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Mathematical Methods II

Orthogonal Curvilinear Coordinates

1 Definitions

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the Cartesian coordinates of a point M with respect to a frame of reference defined by the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots$. We introduce three functions defined by

$$u_j = u_j(x_1, x_2, x_3), \quad j = 1, 2, 3, \quad (1)$$

in a region \mathcal{R} . The equation $u_j = c_j$, where c_j is a constant, represents a surface. The system of two equations $u_2 = c_2$ and $u_3 = c_3$ represent a line γ_1 where the two surfaces intersect. Along γ_1 , only u_1 varies. The system of three equations $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ represent a point where the three surfaces intersect. At every point $M \in \mathcal{R}$, there are three lines $\gamma_i(u_i)$. For Cartesian coordinates, these surfaces are planes. For cylindrical coordinates, we define

$$u_1 = r = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad (2)$$

$$u_2 = \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \quad (3)$$

$$u_3 = x_3. \quad (4)$$

$$(5)$$

Here $r = c_1$ represents a circular cylinder of radius c_1 , $\theta = c_2$ represents a vertical plane, and $x_3 = c_3$ represents a horizontal plane. The two equations $r = c_1$ and $x_3 = c_3$ represent a circle in a horizontal plane, only θ varies as we move along the circle.

The position vector of a point M can be expressed in the Cartesian system as

$$\overrightarrow{OM} = \mathbf{x} = x_i \mathbf{e}_i, \quad (6)$$

where the repeated index implies summation, i.e., $x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$. Note that

$$\frac{\partial \mathbf{x}}{\partial x_i} = \mathbf{e}_i. \quad (7)$$

We now want to use u_j as a new coordinate system. We assume that the Cartesian coordinates x_i are given in terms of the new coordinates u_j ,

$$x_i = x_i(u_1, u_2, u_3), \quad i = 1, 2, 3. \quad (8)$$

Differentiating \mathbf{x} with respect to u_j , we get

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_j} du_j = \mathbf{e}_i \frac{\partial x_i}{\partial u_j} du_j. \quad (9)$$

The vector

$$\hat{\mathbf{E}}_j = \frac{\partial \mathbf{x}}{\partial u_j} = \mathbf{e}_i \frac{\partial x_i}{\partial u_j} \quad (10)$$

is tangent to γ_j . Note that

$$\hat{\mathbf{E}}_j = \frac{\partial \mathbf{x}}{\partial s_j} \frac{ds_j}{du_j}, \quad (11)$$

where ds_j is the elementary arc length along γ_j . We also note that the vector $\mathbf{E}_j = \frac{\partial \mathbf{x}}{\partial s_j}$ is a unit vector. Thus if $h_j = \frac{ds_j}{du_j}$, $ds_j = h_j du_j$ and $\hat{\mathbf{E}}_j = h_j \mathbf{E}_j$. Hence, using (9, 10), we get

$$h_j \mathbf{E}_j = \mathbf{e}_i \frac{\partial x_i}{\partial u_j}, \quad (12)$$

$$d\mathbf{x} = h_j \mathbf{E}_j du_j. \quad (13)$$

Since both \mathbf{e}_i and \mathbf{E}_j are orthonormal vectors,

$$h_j^2 = \sum_{i=1}^3 \left(\frac{\partial x_i}{\partial u_j} \right)^2. \quad (14)$$

Equation(14) defines the three scales associated with the new coordinates system.

