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Mathematical Methods II

One Dimensional Piston Problem

1 Governing equations

Consider a one-dimensional duct aligned with the x-axis. To the right of the piston, the duct is full with a gas at rest. At time t=0 the piston starts moving with a velocity $u_p(t)$. Let ρ, u, and s describe the gas density, velocity and entropy. We assume an isentropic process and therefore the gas properties are governed by the two first order partial differential equations:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \tag{2}$$

where c is the speed of sound. We assume the piston position is given by the function $X(t)$. The initial conditions are

$$t \leq 0, \qquad u = 0 \ \ , c = c_0 \tag{3}$$

and the boundary conditions are,

$$t > 0, \qquad u(X(t), t) = u_p = \dot{X}(t). \tag{4}$$

2 Characteristics Equations

$$I_+ : \quad \frac{dx}{dt} = u + c \ \ C_+ \tag{5}$$

$$I_- : \quad \frac{dx}{dt} = u - c \ \ C_- \tag{6}$$

$$II_+ : \quad du + \frac{c}{\rho}d\rho = 0 \ \text{ on } C_+ \tag{7}$$

$$II_- : \quad du - \frac{c}{\rho}d\rho = 0 \ \text{ on } C_+ \tag{8}$$

Note that since the gas is isentropic, $(\gamma - 1)\frac{d\rho}{\rho} = 2\frac{dc}{c}$. Therefore (7-8) can be rewritten as

$$II_+ : \quad du + \frac{2}{\gamma - 1}dc = 0 \quad \text{on } C_+ \tag{9}$$

$$II_- : \quad du - \frac{2}{\gamma - 1}dc = 0 \quad \text{on } C_- \tag{10}$$

Integrating 9 along C_+ and 10 along C_-, gives

$$II_+ : \quad u + \frac{2}{\gamma - 1}c = r^*, \quad \text{on } C_+ \tag{11}$$

$$II_- : \quad u - \frac{2}{\gamma - 1}c = s^*, \quad \text{on } C_- \tag{12}$$

where r^* and s^* constant along C_+ and C_-, respectively. r^* and s^* are known as the Riemann invariants.

3 Simple Waves

If one of the Riemann invariants is constant throughout the domain, the solution corresponds to a wave motion in *only* one direction. In the present problem,

$$u = \frac{r^* + s^*}{2} \tag{13}$$

$$c = \frac{\gamma - 1}{2}(r^* - s^*) \tag{14}$$

In general, the value of u and c will depend on the two parameters (r^*, c^*) indicating two waves where the information is propagating along the two families of characteristics. On the other hand, if s^* =constant everywhere, then u and c receive information from the characteristic C_+ only through the variation of r^* from one characteristic to another. Problems reduced to simple waves are much simpler to solve.

3.1 Proposition

The solution in a region adjacent to a constant state is always a simple wave solution.

4 Solution

We construct the solution assuming no breaking will occur. This will be examined later.

4.1 Steady State Region: $t > 0$, $x > c_0 t$

Along C_+ originating from the positive x-axis,

$$u + \frac{2}{\gamma - 1} c = \frac{2}{\gamma - 1} c_0 \tag{15}$$

Similarly along C_- originating from the positive x-axis,

$$u - \frac{2}{\gamma - 1} c = -\frac{2}{\gamma - 1} c_0 \tag{16}$$

These equations imply that for the region $x > c_0 t$, we have

$$u = 0 \qquad\qquad c = c_0 \tag{17}$$

The charcteristics in this region are straight lines whose equations are

$$x - x_0^+ = c_0 t \quad \text{along} \;\; C_+ \tag{18}$$

$$x - x_0^- = -c_0 t \quad \text{along} \;\; C_-, \tag{19}$$

where x_0^+ and x_0^- are the points of their intersection with the x-axis. Note in this region there no waves and the gas remain quiescent.

4.2 Simple Wave Solution

All characteristics C_- will originate from the positive x-axis. Thus they have the same Riemann invariant as (16) shows. This means that we have the following relationship

$$u = \frac{2}{\gamma - 1}(c - c_0). \tag{20}$$

Equation 20 is valid everywhere in the field, assuming no shocks. Substituting 20 into 15 show that both u and c are constant along C_+ originating from the piston surface. Hence, we have

$$u = u_p = \dot{X}(t) \tag{21}$$

$$c = c_p \tag{22}$$

Since both u and c are constant along C_+ we can integrate 5,

$$x = X(\tau) + (u_p + c_p)(t - \tau) \tag{23}$$

or

$$x = X(\tau) + (c_0 + \frac{\gamma + 1}{2}\dot{X}(\tau))(t - \tau) \tag{24}$$

5 Piston Moving with a Constant Speed (-V)

Substituting $\dot{X}$ by $-V$, we get

$$\begin{aligned} x &= -Vt + (c_0 - \frac{\gamma+1}{2}V)(t-\tau) && (25) \\ u &= -V && (26) \\ c &= c_0 - \frac{\gamma-1}{2}V && (27) \end{aligned}$$

In the fan region defined by

$$(c_0 - \frac{\gamma+1}{2}V)t < x < tc_0,$$

we have

$$\frac{dx}{dt} = u + c = c_0 + \frac{\gamma+1}{2}u \quad (28)$$

Moreover along C_+, $c + \frac{\gamma-1}{2}u =$ constant. Therefore u and c are constant. As a result,

$$x = (c_0 + \frac{\gamma+1}{2}u)t \quad (29)$$

and

$$\begin{aligned} u &= \frac{2}{\gamma+1}(\frac{x}{t} - c_0) && (30) \\ c &= \frac{2}{\gamma+1}c_0 + \frac{\gamma-1}{\gamma+1}\frac{x}{t} && (31) \end{aligned}$$

6 Breaking

It is easy to show that breaking will occur if $\ddot{X} < 0$.