2}\phi$. Thus the function $\phi \circ h$ will be in $L^2(W(\mathfrak{k}), P)$ and by Theorem 4.1 we will have

$$S(\phi \circ h) = \Phi \circ h_{\mathbb{C}} = F = Sf.$$

Finally, then, the injectivity of S tells us that $f = \phi \circ h$, which is what we want to show. \square

Proof of Theorem 4.1. This theorem was first proven by Gross and Malliavin in Corollary 7.12 of [GM]. Essentially the same argument was used in Sections 2.4 and 2.5 of [HS] and in [AHS]. (The transform we are calling S here is RS in the notion of [HS].)

We consider the restriction to $H(\mathfrak{k})$ of the Segal–Bargmann transform of $\phi \circ h$, which we compute using (3.8) as

$$S(\phi \circ h)(X) = \int_{W(\mathfrak{k})} \phi(h(B + X)) dP(B).$$

Let us think first about the case in which K is commutative. Then $h(B + X) = h(B)h(X)$ and we get

$$\begin{aligned} S(\phi \circ h)(X) &= \int_{W(\mathfrak{k})} \phi(h(B)h(X)) dP(B) \\ (4.1) \qquad &= \int_K \phi(xh(X))\rho(x) dx, \end{aligned}$$

because the distribution of $h(B)$ is given by the measure $\rho(x)dx$ on K . We recognize the last integral as the generalized Segal–Bargmann transform of ϕ , evaluated at the point $g = h(X)$. (See (3.11).) Thus at least on $H(\mathfrak{k}) \subset H(\mathfrak{k}_{\mathbb{C}})$ we have $S(\phi \circ h) = \Phi \circ h = \Phi \circ h_{\mathbb{C}}$.

In the non-commutative case, $h(B + X)$ will not equal $h(B)h(X)$. Nevertheless, these two quantities are equal *in distribution*, which is all that we need for (4.1) to be valid. In fact more generally, $\tilde{\theta}(B + X)$ has the same distribution as $\tilde{\theta}(B)\theta(X)$,

where $\tilde{\theta}(B)$ is the result of applying the stochastic Itô map to the Brownian path B and $\theta(X)$ is the result of applying the pathwise Itô map to the finite-energy path X . To see this we note that (by a standard calculation)

$$\tilde{\theta}^{-1}(\tilde{\theta}(B)\theta(X))_\tau = \int_0^\tau \theta(X)_\sigma^{-1} (dB_\sigma) \theta(X)_\sigma + X_\tau.$$

But the first term on the right above has the same distribution as B itself, since the increments of the Brownian motion are Ad- K -invariant. So $\tilde{\theta}^{-1}(\tilde{\theta}(B)\theta(X))$ has the same distribution as $B + X = \tilde{\theta}^{-1}(\tilde{\theta}(B + X))$ and thus also $\tilde{\theta}(B)\theta(X)$ has the same distribution as $\tilde{\theta}(B + X)$.

We see then that (4.1) is valid even when K is non-commutative. So $S(\phi \circ h)$ and $\Phi \circ h_{\mathbb{C}}$ are equal on $H(\mathfrak{k})$. These two functions are also both holomorphic on $H(\mathfrak{k}_{\mathbb{C}})$ (Proposition 4.5) hence equal on $H(\mathfrak{k}_{\mathbb{C}})$. \square

Proof of Proposition 4.2. Note that if $f \in \bigcup_{p>1} L^p(W(\mathfrak{k}), P)$ then so is f_l , so Sf_l makes sense. Now the action of l on $W(\mathfrak{k})$ is the composition of two maps, first $B \rightarrow B' = \int_0^\tau l_\sigma^{-1} (dB_\sigma) l_\sigma$ and the second $B' \rightarrow B'' = B' + \int_0^\tau l_\sigma^{-1} dl/d\sigma d\sigma$. The first of these is an invertible linear transformation of $W(\mathfrak{k})$ whose restriction to $H(\mathfrak{k})$ is orthogonal. It follows that the map $B \rightarrow B'$ is measure-preserving. The second transformation is just a translation. So now we compute that for $X \in H(\mathfrak{k})$

