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DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

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Mathematical Methods II

Homework 2

I. We have shown in class that for any two differentiable functions φ_1 and φ_1 defined in a region $\mathcal{V}$ bounded by a surface Σ, the Green's theorem states that

$$\int_{\mathcal{V}} (\varphi_1 \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1) dV = \int_{\Sigma} (\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1) \cdot \vec{n}\, d\Sigma. \tag{1}$$

1. Taking $\varphi_2 = \frac{1}{r}$, where $r = |\vec{x} - \vec{y}|$ and using $\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{x} - \vec{y})$, show that

$$\varphi_1(\vec{y}) = -\frac{1}{4\pi} \int_{\mathcal{V}} \frac{\nabla^2 \varphi_1}{r} dV - \frac{1}{4\pi} \int_{\Sigma} (\varphi_1 \frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r} \frac{\partial \varphi_1}{\partial n}) d\Sigma. \tag{2}$$

2. Show that if φ_1 is harmonic, i.e., $\nabla^2 \varphi_1 = 0$, and Σ is a sphere of radius a and centered at $\vec{y}$, equation (2) reduces to

$$\varphi_1(\vec{y}) = \frac{1}{4\pi a^2} \int_{\Sigma} \varphi_1 d\Sigma. \tag{3}$$

Hint:Use the divergence theorem to evaluate $\int_{\Sigma} \frac{\partial \varphi_1}{\partial n} d\Sigma$.

Give the physical interpretation for (3).

3. The temperature field T in a sphere of radius unity and centered at the origin is governed by Laplace equation, i.e., $\nabla^2 T = 0$. The distribution of temperature on the surface of the sphere is given in terms of the polar angle θ and azimuthal angle ϕ,

$$T = cos2\theta sin\phi.$$

Use (3) to obtain the temperature at the center of the sphere.

4. Use the Green's theorem to obtain a particular solution to the inhomogeneous Laplace equation known as Poisson equation

$$\nabla^2 u = f(\vec{x}). \tag{4}$$

5. We define the free-space Green's function of the Helmholtz equation as the solution of

$$\nabla^2 g + k^2 g = -4\pi\delta(\vec{x} - \vec{y}). \tag{5}$$

Following the same method used to derive the Green's function for the Laplace equation, show that

$$g(\vec{x}, \vec{y}) = \frac{e^{\pm ikr}}{r} \tag{6}$$

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and determine a particular solution to

$$\nabla^2 u + k^2 u = f(\vec{x}). \tag{7}$$

II. Problems 103, 104; page 337.

III. Problems 110, 111; page 338.