2 Elementary Quantities

2.1 Elementary Arc Length

The elementary arc length of a line not coinciding with the three lines defining the coordinate system at a point M, the elementary arc length is obtained by taking the magnitude of (13),

$$(ds)^2 = h_j^2 (du_j)^2 \quad (15)$$

2.2 Elementary Surface

The elementary surface of $u_i = c_i$ which contains γ_j and γ_k ($i \neq j \neq k$) is

$$d\sigma_i = ds_j ds_k = h_j h_k du_j du_k \quad (16)$$

2.3 Elementary Volume

The elementary volume

$$dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3 \quad (17)$$

3 Differential Operators

3.1 Gradient

The gradient is defined by

$$df = \nabla f \cdot d\mathbf{x}. \quad (18)$$

We can also express df as

$$df = \frac{\partial f}{\partial u_j} du_j. \quad (19)$$

Using (13), we get

$$\nabla f = \frac{\mathbf{E}_j}{h_j} \frac{\partial f}{\partial u_j}. \quad (20)$$

Or

$$\nabla = \frac{\mathbf{E}_j}{h_j} \frac{\partial}{\partial u_j}. \quad (21)$$

3.1.1 Useful Results

1.

$$\nabla u_j = \frac{\mathbf{E}_j}{h_j} \quad (22)$$

2. Equation(22) implies that

$$\nabla \times \frac{\mathbf{E}_j}{h_j} = 0 \quad (23)$$

Since

$$\nabla \times (f\mathbf{a}) \equiv f\nabla \times \mathbf{a} + \nabla f \times \mathbf{a}, \quad (24)$$

then,

$$\nabla \times \frac{\mathbf{E}_j}{h_j} \equiv \frac{1}{h_j} \nabla \times \mathbf{E}_j + \nabla \frac{1}{h_j} \times \mathbf{E}_j, \quad (25)$$

we deduce

$$\nabla \times \mathbf{E}_j = \frac{\nabla h_j \times \mathbf{E}_j}{h_j} \quad (26)$$

3.2 Divergence

Note that

$$\frac{\mathbf{E}_1}{h_2 h_3} = \frac{\mathbf{E}_2}{h_2} \times \frac{\mathbf{E}_3}{h_3}.$$

Using (22),

$$\frac{\mathbf{E}_1}{h_2 h_3} = \nabla u_2 \times \nabla u_3.$$

Taking the divergence of both sides and noting that

$$\nabla \cdot \mathbf{A} \times \mathbf{B} \equiv \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

we arrive at

$$\nabla \cdot \frac{\mathbf{E}_1}{h_2 h_3} = 0. \quad (27)$$

Or

$$\nabla \cdot \frac{\mathbf{E}_i}{h_j h_k} = 0, \quad (28)$$

where $i \neq j \neq k$.

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \nabla \cdot (F_i \mathbf{E}_i) \\ &= \nabla \cdot \left(\frac{\mathbf{E}_i}{h_j h_k} (h_j h_k F_i) \right) \\ &= \frac{\mathbf{E}_i}{h_j h_k} \cdot \nabla (h_j h_k F_i). \end{aligned} \quad (29)$$

This gives the expression for the divergence

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} (h_j h_k F_i), \quad (30)$$

where $i \neq j \neq k$.

3.3 curl

Using (24, 26), it is readily shown that

$$\nabla \times F_k \mathbf{E}_k = \frac{1}{h_k} \nabla(h_k F_k) \times \mathbf{E}_k, \quad (31)$$

or

$$\nabla \times F_k \mathbf{E}_k = \frac{1}{h_j h_k} \frac{\partial(h_k F_k)}{\partial u_j} (\mathbf{E}_j \times \mathbf{E}_k). \quad (32)$$

Noting that $\mathbf{E}_j \times \mathbf{E}_k = \epsilon_{ijk} \mathbf{E}_i$, where the permutation symbol $\epsilon_{ijk} = 1$ for i, j, k in order but $i \neq j \neq k$, $\epsilon_{ijk} = -1$ for i, j, k not in order but $i \neq j \neq k$, and $\epsilon_{ijk} = 0$ when two indices are equal, we obtain,

$$\nabla \times \mathbf{F} = \epsilon_{ijk} \frac{h_i \mathbf{E}_i}{h_1 h_2 h_3} \frac{\partial}{\partial u_j} (h_k F_k). \quad (33)$$

The expression (33) for the curl can be cast in the familiar form,

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{E}_1 & h_2 \mathbf{E}_2 & h_3 \mathbf{E}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \quad (34)$$