$$\begin{aligned} \int_{W(\mathfrak{k})} f_l(B + X) dP(B) &= \int_{W(\mathfrak{k})} f(l \cdot (B + X)) dP(B) \\ &= \int_{W(\mathfrak{k})} f\left(B' + X' + \int_0^\tau l_\sigma^{-1} \frac{dl}{d\sigma} d\sigma\right) dP(B) \\ &= \int_{W(\mathfrak{k})} f\left(B + X' + \int_0^\tau l_\sigma^{-1} \frac{dl}{d\sigma} d\sigma\right) dP(B) \\ &= Sf(l \cdot X). \end{aligned}$$

In the second-to-last equality we have used that the map $B \rightarrow B'$ is measure-preserving. So now $Sf(l \cdot Z)$ and $Sf_l(Z)$ agree on $H(\mathfrak{k})$. But both quantities are holomorphic functions of Z , so they must be equal on $H(\mathfrak{k}_{\mathbb{C}})$. \square

Proof of Proposition 4.3. This is shown in Theorem 5.13 of [GM]. The idea is as follows. Since $K_{\mathbb{C}}$ is diffeomorphic to $K \times \mathfrak{k}$ [H2], every loop in $K_{\mathbb{C}}$ is homotopic to a loop in K . By standard arguments, every homotopy class of loops in $K_{\mathbb{C}}$ or K has a smooth representative, and the homotopy of two smooth loops can be performed smoothly. Thus we can connect any element of $\mathcal{L}(K_{\mathbb{C}})$ in a smooth way to an element of $\mathcal{L}(K)$. From this it follows that every element L of $\mathcal{L}(K_{\mathbb{C}})$ can be written as $L = l \exp(h_1) \cdots \exp(h_n)$, where l is in $\mathcal{L}(K)$ and h_1, \dots, h_n are in $H(\mathfrak{k}_{\mathbb{C}})$. Suppose $F \in \mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$ is such that $F(l \cdot Z) = F(Z)$ for all $Z \in H(\mathfrak{k}_{\mathbb{C}})$ and all $l \in \mathcal{L}(K)$. Then

$$F((l \exp h_1 \cdots \exp h_n) \cdot Z)$$

is independent of l and of h_1, \dots, h_n , as long as h_1, \dots, h_n are in $H(\mathfrak{k})$. But then the expression above must also, by holomorphicity, be independent of h_1, \dots, h_n for all h_1, \dots, h_n in $H(\mathfrak{k}_{\mathbb{C}})$. \square

Proof of Proposition 4.4. The action of $\mathcal{L}(K_{\mathbb{C}})$ on $H(\mathfrak{k}_{\mathbb{C}})$ is defined so that $\theta_{\mathbb{C}}(l \cdot Z)_\tau = \theta_{\mathbb{C}}(Z)_\tau l_\tau$. So if $\theta_{\mathbb{C}}(Z)_1 = \theta_{\mathbb{C}}(W)_1$ we define $l_\tau = \theta_{\mathbb{C}}(Z)_\tau^{-1} \theta_{\mathbb{C}}(W)_\tau$ and then $\theta_{\mathbb{C}}(l \cdot Z) = \theta_{\mathbb{C}}(Z)l = \theta_{\mathbb{C}}(W)$, which means that $l \cdot Z = W$. \square

Proof of Proposition 4.5. This follows from Theorem 5.7 of [GM]. \square

Proof of Proposition 4.6. We consider $F \in \mathcal{H}(H(\mathfrak{k}_{\mathbb{C}}))$ of the form $F(Z) = \Phi(h_{\mathbb{C}}(Z))$, where Φ is a holomorphic function on $K_{\mathbb{C}}$. To compute the Segal–Bargmann norm of F we consider the subspaces V_n of $H(\mathfrak{k}_{\mathbb{C}})$ consisting of those Z 's in $H(\mathfrak{k}_{\mathbb{C}})$ such that Z_{τ} is a linear function of τ on each interval of the form $[k/2^n, (k+1)/2^n]$. Then $V_n \subset V_{n+1}$ and the union of the V_n 's is dense in $H(\mathfrak{k}_{\mathbb{C}})$. For $Z \in V_n$ the holonomy of Z can be computed exactly as

$$h_{\mathbb{C}}(Z) = \exp(Z_{1/2^n} - Z_0) \exp(Z_{2/2^n} - Z_{1/2^n}) \cdots \exp(Z_1 - Z_{(2^n-1)/2^n}).$$

