Test 1, Version White,
1(b), 2(d), 3(e), 4(c), 5(c), 6(e), 7(c), 8(e), 9(d), 10(b)

Problem 11
Find the Taylor (MacLaurin) series of the following functions. Write your answer in the format of a summation. For example, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$.

(a) $\dfrac{1}{x}\ln\left(1+\dfrac{x}{2}\right)$

Sol.

$$\begin{aligned} \frac{1}{x}\ln\left(1+\frac{x}{2}\right) &= \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{x}{2}\right)^n}{n} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n-1}}{n2^n} \end{aligned}$$

(b) $\displaystyle\int_0^x \cos t^2\,\mathrm{d}t.$

Sol.

$$\begin{aligned} \int_0^x \cos t^2\,\mathrm{d}t &= \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n(t^2)^{2n}}{(2n)!}\mathrm{d}t \\ &= \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n}}{(2n)!}\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)} \end{aligned}$$

Problem 12,

(a) Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{3^n n!}$ is convergent. Indicate the test you use. Show all your work. You may need the formula $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.7$.

Sol. Ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\dfrac{n^n}{3^n n!}} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}\,\frac{3^n n!}{n^n} \\
&= \lim_{n\to\infty} \frac{1}{3}\frac{(n+1)^n}{n^n} \\
&= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1.
\end{aligned}$$

Convergent by Ratio test.

(b) Use Taylor series to find $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-1}{\sin^2(2x)} =$

Sol.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{x^2}-1}{\sin^2(2x)} &= \lim_{n\to\infty} \frac{\left(1+x^2+\dfrac{1}{2!}(x^2)^2+\cdots\right)-1}{\left((2x)-\dfrac{1}{3!}(2x)^3+\cdots\right)^2} \\
&= \lim_{n\to\infty} \frac{x^2+\dfrac{1}{2}x^4+\cdots}{4x^2-\cdots} = \frac{1}{4}.
\end{aligned}$$