1 Fundamental Solutions to the Wave Equation

Physical insight in the sound generation mechanism can be gained by considering simple analytical solutions to the wave equation. One example is to consider acoustic radiation with spherical symmetry about a point $\vec{y} = \{y_i\}$, which without loss of generality can be taken as the origin of coordinates. If t stands for time and $\vec{x} = \{x_i\}$ represent the observation point, such solutions of the wave equation,

$$(\frac{\partial^2}{\partial t^2} - c_o^2 \nabla^2)\phi = 0, \tag{1}$$

will depend only on the $r = |\vec{x} - \vec{y}|$. It is readily shown that in this case (1) can be cast in the form of a one-dimensional wave equation

$$(\frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial r^2})(r\phi) = 0. \tag{2}$$

The general solution to (2) can be written as

$$\phi = \frac{f(t - \frac{r}{c_o})}{r} + \frac{g(t + \frac{r}{c_o})}{r}. \tag{3}$$

The functions f and g are arbitrary functions of the single variables $\tau_\pm = t \pm \frac{r}{c_o}$, respectively. They determine the pattern or the phase variation of the wave, while the factor $1/r$ affects only the wave magnitude and represents the spreading of the wave energy over larger surface as it propagates away from the source. The function $f(t - \frac{r}{c_o})$ represents an outwardly going wave propagating with the speed c_o. The function $g(t + \frac{r}{c_o})$ represents an inwardly propagating wave propagating with the speed c_o.

1.1 Acoustic Energy

Consider a fluid blob at rest of volume V_0 with pressure p_0 and density ρ_0. As a sound wave reaches the blob of fluid, it acquires a velocity v and its volume, pressure and density change to V, p and ρ, respectively. If we write

$$\begin{aligned} V &= V_0 + V' \qquad (4)\\ p &= p_o + p' \qquad (5)\\ \rho &= \rho_o + \rho', \qquad (6) \end{aligned}$$

the incremental changes V'/V_0, p'/p_0 and ρ'/ρ are very small. The blob has the same mass, so $\rho_o V_o = \rho V = m$.

The kinetic energy of the blob is given by

$$\frac{1}{2}\rho v^2 V = \frac{1}{2}\rho_o v^2 V_o$$

and the acoustic kinetic energy density is

$$\frac{1}{2}\rho v^2.$$

The blob is compressed against the background pressure p_0. The compression energy is given by

$$\Delta W = -\int_{V_o}^{V} p' dV \approx -\frac{V_0}{\rho_0}\int_{V_o}^{V} p' d\rho' \qquad (7)$$

If the process is assumed to be isentropic, then $dp'/d\rho' = c_0^2$. Using this relationship and substituting for $d\rho'$ in (7) gives

$$\Delta W = \frac{V_0}{\rho_0 c_0^2}\int_{V_o}^{V} p' dp' = \frac{p'^2}{2\rho_0 c_0^2} V_0 \qquad (8)$$

or, expressed in terms of the acoustic potential energy density,

$$\frac{p'^2}{2\rho_o {c_o}^2}$$

.
The total energy density is

$$E = \frac{1}{2}\rho_o v^2 + \frac{p'^2}{2\rho_o {c_o}^2}. \qquad (9)$$

The acoustic energy flux or *acoustic intensity* is

$$\vec{I} = p'\vec{v}. \qquad (10)$$

Note that the conservation of energy implies

$$\int_V \frac{\partial E}{\partial t} dV + \int_S \vec{I}\cdot\vec{n}\, dS = 0,$$

which using the divergence theorem gives

$$\frac{\partial E}{\partial t} + \nabla\cdot\vec{I} = 0. \qquad (11)$$

Equation (11) can be verified by substituting the expressions for $\vec{I}$ and E into the mass and momentum equations, namely

$$\frac{\partial \rho'}{\partial t} + \rho_o \nabla\cdot\vec{v} = 0$$
$$\rho_o \frac{\partial \vec{v}}{\partial t} = -\nabla p'$$

2 Plane Waves

Plane waves are solutions to the wave equation (1) of the form

$$p' = \bar{p}e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{12}$$

where the wave number $k = |\vec{k}| = \omega/c_0$. Note that if we define the unit vector $\vec{u} = \vec{k}/k$, the wave propagates in the direction $\vec{u}$ and

$$\vec{v} = \frac{p'}{\rho_0 c_0}\vec{u}. \tag{13}$$

Thus for plane waves the relationship between pressure and velocity is given by

$$p' = \rho_0 c_0 v. \tag{14}$$

Note also that the energy of a plane wave is equally divided between kinetic and potential energy and we have

$$E = \rho_0 v^2 = \frac{p'^2}{\rho_0 c_0^2} \tag{15}$$

3 The Pulsating Sphere

Consider a sphere centered at the origin and having a small pulsating motion so that the equation of its surface is

$$r = a(t) = a_0 + a_1(t), \tag{16}$$

where $|a_1(t)| << a_0$. The fluid velocity at the sphere surface is

$$u_r = \frac{dr}{dt} = \dot{a}(t). \tag{17}$$

At the surface of the sphere

$$(\frac{\partial \phi}{\partial r})_a = \dot{a}(t). \tag{18}$$

A Taylor expansion of (18) gives

$$(\frac{\partial \phi}{\partial r})_a = (\frac{\partial \phi}{\partial r})_{a_0} + (a - a_0)(\frac{\partial^2 \phi}{\partial r^2})_{a_0} + \cdots \tag{19}$$