Let $P^{(n)}$ denote the standard Gaussian measure on V_n and let $\mu^{(n)}$ denote the push-forward to $K_{\mathbb{C}}$ of $P^{(n)}$ under $h_{\mathbb{C}}$. (That is, $\mu^{(n)}(E) := h_{\mathbb{C}}^{-1}(E)$, for $E \subset K_{\mathbb{C}}$.) Then by the abstract change-of-variables theorem $\|F\|_{L^2(V_n, P^{(n)})} = \|\Phi\|_{L^2(K_{\mathbb{C}}, \mu^{(n)})}$.

Applying results of Stroock and Varadhan [SV, Thm. 2.4] as in [G2, Prop. 3.1] shows that $\mu^{(n)}$ converges weakly to the heat kernel measure μ on $K_{\mathbb{C}}$, as n tends to infinity. This also follows from results of Driver and Hu [D2, DHu]. Then let f be any continuous function of compact support on $K_{\mathbb{C}}$ for which $0 \leq f(g) \leq 1$ for all $g \in K_{\mathbb{C}}$. We have

$$\begin{aligned} \int_{K_{\mathbb{C}}} |\Phi(g)|^2 f(g) d\mu(g) &= \lim_{n \rightarrow \infty} \int_{K_{\mathbb{C}}} |\Phi(g)|^2 f(g) d\mu^{(n)}(g) \\ &\leq \lim_{n \rightarrow \infty} \int_{K_{\mathbb{C}}} |\Phi(g)|^2 d\mu^{(n)}(g) \\ &= \lim_{n \rightarrow \infty} \int_{V_n} |F(Z)|^2 dP^{(n)}(Z) \\ &= \|F\|_{SB}^2. \end{aligned}$$

Since this estimate holds independently of f it follows that

$$\int_{K_{\mathbb{C}}} |\Phi(g)|^2 d\mu(g) \leq \|F\|_{SB}^2.$$

Thus if $F = \Phi \circ h_{\mathbb{C}}$ is in the Segal–Bargmann space, Φ must be in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu)$ with $\|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)} \leq \|F\|_{SB}$.

We are trying to prove that in fact $\|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)} = \|F\|_{SB}$. I do not know how to prove this directly from the definition of the Segal–Bargmann norm. Nevertheless, equality can be shown indirectly as follows. By the surjectivity of the transform S_K there exists $\phi \in L^2(K, \rho)$ with $S_K \phi = \Phi$. Then by the isometricity of the maps S_K and S and by Theorem 4.1 we have

$$\|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)} = \|\phi\|_{L^2(K, \rho)} = \|\phi \circ h\|_{L^2(W(\mathfrak{k}), P)} = \|F\|_{SB}.$$

This completes the proof of Proposition 4.6. \square

Using results of Driver and Hu [D2, DHu] we can give a more direct proof of the equality of the norms of Φ and $\Phi \circ h_{\mathbb{C}}$, though still not as direct as one would like. Consider first the case in which Φ is a finite linear combination of matrix entries for finite-dimensional holomorphic representations of $K_{\mathbb{C}}$. Let V_n be as above the space of continuous $\mathfrak{k}_{\mathbb{C}}$ -valued functions that are linear on each interval of the form $[k/2^n, (k+1)/2^n]$. For any continuous $\mathfrak{k}_{\mathbb{C}}$ -valued path Z , let $Z^{(n)}$ denote the “projection” of Z onto V_n , that is, the unique element of V_n such that $Z_{\tau}^{(n)} = Z_{\tau}$ for each τ of the form $k/2^n$. Let F_n be the function on the *continuous* pathspace given by $F_n(Z) = \Phi(h_{\mathbb{C}}(Z^{(n)}))$. Let M be the Wiener measure on $W(\mathfrak{k}_{\mathbb{C}})$. Then if

