Acoustic Waves in a Circular Duct

Consider a circular duct of radius a. We take a cylindrical coordinate system $\{x, r, \theta\}$, where the x axis is along the duct axis. The acoustic pressure is governed by the wave equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0. \tag{1}$$

The pressure must satisfy an initial condition at $x = x_0$ and a wall boundary condition at $r = a$. We use the method of separation of variables and assume

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t). \tag{2}$$

Substituting (2) into (1) and dividing by $X(x)R(r)\Theta(\theta)T(t)$, gives

$$\frac{X''}{X} + \frac{R'' + R'/r}{R} + \frac{\Theta''}{r^2\Theta} - \frac{1}{c^2}\frac{T''}{T} = 0. \tag{3}$$

If we take

$$\frac{\Theta''}{\Theta} = -m^2, \tag{4}$$

$$\frac{X''}{X} = -k^2, \tag{5}$$

$$\frac{T''}{T} = -\omega^2, \tag{6}$$

where m is an integer. This implies a solution of the form

$$p_{mk\omega} = R_m(r)e^{i(kx+m\theta-\omega t)}. \tag{7}$$

The function R_m satisfies the equation

$$r^2 R_m'' + rR_m' + (\mu^2 r^2 - m^2)R_m = 0, \tag{8}$$

where we have introduced the eigenvalue $\mu^2 = \omega^2/c^2 - k^2$. For a rigid duct, this equation must satisfy an impermeability condition

$$(\frac{dR_m}{dr})_{r=a} = 0. \tag{9}$$

Introducing the non-dimensional variable $\tilde{r} = \mu r$, equation (9)becomes

$$\tilde{r}^2\frac{d^2 R_m}{d\tilde{r}^2} + \tilde{r}\frac{dR_m}{dr} + (\tilde{r}^2 - m^2)R_m = 0. \tag{10}$$

We recognize the Bessel equation and since the pressure is finite along the axis $R_m = J_m(\tilde{r})$. The wall condition (9) implies

$$J_m'(\mu a) = 0. \tag{11}$$

The boundary-value problem (10, 11) is a Sturm-Liouville problem whose solutions form a complete set. The derivative of the Bessel function has an infinite number of zeros which we denote as $\{\alpha_{mn}\}$,

$$J'_m(\alpha_{mn}) = 0, \;\; m = 0, 1, \cdots . \tag{12}$$

Hence, the eigenvalues are

$$\mu_{mn} = \frac{\alpha_{mn}}{a}. \tag{13}$$

This defines the axial wave number as

$$k_{mn} = \sqrt{(\frac{\omega}{c})^2 - \mu_{mn}^2}. \tag{14}$$

The eigenfunction

$$p_{mn} = J_m(\frac{\alpha_{mn} r}{a}) e^{i(k_{mn}x + m\theta - \omega t)} \tag{15}$$

is called the $\{mn\}$ mode. For every frequency ω, the solution is then

$$p_\omega = \sum_{m=-\infty}^{m=+\infty} \sum_{n=0}^{n=+\infty} c_{mn} p_{mn}. \tag{16}$$

The expression for the coefficients c_{mn} is determined using the initial condition

$$p_\omega(0, r, \theta, t) = f_\omega(r, \theta) e^{-i\omega t} \tag{17}$$

and the orthogonality of the Bessel functions,

$$c_{mn} = \frac{1}{\pi a^2} \frac{\alpha_{mn}^2}{(\alpha_{mn}^2 - m^2) J_m^2(\alpha_{mn})} \int_0^{2\pi} \int_0^a f_\omega(r, \theta) J_m(\alpha_{mn} \frac{r}{a}) e^{-im\theta} r dr d\theta, \tag{18}$$

where we have used (see Hildebrand, p. 229)

$$\int_0^a r J_m^2(\alpha_{mn} \frac{r}{a}) dr = \frac{a^2(\alpha_{mn}^2 - m^2)}{2\alpha_{mn}^2} J_m^2(\alpha_{mn}). \tag{19}$$

Note the condition for propagation of an acoustic mode is that the wave number k_{mn} must be real. Otherwise the wave will decay exponentially and is known as an evanescent wave. Therefore an $\{mn\}$ mode propagates if

$$\frac{\omega a}{c} > \alpha_{mn}. \tag{20}$$

At low frequencies, only the fundamental mode

$$p_{00} = e^{i[(\omega/c)x - \omega t]} \tag{21}$$

propagates. As ω increases an additional mode propagates. The frequency at which a mode $\{mn\}$ begins to propagate is known as the cutoff frequency of the mode. As the frequency increases (decreases) and is equal to the cutoff frequency of a mode $\{mn\}$, the mode $\{mn\}$ is said to cut on (cut off).

As an example, consider a duct of radius $a = 0.5m$, $c = 340m/s$, and the sound frequency is $3000rpm$. $a\omega/c = 0.462$. From the tables of zeros of Bessel functions, the lowest zero is $\alpha_{11} = 1.8412$. hence only the fundamental mode will propagate.

Bessel Function Zeros

The first k roots $x_1, \ldots, x_k$ of the Bessel function $J_n(x)$ are given in the following table. They can be found in *Mathematica* using the command `BesselJZeros[n, k]` in the *Mathematica* add-on package ``NumericalMath`BesselZeros` `` (which can be loaded with the command ``<<NumericalMath` ``).

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

The first k roots $x_1, \ldots, x_k$ of the derivative of the Bessel function $J_n'(x)$ can be found in *Mathematica* using the command `BesselJPrimeZeros[n, k]` in the *Mathematica* add-on package ``NumericalMath`BesselZeros` `` (which can be loaded with the command ``<<NumericalMath` ``). The first few such roots are given in the following table.

zero	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

SEE ALSO: Bessel Function, Bessel Function of the First Kind. [Pages Linking Here]

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