FINITE WING THEORY

At every positition y_0 along the span :

$$\alpha(y_0) - \alpha_{L=0}(y_0) - \alpha_i(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)}, \tag{1}$$

where the induced angle of attack α_i is defined by

$$\alpha_i = \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy \tag{2}$$

Let

$$y_0 = -\frac{b}{2} cos\theta_0, \qquad \overline{\alpha}(\theta_0) = \alpha(\theta_0) - \alpha_{L=0}(\theta_0)$$

Then

$$\overline{\alpha}(\theta_0) = \alpha(y_0) - \alpha_{L=0}(y_0) = \frac{\Gamma(\theta_0)}{\pi V_\infty c(\theta_0)} + \frac{1}{2\pi V_\infty b} \int_0^\pi \frac{\frac{d\Gamma}{d\theta}}{cos\theta - cos\theta_0} d\theta \tag{3}$$

We assume the following expansion for Γ :

$$\Gamma(\theta) = 2bV_\infty \sum_1^N A_n sinn\theta \tag{4}$$

$A_1, A_2, \ldots, A_N$ are constantants to be determined.

We note that

$$\int_0^\pi \frac{cosn\theta}{cos\theta - cos\theta_0} d\theta = \pi \frac{sinn\theta_0}{sin\theta_0} \tag{5}$$

Substituting (4) into (2 and 3) and using (5), we obtain the following expression for $\alpha_i(\theta_0)$

$$\alpha_i(\theta_0) = \sum_1^N nA_n \frac{sinn\theta_0}{sin\theta_0}, \tag{6}$$

and the fundamental equation (3) for the finite wing becomes

$$\overline{\alpha}(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_1^N A_n sinn\theta_0 + \sum_1^N nA_n \frac{sinn\theta_0}{sin\theta_0} \tag{7}$$

Equation (5) must be satisfied at N locations of the span. This gives N equations for determining $A_1, A_2, \ldots, A_N$. All aerodynamic quantities can now be calculated :

$$C_L = \pi \mathcal{AR} A_1$$

$$C_{D,i} = \pi \mathcal{AR} A_1^2 [1 + \sum_2^N n (\frac{A_n}{A_1})^2]$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathcal{AR}} (1 + \delta)$$

For a wing with no geometric twist

$$C_L = a(\alpha - \alpha_{L=0})$$

$$a = \frac{a_0}{1 + (\frac{a_0}{\pi \mathcal{AR}})(1 + \tau)}$$

For a thin airfoil, $a_0 = 2\pi$.

ELLIPTIC WING

For a wing of uniform cross-section and no geometric twist, $\overline{\alpha}(\theta)$ is constant. We further assume the wing to have an elliptic planform, i.e.,

$$c = c_0 \sqrt{1 - (\frac{2y}{b})^2} \qquad or \quad c(\theta) = c_0 sin\theta$$

Substituting (11) into (5), we find the following solution

$$A_1 = \frac{\overline{\alpha}}{1 + \frac{2b}{\pi c_0}} = \frac{\overline{\alpha}}{1 + \frac{\mathcal{AR}}{2}}$$
$$A_2 = A_3, \ldots, = A_N = 0.$$

All aerodynamic quantities can now be calculated :

$$\Gamma(\theta) = 2bV_\infty \frac{\overline{\alpha}}{1 + \frac{\mathcal{AR}}{2}} sin\theta$$

$$\alpha_i = A_1 = \frac{\overline{\alpha}}{1 + \frac{\mathcal{AR}}{2}}$$

$$C_L = \pi \mathcal{AR} \alpha_i = \frac{2\pi\overline{\alpha}}{1 + \frac{2}{\mathcal{AR}}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathcal{AR}}$$

$$a = \frac{\pi \mathcal{AR}}{1 + \frac{\mathcal{AR}}{2}} = \frac{a_0}{1 + \frac{a_0}{\pi \mathcal{AR}}}$$