Sample Final Exam,

1(b), 2(a), 3(e), 4(b), 5(c), 6(e), 7(e), 8(d),

9(a), 10(a), 11(d), 12(e), 13(a), 14(c), 15(c)

Problem 16.

(a) Find the Taylor series for $\sin(x^2)$ about $x = 0$.

Sol.

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

(b) Find the Taylor series for the function $G(x) = \int_0^x \sin(t^2)dt$ about $x = 0$. **Sol.**

$$G(x) = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{4n+2}}{(2n+1)!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!(4n+3)}$$

Probelm 17. (a) Find the minimum distance from the origin to the plane $x + 2y + 3z = 14$.

Sol. Lagrange method: Minimizing $\sqrt{x^2 + y^2 + z^2}$ is the same as minimizing $x^2 + y^2 + z^2$.

$$f(x) = x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 14)$$

$$\frac{\partial f}{\partial x} = 2x + \lambda = 0 \quad \Rightarrow \quad x = -\frac{\lambda}{2}$$

$$\frac{\partial f}{\partial y} = 2y + 2\lambda = 0 \quad \Rightarrow \quad y = -\lambda$$

$$\frac{\partial f}{\partial z} = 2z + 3\lambda = 0 \quad \Rightarrow \quad z = -\frac{3\lambda}{2}$$

$$\Big(-\frac{\lambda}{2}\Big) + 2(-\lambda) + 3\Big(-\frac{3\lambda}{2}\Big) = 14 \quad \Rightarrow \quad \lambda = -2$$

$$x = 1, \quad y = 2, \quad z = 3.$$

The minimum distance: $\sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

(b) Find the equation of the normal line of the surface $x^2+2y^2+3z^2=4$ at the point $(1,1,1)$.
Sol. Let $f(x)=x^2+2y^2+3z^2-4$

$$\nabla f=(2x,4y,6z)=2(1,2,3) \text{ at } (x,y,z)=(1,1,1).$$

The normal line is

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$$

Problem 18 (a) What extension is needed for a function $f(x)$ defined on $(0,l)$ in order to have a Fourier Sine series?
Sol. Odd extension.

(b) If $f(x)=2$ for $0<x<l$, find the Fourier Sine series.

$$\begin{aligned} a_n &= 0 \\ b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{4}{l}\int_0^l \sin\frac{n\pi x}{l}dx = \frac{4}{l}\Big[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big]_{x=0}^{l} \\ &= \frac{4}{n\pi}\Big(1-\cos(n\pi)\Big) = \frac{4}{n\pi}\Big(1-(-1)^n\Big) = \begin{cases} \dfrac{8}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Thus

$$f(x)=\frac{8}{\pi}\Big(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\cdots\Big)$$