Scalar and Vector Fields:

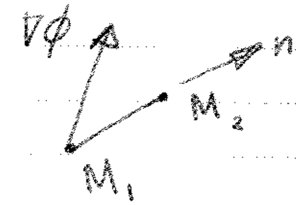
Let $\phi(\vec{x},t)$, a $\vec{V}(\vec{x},t)$ $\qquad \vec{x} = \{x_1, x_2, x_3\}$ position vector.

1. **Gradient:** $\nabla\phi = \left\{\frac{\partial\phi}{\partial x_i}\right\}$

Interpretation: $\phi(M_2) - \phi(M_1)$, where M_1, M_2 are two close points.

$$\phi(M_2) - \phi(M_1) = (x_i^{(2)} - x_i^{(1)})\frac{\partial\phi}{\partial x_i} + \ldots$$

$$= \nabla\phi \cdot d\vec{M}$$

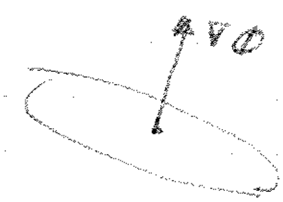


Directional Derivative:

$$\frac{d\phi}{ds} = \nabla\phi\cdot\vec{n}$$

A scalar function has its maximum directional derivative in the direction of its gradient.
A scalar function has zero directional derivative normal to its gradient.

Application:



$\phi = c$ is a surface
$\nabla\phi$ is orthogonal to the surface.

$$d\phi = \nabla\phi\cdot d\vec{M} = 0 \implies \nabla\phi \perp d\vec{M}.$$

Ex: $\quad x^2 + y^2 - R^2 = 0 \qquad \nabla\phi = \{2x, 2y\}$

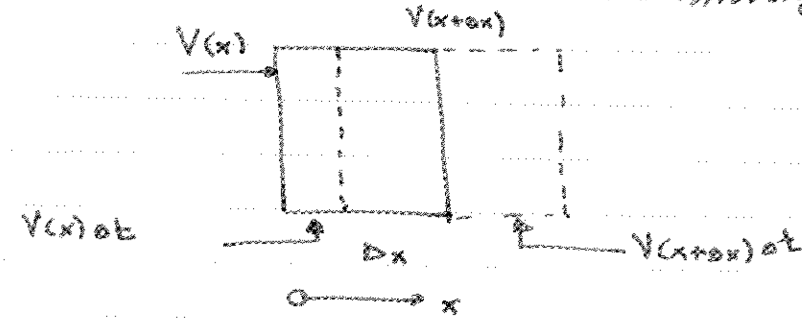
$$(x-x_0)^2 + (y-y_0)^2 - R^2 = 0 \quad \nabla\phi = \{2(x-x_0), 2(y-y_0)\}$$

Divergence:

$$\nabla \cdot \vec{V} = \frac{\partial V_i}{\partial x_i}$$

Interpretation:

Consider a one-dimensional ~~flux~~ field $V(x)$ and a volume of cross-section A $\perp$ to V and width Δx moving with V



Initially, the volume is $A\Delta x$, after a time Δt, the volume has expanded along the dashed lines to

$$A\Delta x - V(x)\Delta t + V(x+\Delta x)\Delta t \sim A\Delta x + A\frac{\partial V}{\partial x}\Delta x\,\Delta t$$

The change in volume is $A\frac{\partial V}{\partial x}\Delta x\,\Delta t$

The rate of chang in volume is $(\Delta x A)\frac{\partial V}{\partial x}$

The relative rate of volume change is $\frac{\partial V}{\partial x}$

Hence divergence represents the relative ~~volume~~ rate of change of a volume moving with the field V

IF $\vec{V}$ is the velocity field of a liquid, then since liquid density is constant, conservation of mass implies conservation of volume, Hence $\nabla \cdot \vec{V} = 0$ for a liquid or incompressible fluid.