We assume $|(a-a_0)(\frac{\partial^2\phi}{\partial r^2})_{a_0}| << |\dot{a}|$. This allows us to linearize the boundary condition along the sphere by transferring it to the mean position at a_0,

$$(\frac{\partial \phi}{\partial r})_{a_0} = \dot{a}(t). \tag{20}$$

The velocity potential can be expressed as in (3). Moreover since the sphere pulsating motion is the source of acoustic waves, the principle of causality suggests that $g \equiv 0$. Thus

$$\phi = \frac{f(t - \frac{r}{c_o})}{r}. \tag{21}$$

Applying the condition (20) at the sphere mean location,

$$\frac{\partial \phi}{\partial r} = -\frac{f(t-\frac{a_0}{c_o})}{a_0^2} - \frac{\dot{f}(t-\frac{a_0}{c_o})}{a_0 c_o} = \dot{a}(t) \tag{22}$$

Integration of (22) gives

$$f(t) = -a_0 c_o \int_{-\infty}^{t} \dot{a}(t' + \frac{a_0}{c_0}) e^{-\frac{c_o}{a_0}(t-t')} dt'. \tag{23}$$

Note that if T is a representative period of the sphere pulsation, $c_o T/a_0 = \lambda/a_0$, where λ is a representative of the sound wave length. If $\lambda/a_0 >> 1$, then most of the contribution to the integral (23) is when $t' \approx t$. Neglecting terms of $O(a_0/\lambda)$, we get

$$f(t) = -a_0^2 \dot{a}(t), \tag{24}$$

and the acoustic field potential function is given by

$$\phi = -\frac{a_0^2 \dot{a}(t-\frac{r}{c_o})}{r}. \tag{25}$$

The expression for the acoustic pressure is

$$p' = \rho_0 a_0^2 \frac{\ddot{a}(t-\frac{r}{c_o})}{r} \tag{26}$$

It is convenient to cast (25, 26) in terms of the mass flow rate crossing the sphere of radius a_0, $m(t) = 4\pi a_0^2 \dot{a} \rho_0$. $f(t) = -\frac{m}{4\pi\rho_0}$ and

$$\phi = -\frac{m(t-\frac{r}{c_o})}{4\pi\rho_0 r}, \tag{27}$$

$$p' = \frac{\dot{m}(t-\frac{r}{c_o})}{4\pi r}. \tag{28}$$

3.1 Harmonic Motion

If we have a harmonic motion

$$\dot{a} = \bar{v} e^{i\omega t}, \tag{29}$$

where $\bar{v}$ is the amplitude of the pulsation velocity and ω its frequency. Substituting (29) into (23) and carrying out the integration, we get

$$f(t) = -a_0 c_0 \bar{v} \frac{e^{i\omega(t+\frac{a_0}{c_0})}}{\frac{c_0}{a_0} + i\omega}. \tag{30}$$

The expressions for the potential function and the pressure can be readily obtained by substituting (30) into (21),

$$\phi = -\frac{\bar{m}}{4\pi\rho_0 r\sqrt{1+\tilde{\omega}^2}}\; e^{i(\omega t - k(r-a_0)+\varphi)}, \tag{31}$$

$$p' = \frac{i\omega\bar{m}}{4\pi r\sqrt{1+\tilde{\omega}^2}}\; e^{i(\omega t - k(r-a_0)+\varphi}. \tag{32}$$

where we have introduced $\tilde{\omega} = \omega a_0/c_0$, $\varphi = -tan^{-1}\tilde{\omega}$, $k = \omega/c_0$, and $\bar{m} = 4\pi a_0^2 \bar{v}\rho_0$.

The average acoustic intensity and power can be calculated and we have,

$$\bar{I} = \frac{1}{8\pi}\bar{m}c_0\bar{v}\frac{\tilde{\omega}^2}{1+\tilde{\omega}^2}\frac{1}{r}, \tag{33}$$

$$\mathcal{P} = \frac{1}{2}\bar{m}c_0\bar{v}\frac{\tilde{\omega}^2}{1+\tilde{\omega}^2}. \tag{34}$$

4 The Simple Source

The limit of the pulsating sphere solution as the sphere radius vanishes represents the simple source or *monopole* solution. In this case, the source is characterized by the source mass flow rate

$$m(t) = \lim_{Ra_0 \to 0} 4\pi a_0^2 u_r = 4\pi a_0^2 \dot{a}(t),$$

and the exact solution is the same as for the low frequency case (27). If the source is located at the point $|\vec{y}|$, then

$$\phi = -\frac{m(t-\frac{r}{c_o})}{4\pi\rho_0 r}, \tag{35}$$

where $r = |\vec{x} - \vec{y}|$. Equation (35) states that at the observation point $\vec{x}$ and time t the sound signal received was emitted from the source point $\vec{y}$ at the *retarded time* $\tau = t - \frac{r}{c_o}$.

