Fundamental Equations

1. Continuity : Conservation of Mass

Consider a surface Σ enclosing a volume V of fluid. Let $\vec{n}$ be the outward unit normal to Σ, and m be the distribution of sources in V.

The conservation of mass:

$$\frac{\partial}{\partial t}\int_V \rho \, dv = -\int_\Sigma (\rho \vec{V}\cdot\vec{n})\, d\Sigma + \int_V m \, dv$$

Using the divergence theorem

$$\int_\Sigma (\rho\vec{V}\cdot\vec{n})\, d\Sigma = \int_V \nabla\cdot(\rho\vec{V})\, dv$$

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) - m\right] dv = 0$$

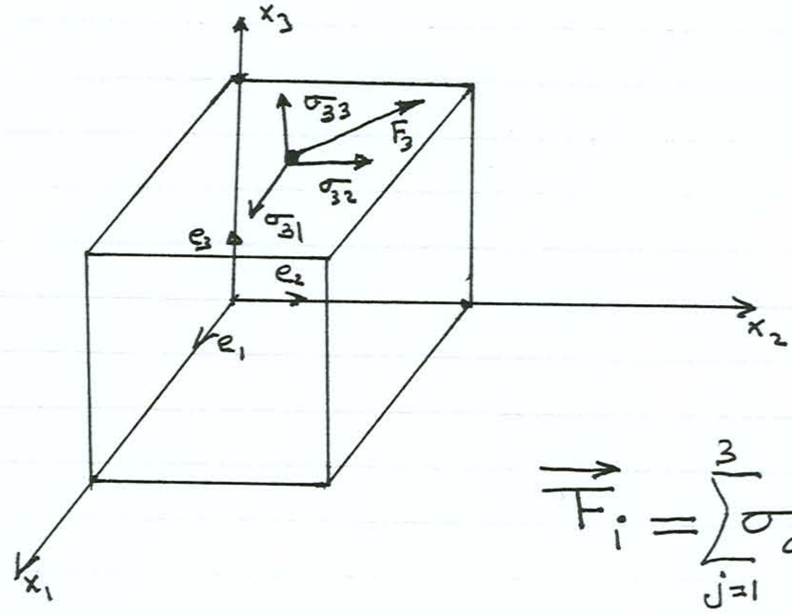
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = m$$

$$\frac{D}{Dt}\rho + \rho\,\nabla\cdot\vec{V} = m$$

Incompressible, no source,

$$\nabla\cdot\vec{V} = 0$$

2. The stress tensor



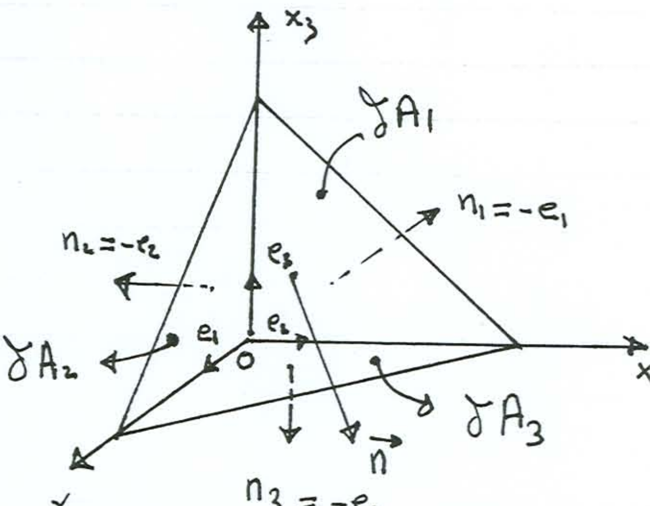
$$\vec{F}_i = \sum_{j=1}^{3} \sigma_{ij}\, \vec{e}_j$$

σ_{ij} = force per unit area in the direction j acting on a surface whose outward normal is in the direction i.