IF $\vec{V}$ is the velocity field of a gas, then $\nabla \cdot \rho\vec{V}$ represents the mass flux outward, per unit volume

Curl of a vector $\quad \nabla \times \vec{V}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \vec{i}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \vec{i}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \vec{i}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$= \varepsilon_{ijk}\, \vec{e_i}\, \frac{\partial V_k}{\partial x_j}$$

ε_{ijk} is the permutation index $= \pm 1$

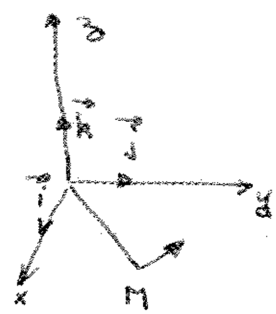
Significance:

① RBR

$u = -y\omega$

$v = x\omega$

$w = 0$



$u_r = 0$

$u_\theta = r\omega$

$$\nabla \times \vec{V} = \vec{k}(\omega + \omega) = 2\omega\vec{k}$$

The curl is twice the angular velocity of a rotating rigid body

② Vortex Field

$$u = \frac{-\Gamma y}{x^2+y^2}$$

$$v = \frac{\Gamma x}{x^2+y^2}$$

$u_r = 0$

$u_\theta = \frac{\Gamma}{r}$

$$\frac{\partial v}{\partial x} = \Gamma \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad \frac{\partial u}{\partial y} = -\Gamma \frac{+x^2-y^2}{(x^2+y^2)^2} \Rightarrow \nabla \times \vec{V} = 0$$

Results:

1. If $\vec{V} = \nabla \phi$ then $\nabla \times \vec{V} = 0$

$$\nabla \times (\nabla \phi) = 0$$

and the reciprocal is true, if $\nabla \times \vec{V} = 0$

then $\exists\, \phi$ such that $\vec{V} = \nabla \phi$ is potential or irrotational

2. If $\nabla \cdot \vec{V} = 0$, the field is said to be solenoidal or divergence-free.

3. if $\vec{V}$: $\nabla \cdot \vec{V} = 0$ and $\nabla \times \vec{V} = 0$, then $\exists\, \phi$

$\vec{V} = \nabla \phi$ $\quad \nabla \cdot \vec{V} = \nabla^2 \phi = 0$ Laplace Equation.
solutions to Laplace equations are called harmonic functions.

Concept of a circulation

Let $\vec{V}$ be a field and C be a simply connected piecewise smooth curve, then

$$\Gamma = \int_C \vec{V} \cdot d\vec{s} \quad \text{is the circulation of } \vec{V} \text{ along } C$$

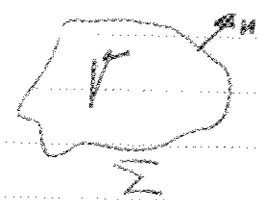
$d\vec{s} = ds\,\vec{t}$ where $\vec{t}$ is the unit tangent to C, and ds is the elemental lengths of the arc along C. The line integral is calculated by moving along C in a given direction. If C is a closed curve, the positive direction is determined by the right-hand screw rule.

General Theorems:

1. Divergence Theorem.

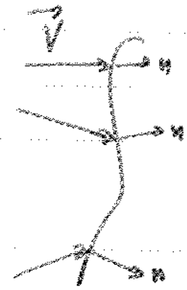
Consider a volume V surrounded by a surface Σ. Let $\vec{n}$ be the unit outward normal to Σ, then

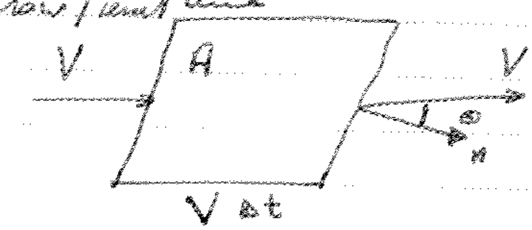
$$\int_V \nabla\cdot\vec{V}\, dV = \int_\Sigma \vec{V}\cdot\vec{n}\, d\Sigma$$