The velocity and pressure are given by

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{4\pi\rho_0}\left[\frac{\dot{m}(t-\frac{r}{c_0})}{rc_0} + \frac{m(t-\frac{r}{c_0})}{r^2}\right] \tag{36}$$

$$p' = -\rho_0\frac{\partial\phi}{\partial t} = \frac{\dot{m}(t-\frac{r}{c_0})}{4\pi r} \tag{37}$$

Harmonic sources:

In this case

$$m = \bar{m}e^{i\omega t} \tag{38}$$

$$\phi = -\frac{\bar{m}}{4\pi\rho_0 r}e^{i\omega(t-\frac{r}{c_0})} \tag{39}$$

$$u_r = \frac{\bar{m}}{4\pi\rho_0}\left[\frac{i\omega}{rc_0} + \frac{1}{r^2}\right] e^{i\omega(t-\frac{r}{c_0})} \tag{40}$$

Noting that $\frac{\omega}{c_0} = \frac{2\pi}{\lambda}$,

$$u_r = \frac{\bar{m}}{4\pi\rho_0}\left[\frac{i2\pi}{r\lambda} + \frac{1}{r^2}\right] e^{i\omega(t-\frac{r}{c_0})} \tag{41}$$

$$p' = \frac{i\omega\bar{m}}{4\pi r} e^{i(t-\frac{r}{c_0})} \tag{42}$$

At large distance, $r \gg \lambda$, the acoustic intensity is given by

$$I = p'u_r = \frac{\bar{m}^2\omega^2}{16\pi^2 r^2 c_0\rho_o}\sin^2(\omega t - \frac{r}{c_0}) \tag{43}$$

and

$$\bar{I} = p'u_r = \frac{\bar{m}^2\omega^2}{32\pi^2 r^2 c_0\rho_o} \tag{44}$$

$$\bar{\mathcal{P}} = \frac{\bar{m}^2\omega^2}{8\pi c_0\rho_o} \tag{45}$$

Simple source distribution:

Suppose we have N sources located at $\vec{y_i}$ with strength m_i, then the principle of superposition states that:

$$\phi = \sum_{i=1}^{N}\phi_i = -\frac{1}{4\pi}\sum_{i=1}^{N}\frac{m_i(t-\frac{r_i}{c_0})}{r_i} \tag{46}$$

where $r_i = |\vec{x} - \vec{y_i}|$.

The Dipole:

Consider two sources of equal and opposite strength $\pm m_i$ located at $\pm\vec{l}$.

$$\phi_\pm = \pm m(t - \frac{r_\pm}{c_0})/(4\pi\rho_o r_\pm) \tag{47}$$

We further assume $|\vec{l}| \ll |\vec{x}|$, then

$$r_\pm = r_\mp\frac{\vec{l}\vec{x}}{r} + \ldots = r_\mp l\cos\theta + \ldots \tag{48}$$

$$\phi = \phi_+ + \phi_- = \frac{-1}{4\pi\rho_0}\left[\frac{m(t-\frac{r_+}{c_0})}{r_+} - \frac{m(t-\frac{r_-}{c_0})}{r_-}\right] \tag{49}$$

$$\frac{m(t-\frac{r_\pm}{c_0})}{r_\pm} = \frac{m(t-\frac{r}{c_0})}{r} \pm l\cos\theta\left[\frac{\dot{m}(t-\frac{r}{c_0})}{rc_0} + \frac{m(t-\frac{r}{c_0})}{r^2}\right] \tag{50}$$

$$\phi = \frac{2l\cos\theta}{4\pi\rho_0}\left[\frac{\dot{m}(t-\frac{r}{c_0})}{rc_0} + \frac{m(t-\frac{r}{c_0})}{r^2}\right] \tag{51}$$

If we assume l to be small and consider the field at a distance $r >> \lambda$, then

$$\phi = \frac{2l\dot{m}\left(t-\frac{r}{c_0}\right)}{4\pi\rho_0 rc_0}\cos\theta \tag{52}$$

let $\mu = -2l\dot{m}$,

$$p' = \frac{1}{4\pi rc_0}\dot{\mu}\left(t-\frac{r}{c_0}\right)\cos\theta \tag{53}$$

μ is called the strength of the dipole. The dipole strength has the dimension of a force. We define the pressure directivity by:

$$(p'r) = \frac{1}{4\pi c_0}\dot{\mu}\left(t-\frac{r}{c_0}\right)\cos\theta \tag{54}$$