$$\delta A\, \vec{F}_n + \sum_{i=1}^{3} \delta A_i \vec{F}_i = 0$$

$$\vec{F}_n = - \sum_{i=1}^{3} \frac{\delta A_i}{\delta A} \vec{F}_i$$

$$\vec{F}_n = \sum_{i=1}^{3} (\vec{n}\,\vec{e}_i) \sum_{j=1}^{3} \sigma_{ij}\, \vec{e}_j$$



$$\vec{F}_n = \sum_{i=1}^{3} \sum_{j=1}^{3} (\vec{n}\,\vec{e}_i)\, \sigma_{ij}\, \vec{e}_j$$

$$\vec{F}_n = n_i\, \sigma_{ij}\, \vec{e}_j = (\vec{e}_1\, \vec{e}_2\, \vec{e}_3)\, \sigma_{ij}^{T} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

3. The Momentum Equation

Consider a surface Σ enclosing a volume V of fluid and moving with it. Let $\vec{n}$ be the unit outward normal to Σ and $\vec{f}$ a distribution of forces per unit volume. The Newton Second Law of motion states

$$\frac{D}{Dt}\int_V \rho\vec{V}\,dv = \int_V \rho\vec{f}\,dv + \int_\Sigma \sum_{i=1}^{3}\sum_{j=1}^{3} n_i\sigma_{ij}e_j\,d\Sigma$$

or since $\frac{D}{Dt}\rho dv = 0$

$$\int_V \rho\frac{D}{Dt}v_i\,dv = \int_V \rho f_i\,dv + \int_\Sigma \sum_{i=1}^{3} n_j\sigma_{ji}\,d\Sigma$$

$$\int_\Sigma n_j\sigma_{ji}\,d\Sigma = \int_V \frac{\partial\sigma_{ji}}{\partial x_j}\,dv$$

$$\int_V \left[\rho\frac{D}{Dt}v_i - \rho f_i - \frac{\partial\sigma_{ji}}{\partial x_j}\right]dv = 0$$

$$\rho\frac{D}{Dt}v_i = \rho f_i + \frac{\partial\sigma_{ji}}{\partial x_j}$$

For a Newtonian fluid, the constitutive relations are

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\tau_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}e_{kk}\,\delta_{ij}\right)$$

μ = coefficient of Viscosity

The strain tensor $\quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

$$\frac{D}{Dt}v_i = f_i - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}$$

B C

Euler's Equation: $\mu = 0$

$$\frac{D}{Dt}v_i = f_i - \frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

$$\frac{D}{Dt}\vec{V} = \vec{f} - \frac{1}{\rho}\nabla P$$

- D'Alembert Paradox.
- Prandtl's concept of the boundary layer
- Bluff and streamlined bodies

Inviscid Flows

$$\sigma_{ij} = -p\,\delta_{ij} \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq 0 \end{cases}$$

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{V} = 0$$

$$\frac{D}{Dt}\vec{V} = \vec{f} - \frac{1}{\rho}\nabla p \qquad \text{Euler Equations}$$

$$\frac{D}{Dt} \equiv \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

$$\equiv \frac{\partial\vec{V}}{\partial t} + \frac{1}{2}\nabla(\vec{V}^2) + (\nabla\times\vec{V})\times\vec{V} \qquad \vec{\xi} = \nabla\times\vec{V}$$

$$\nabla\times\vec{V} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} e_x & e_r & e_\theta \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ V_x & V_r & (rV_\theta) \end{vmatrix}$$

Assumptions:

① Conservative force: $\vec{f} = -\nabla\Omega$

$\Omega = gz$

② Barotropic fluid: $p = f(\rho)$ or $\rho = g(p)$

$$\frac{\nabla p}{\rho} = \nabla\int^{p}\frac{dp}{\rho}$$

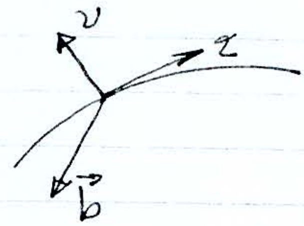
$$\frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{1}{2}\vec{V}^2 + \int\frac{dp}{\rho} + \Omega\right) = \vec{V}\times\vec{\xi}$$

