

Permanence of Vorticity

The acceleration \mathbf{a} of a fluid particle is given in terms of the material derivative

$$\mathbf{a} = \frac{D}{Dt} \mathbf{V}. \quad (1)$$

Note that

$$\frac{D}{Dt} \mathbf{V} \equiv \frac{\partial \mathbf{V}}{\partial t} + \zeta \times \mathbf{V} + \frac{1}{2} \nabla \mathbf{V}^2. \quad (2)$$

Taking the curl of both sides of (1), noting that

$$\nabla \times (\zeta \times \mathbf{V}) \equiv (\mathbf{V} \cdot \nabla) \zeta - (\zeta \cdot \nabla) \mathbf{V} + \zeta (\nabla \cdot \mathbf{V}) - \mathbf{V} (\nabla \cdot \zeta), \quad \nabla \cdot \zeta = 0, \quad \text{and} \quad \nabla \cdot \mathbf{V} = -(D\rho)/(\rho Dt),$$

we get

$$\frac{D}{Dt} \frac{\zeta}{\rho} = \frac{\zeta}{\rho} \cdot \nabla \mathbf{V} + \frac{1}{\rho} \nabla \times \mathbf{a}. \quad (3)$$

For conservative forces and barotropic fluids, $\nabla \times \mathbf{a} = 0$, and we get

$$\frac{D}{Dt} \frac{\zeta}{\rho} = \left(\frac{\zeta}{\rho} \cdot \nabla \right) \mathbf{V} \quad (4)$$

To integrate this equation, we first write (4) in tensor notation

$$\frac{D}{Dt} \frac{\zeta_i}{\rho} = \frac{\zeta_j}{\rho} \cdot \frac{\partial V_i}{\partial x_j} \quad (5)$$

We introduce the Lagrangian coordinates $\mathbf{x}^{(0)} = \{x_i^{(0)}\}$ and note that $\partial/\partial x_j = (\partial x_k^{(0)}/\partial x_j) \partial/\partial x_k^{(0)}$. Equation (5) can then be written as

$$\frac{D}{Dt} \frac{\zeta_i}{\rho} = \frac{\zeta_j}{\rho} \frac{\partial x_k^{(0)}}{\partial x_j} \frac{\partial V_i}{\partial x_k^{(0)}}. \quad (6)$$

We also note that

$$\frac{\partial x_k^{(0)}}{\partial x_j} \frac{\partial x_i}{\partial x_k^{(0)}} = \delta_{ij}, \quad (7)$$

and therefore,

$$\frac{D}{Dt} \left\{ \frac{\partial x_k^{(0)}}{\partial x_j} \frac{\partial x_i}{\partial x_k^{(0)}} \right\} = 0. \quad (8)$$

Equation (8) can be written as

$$\frac{\partial x_i}{\partial x_k^{(0)}} \frac{D}{Dt} \left(\frac{\partial x_k^{(0)}}{\partial x_j} \right) + \frac{\partial x_k^{(0)}}{\partial x_j} \frac{\partial V_i}{\partial x_k^{(0)}} = 0, \quad (9)$$

since $V_i = (Dx_i)/(Dt)$.

Substituting (9) into (6), we get

$$\frac{D}{Dt} \frac{\zeta_i}{\rho} + \frac{\zeta_j}{\rho} \frac{D}{Dt} \left(\frac{\partial x_k^{(0)}}{\partial x_j} \right) \frac{\partial x_i}{\partial x_k^{(0)}} = 0. \quad (10)$$

Multiplying by $\partial x_\ell^{(0)}/\partial x_i$ and using (7), we get

$$\frac{\partial x_\ell^{(0)}}{\partial x_i} \frac{D}{Dt} \frac{\zeta_i}{\rho} + \frac{\zeta_j}{\rho} \frac{D}{Dt} \left(\frac{\partial x_k^{(0)}}{\partial x_j} \right) \delta_{k\ell} = 0. \quad (11)$$

Or

$$\frac{D}{Dt} \left(\frac{\zeta_i}{\rho} \frac{\partial x_\ell^{(0)}}{\partial x_i} \right) = 0. \quad (12)$$

The Lagrangian coordinates were defined at some location upstream $\mathbf{x}^{(0)}$. At this location the tensor $\partial x_\ell^{(0)}/\partial x_i = \delta_{i\ell}$, hence

$$\frac{\zeta_i}{\rho} \frac{\partial x_\ell^{(0)}}{\partial x_i} = \left(\frac{\zeta_\ell}{\rho} \right)_0. \quad (13)$$

Multiplying by $\partial x_j/\partial x_\ell^{(0)}$ and using (7), we get

$$\frac{\zeta_i}{\rho} \delta_{ij} = \left(\frac{\zeta_\ell}{\rho} \right)_0 \frac{\partial x_j}{\partial x_\ell^{(0)}}. \quad (14)$$

Or

$$\frac{\zeta_j}{\rho} = \left(\frac{\zeta_\ell}{\rho} \right)_0 \frac{\partial x_j}{\partial x_\ell^{(0)}}. \quad (15)$$

Equation (15) gives the vorticity at \mathbf{x} in terms of the vorticity at the previous location $\mathbf{x}^{(0)}$. Therefore, if the vorticity associated with a fluid particle is zero at some initial location \mathbf{x}_0 , it will be zero at all subsequent locations. On the other hand, if the vorticity is not zero at the initial location \mathbf{x}_0 , it will never be zero. This result is known as the *permanence of vorticity*.