Z is distributed according to M , $Z^{(n)}$ will be distributed according to the standard Gaussian measure on V_n . Thus

$$\int_{V_n} |\Phi(h_{\mathbb{C}}(Z))|^2 \frac{e^{-\|Z\|^2}}{\pi^{d_n}} dZ = \int_{W(\mathfrak{k}_{\mathbb{C}})} \left| \Phi(h_{\mathbb{C}}(Z^{(n)})) \right|^2 dM(Z).$$

Now results of [D2, DHu] show (if Φ is a linear combination of matrix entries) that $\Phi(h_{\mathbb{C}}(Z^{(n)}))$ converges in $L^2(W(\mathfrak{k}_{\mathbb{C}}), M)$ to $\Phi(\tilde{\theta}(Z)_1)$, where $\tilde{\theta}$ is the stochastic Itô map for $W(\mathfrak{k}_{\mathbb{C}})$. Since $\tilde{\theta}(Z)_1$ is distributed according to the heat kernel measure μ on $K_{\mathbb{C}}$, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{V_n} |\Phi(h_{\mathbb{C}}(Z))|^2 \frac{e^{-\|Z\|^2}}{\pi^{d_n}} dZ &= \int_{W(\mathfrak{k}_{\mathbb{C}})} \left| \Phi(\tilde{\theta}(Z)_1) \right|^2 dM(Z) \\ &= \int_{K_{\mathbb{C}}} |\Phi(g)|^2 \mu(g) dg. \end{aligned}$$

This establishes the equality of norms in the case Φ is a linear combination of matrix entries.

Now suppose that Φ is any holomorphic function on $K_{\mathbb{C}}$ for which $\|\Phi \circ h_{\mathbb{C}}\|_{SB} < \infty$. Then the argument given above shows that $\Phi \in L^2(K_{\mathbb{C}}, \mu)$. The density results of [H1, Sect. 8] then show that there exist Φ_n which are linear combinations of holomorphic matrix entries that converge to Φ in $L^2(K_{\mathbb{C}}, \mu)$. Note that $\Phi_n - \Phi_m$ is also a linear combination of matrix entries. So by the result of the previous paragraph,

$$\|\Phi_n \circ h_{\mathbb{C}} - \Phi_m \circ h_{\mathbb{C}}\|_{SB} = \|\Phi_n - \Phi_m\|_{L^2(K_{\mathbb{C}}, \mu)}.$$

This means that the sequence $\{\Phi_n \circ h_{\mathbb{C}}\}$ is Cauchy in the Segal–Bargmann space, hence convergent to some F in the Segal–Bargmann space. Then since norm convergence implies pointwise convergence both in the Segal–Bargmann space and in $\mathcal{HL}^2(K_{\mathbb{C}}, \mu)$, $F(Z)$ must equal $\Phi(h_{\mathbb{C}}(Z))$. So finally,

$$\|\Phi \circ h_{\mathbb{C}}\|_{SB} = \lim_{n \rightarrow \infty} \|\Phi_n \circ h_{\mathbb{C}}\|_{SB} = \lim_{n \rightarrow \infty} \|\Phi_n\|_{L^2(K_{\mathbb{C}}, \mu)} = \|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)}.$$

5. Discussion

One purpose of this paper is to demonstrate the advantages of working in the Segal–Bargmann space. The proof of the ergodicity theorem given here is similar to the original one of Gross except that it uses the Segal–Bargmann transform where Gross uses the expansion into multiple Itô integrals. The identification of the loop-group-invariant elements in the Segal–Bargmann space is easy and geometrically transparent, whereas the calculation of the loop-group-invariant multiple Itô expansions is a bit arduous. (See [G3, Sect. 5].) (Of course after computing the $\mathcal{L}(K)$ -invariant elements in the Segal–Bargmann space we still have to work back to the $\mathcal{L}(K)$ -invariant elements in $L^2(W(\mathfrak{k}), P)$, and this requires the surjectivity of the generalized Segal–Bargmann transform for K .)