1. Steady Flow

$$\nabla\left(\frac{1}{2}\vec{V}^2 + \int \frac{dP}{\rho} + \Omega\right) = \vec{V} \times \vec{\xi}$$

$$\vec{\tau} = \frac{\vec{V}}{|\vec{V}|} \qquad \frac{d}{ds}\left(\frac{1}{2}\vec{V}^2 + \int \frac{dP}{\rho} + \Omega\right) = 0$$

$$\vec{V} \times \vec{\xi} = \begin{vmatrix} \vec{\tau} & \vec{\nu} & \vec{b} \\ V & 0 & 0 \\ \xi_\tau & \xi_\nu & \xi_b \end{vmatrix}$$



$$= -V\xi_b\,\vec{\nu} + V\xi_\nu\,\vec{b}$$

$\frac{1}{2}\vec{V}^2 + \int \frac{dP}{\rho} + \Omega$ = *constant along a streamline.*

2. Irrotational Flow

$$\vec{\xi} = 0, \ \exists\ \phi, \quad \vec{V} = \nabla\phi$$

$$\frac{\partial \phi}{\partial t} + \left(\frac{1}{2}\vec{V}^2 + \int \frac{dP}{\rho} + \Omega\right) = C(t)$$

everywhere in the flowfield.

- Steady flow $\quad \frac{1}{2}V^2 + \int \frac{dP}{\rho} + \Omega = C$
  - Incompressible

$$\frac{1}{2}V^2 + \frac{P}{\rho} + \Omega = C$$

For a steady flow

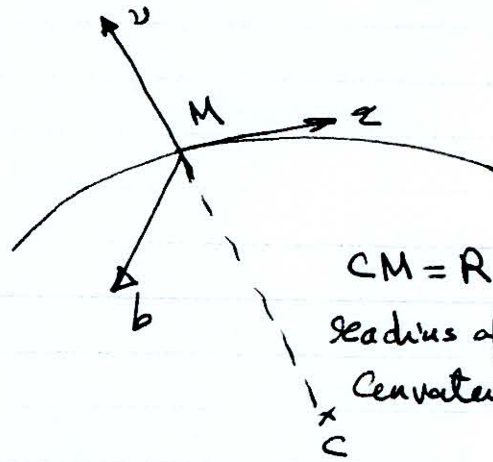
$$\nabla\left(\tfrac{1}{2}V^2 + \int \frac{dP}{\rho} + \Omega\right) = \vec{V} \times \vec{\zeta}$$

$$\frac{d}{d\nu}\left(\tfrac{1}{2}V^2 + \int \frac{dP}{\rho} + \Omega\right) = -V\zeta_b$$

$$\frac{d}{db}\left(\tfrac{1}{2}V^2 + \int \frac{dP}{\rho} + \Omega\right) = V\zeta_n$$

Intrinsic Coordinates:
2D - Flow

$\nabla \times \vec{V} \perp \vec{V}$ and to the plane.



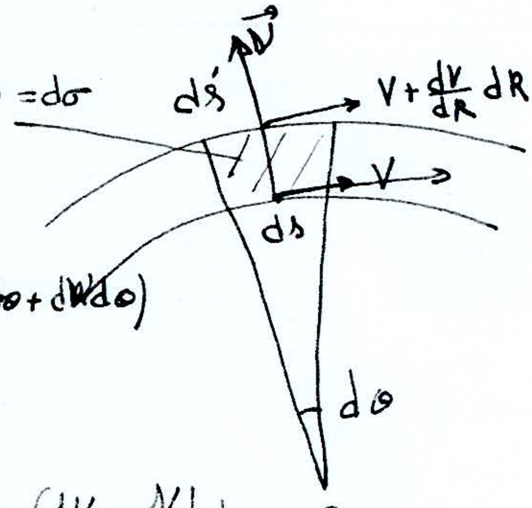
CM = R
radius of curvature

$$\vec{\zeta} = \zeta\,\vec{b}$$
$$\vec{V} = V\,\vec{\tau}$$
$$\vec{V} \times \vec{\zeta} = \zeta_b V\; \vec{\tau} \times \vec{b} = -\zeta_b V \vec{\nu}$$

$$\zeta_\tau = \zeta_n = 0$$

What is ζ_b in this case:

