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Mathematical Methods II

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## Homework 2

I. Consider the spherical coordinates $(r, \phi, \theta)$ defined by

$$x = rsin\phi cos\theta \quad (1)$$
$$y = rsin\phi sin\theta \quad (2)$$
$$z = rcos\phi \quad (3)$$

Sketch the coordinate system and the unit vectors associated with the variables $(r, \phi, \theta)$. Write the expressions for the unit vectors associated with the spherical coordinates in terms of the Cartesian coordinates unit vectors. Give the expressions for the gradient, divergence and curl in spherical coordinates.

II. Consider the vector field defined by its components in Cartesian coordinates$\vec{V} = \{v_1, v_2, v_3\}$:

$$v_1 = \frac{x}{x^2+y^2+z^2}, \quad v_2 = \frac{y}{x^2+y^2+z^2}, \quad v_3 = \frac{z}{x^2+y^2+z^2} \quad (4)$$

1. Write the expressions for the components of $\vec{V}$ in spherical coordinates.
2. Calculate the divergence and curl of $\vec{V}$ using spherical coordinates.

III. A rigid body is rotating about the z-axis with angular velocity $\Omega$. If the origin of the coordinates, $O$, is along the z axis, show that the trajectory of a point M(x, y, z), is given by

$$x = rcos(\Omega t), \quad (5)$$
$$y = rsin(\Omega t), \quad (6)$$
$$z = z_0, \quad (7)$$

where $r = OM$. Write the expression for the velocity of M in Cartesian and cylindrical coordinates. Evaluate $\nabla \cdot \vec{V}$ and $\nabla \times \vec{V}$ using both Cartesian and cylindrical coordinates..

IV. We have shown in class that for any two functions $\varphi_1$ and $\varphi_1$ defined in a region $\mathcal{R}$ bounded by a surface $S$, the Green's theorem states that

$$\int_{\mathcal{R}} (\varphi_1 \nabla^2 \varphi_2 - \varphi_2 \nabla^2 \varphi_1) dV = \int_S (\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1) \vec{n} d\sigma. \tag{8}$$

We have also shown that by taking
$\varphi_2 = \frac{1}{r}$, where $r = |\vec{x} - \vec{y}|$

$$\varphi_1(\vec{y}) = -\frac{1}{4\pi} \int_{\mathcal{R}} \frac{\nabla^2 \varphi_1}{r} dV - \frac{1}{4\pi} \int_S (\varphi_1 \frac{\partial}{\partial n} (\frac{1}{r}) - \frac{1}{r} \frac{\partial \varphi_1}{\partial n}) d\sigma. \tag{9}$$

1. Show that if $\varphi_1$ is harmonic and $S$ is a sphere of radius R, equation (9) reduces to

   $$\varphi_1(\vec{y}) = \frac{1}{4\pi R^2} \int_S \varphi_1 d\sigma. \tag{10}$$

   *Hint: Use the divergence theorem to evaluate* $\int_S \frac{\partial \varphi_1}{\partial n} d\sigma$.

   Give the physical interpretation for (10).

2. The temperature field T in a sphere of radius unity is governed by Laplace equation, i.e., $\nabla^2 T = 0$. The distribution of temperature on the surface of the sphere is given in terms of the polar angle $\theta$ and azimuthal angle $\phi$,

   $$T = cos 2\theta sin\phi.$$

   Use (10) to obtain the temperature at the center of the sphere.

V. Let $r = |\vec{x} - \vec{y}|$, where $\vec{x} = \{x_1, x_2, x_3\}$ is the position vector and $\vec{y} = \{y_1, y_2, y_3\}$ is a constant vector. The gradient of a scalar function $u(x_1, x_2, x_3)$ is defined in Cartesian coordinates as $\nabla u = \{\partial u/\partial x_1, \partial u/\partial x_2, \partial u/\partial x_3\}$.

1. Show that $\nabla^2(\frac{1}{r}) = 0$, except at $r = 0$.
2. Use the divergence theorem to show that

   $$\int_\Sigma \nabla^2 (\frac{1}{r}) d\vec{x} = -4\pi \tag{11}$$

   where $\Sigma$ is the sphere centered at the point $\vec{y}$ and of radius R. The function $G(r) = 1/r$ is known as the free–space Green function for Laplace equation in a three-dimensional space.

3. $\delta(\vec{x}-\vec{y}) = -\frac{1}{4\pi}\nabla^2(\frac{1}{r})$ is a "function" which vanishes everywhere except at $\vec{x}=\vec{y}$, and is such that its integral in any sphere containing $\vec{y}$ is finite and equal to unity. Sketch $\delta$ and show that

$$\int_\Sigma \delta(\vec{x}-\vec{y})f(\vec{x})d\vec{x} = f(\vec{y}). \tag{12}$$

4. Use these results with Green's theorem to obtain a particular solution to the inhomogeneous Laplace equation known as Poisson equation

$$\nabla^2 u = f(\vec{x}). \tag{13}$$

5. Extend the above results to find the Green function of the Helmholtz equation,

$$\nabla^2 u + k^2 u = 0. \tag{14}$$

and determine a particular solution to

$$\nabla^2 u + k^2 u = f(\vec{x}). \tag{15}$$

VI. Problems 56, 57, 61; page 329.

VII. Problems 96, 98; pages 335 and 336.

VIII. Problems 103, 104; page 337.

IX. Problems 110, 111; page 338.