So the use of the Segal–Bargmann transform in place of the multiple Itô expansion simplifies certain parts of the proof of the ergodicity theorem. Nevertheless, there is a big benefit to Gross’s approach, namely that it gives rise to the generalized Hermite expansion for K . Perhaps the best approach is to use *both* the Segal–Bargmann transform and the expansion into Itô integrals. (This is the approach used in [DH] in a slightly more general setting. See also Section 4 of [H4].)

So we could consider a loop-invariant element f in $L^2(W(\mathfrak{k}), P)$ and Segal–Bargmann transform it to a holomorphic function F on $H(\mathfrak{k}_{\mathbb{C}})$. Then F must be of the form $F(Z) = \Phi(h_{\mathbb{C}}(Z))$ for some holomorphic function Φ on $K_{\mathbb{C}}$. We may now compute the (complex form of) the Itô expansion of $\Phi(h_{\mathbb{C}}(Z))$. This is fairly easy to do because we have a concrete form to work with, rather than having just an unspecified loop-invariant function. The expansion of $\Phi(h_{\mathbb{C}}(Z))$ is computed in [GM, Lem. 6.4] as

$$(5.1) \quad \Phi(h_{\mathbb{C}}(Z)) = \sum_{n=0}^{\infty} \sum_{k_1, \dots, k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} (X_{k_1} X_{k_2} \cdots X_{k_n} \Phi)(e) dZ_{\tau_1}^{k_1} dZ_{\tau_2}^{k_2} \cdots dZ_{\tau_n}^{k_n}.$$

Here Δ_n is the set of all n -tuples (τ_1, \dots, τ_n) with $0 < \tau_1 < \dots < \tau_n < 1$ and $X_1, \dots, X_{\dim \mathfrak{k}}$ are an orthonormal basis for \mathfrak{k} with respect to the given Ad-invariant inner product. The X_k 's are thought of as left-invariant differential operators on K (applied to the restriction of Φ to K). The same vectors form a basis for $\mathfrak{k}_{\mathbb{C}}$ as a complex vector space, and $Z_{\tau}^1, \dots, Z_{\tau}^{\dim \mathfrak{k}}$ are the coordinates of Z_{τ} with respect to this basis. The expansion of $\phi \circ h$ into multiple Itô integrals is given by a very similar expression, namely,

$$(5.2) \quad \phi(h(B)) = \sum_{n=0}^{\infty} \sum_{k_1, \dots, k_n=1}^{\dim \mathfrak{k}} \int_{\Delta_n} (X_{k_1} X_{k_2} \cdots X_{k_n} \Phi)(e) dB_{\tau_1}^{k_1} dB_{\tau_2}^{k_2} \cdots dB_{\tau_n}^{k_n}.$$

Here the integrals are Itô stochastic integrals. Note that it is not ϕ but rather $\Phi = \exp(\Delta_K/2)\phi$ appearing on the right of (5.2). The expression (5.2) is a consequence of Gross's results in [G3] and the explicit formula of Hijab [Hi1] for the generalized Hermite expansion. Equation (5.2) is derived directly in Lemma 5.7 of [DH].

Applying the isometricity of the complex Itô expansion to this function gives (using Proposition 4.6)

$$\|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)}^2 = \|\Phi \circ h_{\mathbb{C}}\|_{SB}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^{\dim \mathfrak{k}} |(X_{k_1} X_{k_2} \cdots X_{k_n} \Phi)(e)|^2.$$

Instead of using Proposition 4.6, one can directly prove the equality of the first and third items above [DG] and then use that result to prove the equality of the first and second items. (Recall that the equality of the first two items, Proposition 4.6, is proved in an indirect way.) Applying the isometricity of the stochastic Itô expansion for $W(\mathfrak{k})$ gives

$$\|\phi\|_{L^2(K, \rho)}^2 = \|\phi \circ h\|_{L^2(W(\mathfrak{k}), P)}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^{\dim \mathfrak{k}} |(X_{k_1} X_{k_2} \cdots X_{k_n} \Phi)(e)|^2$$