<u>Concept of a flux</u> = rate of flow / unit time

$A(\vec{V}\cdot\vec{n})$



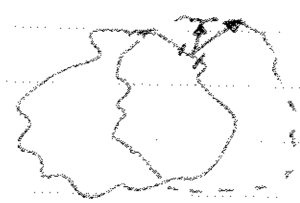


<u>Generalize to</u>

$$\int_\Sigma \vec{V}\cdot\vec{n}\, d\Sigma$$

<u>Physical interpretation of the divergence theorem</u>

$\int_\Sigma \vec{V}\cdot\vec{n}\, d\Sigma$ = rate of expansion of V



$(\nabla\cdot\vec{V})dv$ = rate of expansion of V

For a gas, $\nabla\cdot\rho\vec{V}$ represents the mass flux per unit volume. This is equal to the change in time of the density. Hence,

$\vec{V}\cdot\vec{n}\, d\Sigma$

$$-\nabla\cdot\rho\vec{V} = +\frac{\partial\rho}{\partial t} + Q \text{ or}$$

$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = Q$ expresses the conservation of mass where Q is a source distribution.

2 Green's Theorem

$$\vec{V} = \varphi_1 \nabla \varphi_2$$

$$\int_V \nabla \cdot \vec{V} \, dv = \int_\Sigma \vec{n} \cdot \vec{V} \, d\Sigma$$

$$\int_V \nabla \cdot (\varphi_1 \nabla \varphi_2) dv = \int_\Sigma \varphi_1 \vec{n} \nabla \varphi_2 \, d\Sigma$$

$$\nabla \cdot (\varphi_1 \nabla \varphi_2) = \varphi_1 \nabla^2 \varphi_2 + \nabla \varphi_1 \nabla \varphi_2$$

First Green's theorem

$$\int_V \left(\varphi_1 \nabla^2 \varphi_2 + \nabla \varphi_1 \nabla \varphi_2\right) dv = \int_\Sigma \varphi_1 \vec{n} \nabla \varphi_2 \, d\Sigma$$

or Second Green's theorem

$$\int_V \left(\varphi_1 \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1\right) dv = \int_\Sigma \vec{n} \left(\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1\right) d\Sigma$$

Special Cases:

① $\varphi_1 = \varphi_2 = \varphi$

$$\int \left[\varphi \nabla^2 \varphi + (\nabla \varphi)^2\right] dv = \int_\Sigma \varphi \nabla \varphi \cdot \vec{n} \, d\Sigma$$

② $\varphi_1 = \varphi, \quad \varphi_2 = 1$

$$\int_V \nabla^2 \varphi \, dv = \int_\Sigma \frac{\partial \varphi}{\partial n} \, d\Sigma$$

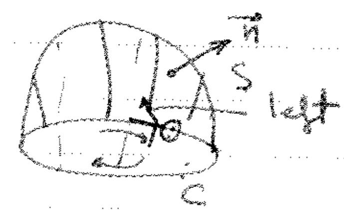
$\varphi_1 = \varphi, \quad \varphi_2 = 1$

$$\int_R \nabla^2 \varphi \, dv = \int_\Sigma \nabla \varphi \cdot \vec{n} \, d\Sigma$$

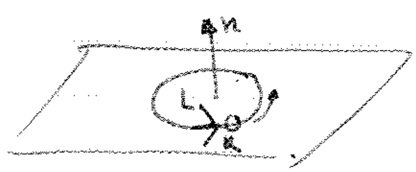
Stokes theorem:

Consider a surface S having a closed curve C as its boundary then

$$\int_S \vec{n} \cdot (\nabla \times \vec{V}) \, d\sigma = \int_C \vec{V} \cdot d\vec{\ell}$$



Ex.



$$\int_R \nabla \cdot (\vec{V} \times \vec{n}) \, dv = \int \vec{n} \cdot (\vec{V} \times \vec{n}) \, d\Sigma$$

$$\vec{V} \times \vec{n} = \vec{u}$$

$\nabla \cdot (\vec{V} \times \vec{n}) = \vec{n} \cdot \nabla \times \vec{V}$