In any case, we obtain in the end the equalities

$$(5.3) \quad \|\phi\|_{L^2(K, \rho)}^2 = \|\Phi\|_{L^2(K_{\mathbb{C}}, \mu)}^2 = \sum_{n=0}^{\infty} \sum_{k_1, \dots, k_n=1}^{\dim \mathfrak{k}} |(X_{k_1} X_{k_2} \cdots X_{k_n} \Phi)(e)|^2.$$

Although these equalities can be proved by purely finite-dimensional means [H1, Hi1, Hi2, D1] we now can understand them as special cases of the isometricity of the Segal–Bargmann transform and the multiple Itô expansions. Specifically, (5.3) is the result of applying those isometricity results to the $\mathcal{L}(K)$ -invariant elements in the respective spaces.

References

- [AHS] S. Albeverio, B. C. Hall, and A. N. Sengupta, *The Segal–Bargmann transform for two-dimensional Euclidean quantum Yang–Mills*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2** (1999), 27–49.
- [BSZ] J. C. Baez, I. E. Segal, and Z. Zhou, “Introduction to Algebraic and Constructive Quantum Field Theory,” Princeton Univ. Press, Princeton, NJ, 1992.
- [B] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, Part I*, *Comm. Pure Appl. Math.* **14** (1961), 187–214.
- [D1] B. K. Driver, *On the Kakutani–Itô–Segal–Gross and Segal–Bargmann–Hall isomorphisms*, *J. Funct. Anal.* **133** (1995), 69–128.
- [D2] B. K. Driver, *Analysis of Wiener measure on path and loop groups*, in this volume.
- [DG] B. K. Driver and L. Gross, *Hilbert spaces of holomorphic functions on complex Lie groups*, In: *New trends in stochastic analysis* (K. Elworthy, S. Kusuoka, and I. Shigekawa, Eds.), World Scientific, Singapore, 1997, pp. 76–106.
- [DH] B. K. Driver and B. C. Hall, *Yang–Mills theory and the Segal–Bargmann transform*, *Commun. Math. Phys.* **201** (1999), 249–290.
- [DHu] B. K. Driver and Y. Hu, *Convergence rates for Wong–Zakai approximations*, in preparation.
- [Go1] M. Gordina, *Holomorphic functions and the heat kernel measure on an infinite-dimensional complex orthogonal group*, *Potential Anal.* **12** (2000), 325–357.
- [Go2] M. Gordina, *Heat kernel analysis and Cameron–Martin subgroup for infinite dimensional groups*, *J. Funct. Anal.* **171** (2000), 192–232.
- [Go3] M. Gordina, *Quasi-invariance for the pinned Brownian motion on a Lie group*, preprint.
- [G1] L. Gross, *Abstract Wiener spaces*, In: “Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II,” Univ. of California Press, 1967.
- [G2] L. Gross, *Logarithmic Sobolev inequalities on Lie groups*, *Illinois J. Math.* **36** (1992), 447–490.
- [G3] L. Gross, *Uniqueness of ground states for Schrödinger operators over loop groups*, *J. Funct. Anal.* **112** (1993) 373–441.
- [G4] L. Gross, *Analysis on loop groups*, In: “Stochastic analysis and applications in physics (Funchal, 1993),” NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 449, Kluwer Acad. Publ., Dordrecht, 1994, pp. 99–118.
- [G5] L. Gross, *Harmonic functions on loop groups*, In: *Séminaire Bourbaki* Vol. 1997/98, Astérisque No. 252, (1998), Exp. No. 846, 5, 271–286.
- [G6] L. Gross, *Heat kernel analysis on Lie groups*, preprint.
- [G7] L. Gross, *Irving Segal’s work on infinite dimensional integration theory*, *J. Funct. Anal.* **190** (2002), 1924.
- [GM] L. Gross and P. Malliavin, *Hall’s transform and the Segal–Bargmann map*, In: “Itô’s stochastic calculus and probability theory,” (M. Fukushima, N. Ikeda, H. Kunita, and S. Watanabe, Eds.), Springer-Verlag, Berlin/New York, 1996, pp. 73–116.
- [H1] B. C. Hall, *The Segal–Bargmann “coherent state” transform for compact Lie groups*, *J. Funct. Anal.* **122** (1994), 103–151.
- [H2] B. C. Hall, *Phase space bounds for quantum mechanics on a compact Lie group*, *Comm. Math. Phys.* **184** (1997), 233–250.
- [H3] B. C. Hall, *Holomorphic methods in analysis and mathematical physics*, In: “First Summer School in Analysis and Mathematical Physics” (S. Pérez Esteva and C. Villegas Blas, Eds.), *Contemp. Math.* Vol. 260, Amer. Math. Soc., Providence, RI, 2000, pp. 1–59.
- [H4] B. C. Hall, *Harmonic analysis with respect to heat kernel measure*, *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), 43–78.
- [H5] B. C. Hall, *Coherent states and the quantization of (1+1)-dimensional Yang–Mills theory*, *Rev. Math. Phys.* **13** (2001), 1281–1305.
- [H6] B. C. Hall, *Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type*, *Comm. Math. Phys.*, **226** (2002), 233–268.
- [HS] B. C. Hall and A. N. Sengupta, *The Segal–Bargmann transform for path-groups*, *J. Funct. Anal.* **152** (1998), 220–254.
- [Hi1] O. Hijab, *Hermite functions on compact Lie groups. I*, *J. Funct. Anal.* **125** (1994), 480–492.
- [Hi2] O. Hijab, *Hermite functions on compact Lie groups. II*, *J. Funct. Anal.* **133** (1995), 41–49.

- [HP] E. Hille and R. Phillips, “Functional analysis and semi-groups.” Third printing of the revised edition of 1957. American Mathematical Society Colloquium Publications, Vol. XXXI. American Mathematical Society, Providence, R. I., 1974.
- [K] H.-H. Kuo, “Gaussian measures in Banach spaces.” Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin-New York, 1975.
- [MM] M.-P. Malliavin and P. Malliavin, *Integration on loop groups. I. Quasi invariant measures*, J. Funct. Anal. 93 (1990), 207–237.
- [M1] J. J. Mitchell, *Short time behavior of Hermite functions on compact Lie groups*, J. Funct. Anal. **164** (1999), 209–248.
- [M2] J. J. Mitchell, *Small-time asymptotics of Hermite functions on compact symmetric spaces*, J. Math. Anal. Appl. **263** (2001), 165–181.
- [Sa] G. Sadasue, *Equivalence-singularity dichotomy for the Wiener measures on path groups and loop groups*, J. Math. Kyoto Univ. **35** (1995), 653–662.
- [S1] I. E. Segal, *Mathematical problems of relativistic physics, Chap. VI*, In: Proceedings of the Summer Seminar, Boulder, Colorado, 1960, Vol. II. (M. Kac, Ed.), Lectures in Applied Mathematics, American Math. Soc., Providence, Rhode Island, 1963.
- [S2] I. E. Segal, *Mathematical characterization of the physical vacuum for a linear Bose-Einstein field*, Illinois J. Math. **6** (1962), 500-523.
- [S3] I. E. Segal, *The complex wave representation of the free Boson field*, In: “Topics in functional analysis: Essays dedicated to M.G. Krein on the occasion of his 70th birthday” (I. Gohberg and M. Kac, Eds), Advances in Mathematics Supplementary Studies, Vol. 3, Academic Press, New York, 1978, pp. 321-343.
- [Sh] I. Shigekawa, *Itô-Wiener expansions of holomorphic functions on the complex Wiener space*, In: “Stochastic analysis,” Academic Press, Boston, MA, 1991, pp. 459-473.
- [SV] D. W. Stroock and S. R. S. Varadhan, *Limit theorems for random walks on Lie groups*, Sankhyā Ser. A **35** (1973), 277–294.
- [Su] H. Sugita, *Holomorphic Wiener function*, In: New trends in stochastic analysis (Charingworth, 1994) (K. D. Elworthy, S. Kusuoka, and I. Shigekawa, Eds.), World Sci. Publishing, River Edge, NJ, 1997, pp. 399–415.

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