$R\,dR\,d\theta = d\sigma$



$$ds = R\,d\theta$$
$$ds' = R\,d\theta + dR\,d\theta$$

Circulation: $V\,ds - \left(V + \frac{dV}{dR}dR\right)\left(R\,d\theta + dR\,d\theta\right)$

$$= -\left(\frac{dV}{dR} R\,dR\,d\theta + V\,dR\,d\theta\right)$$

$$= -\left(R\frac{dV}{dR} + \frac{V}{R}\right) R\,d\theta\,dR = -\left(\frac{dV}{dR} + \frac{V}{R}\right) d\sigma$$

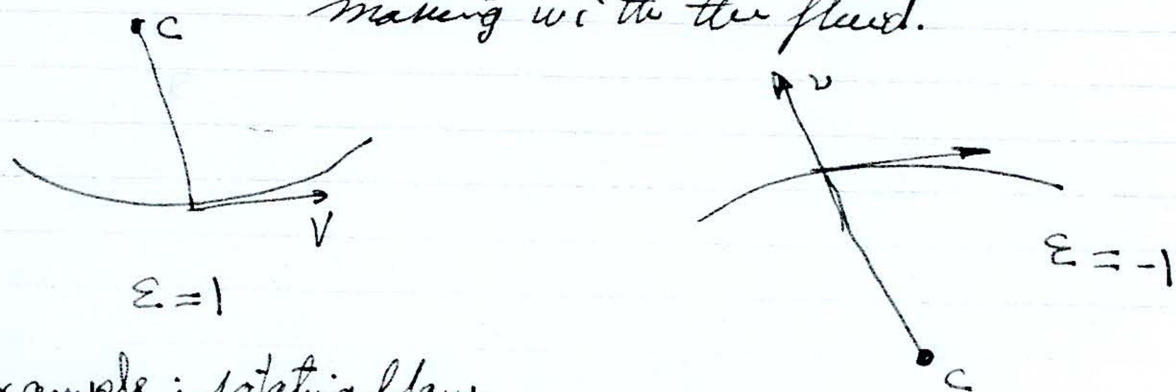
$$= \zeta_b\,d\sigma$$

$$\xi_b = -\left(\frac{V}{R} + \frac{\partial V}{\partial \nu}\right)$$

$$\xi_b = \left(\frac{\varepsilon V}{R} - \frac{\partial V}{\partial \nu}\right)$$

$\varepsilon = +1$, center of curvature to the left of an observer moving with the fluid.

$\varepsilon = -1$, center of curvature to the right of an observer moving with the fluid.

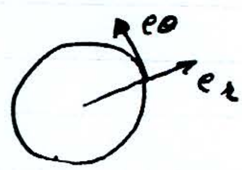


Example: rotating flow.

$$\vec{V} = V(r)\,\vec{e}_\theta$$

$$\vec{\xi} = \frac{1}{r}\begin{vmatrix} e_x & e_r & e_\theta \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ 0 & 0 & rV(r) \end{vmatrix} = \frac{d}{r\,dr}(rV)\,\vec{e}_x = \left(\frac{V}{r} + \frac{dV}{dr}\right)\vec{e}_x$$

Back to Bernoulli:

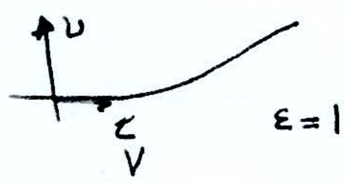


$$\frac{d}{d\nu}\left(\frac{1}{2}V^2 + \int\frac{dP}{\rho} + \Omega\right) = +\varepsilon V\left(\frac{V}{R} + \frac{\partial V}{\partial \nu}\right)$$

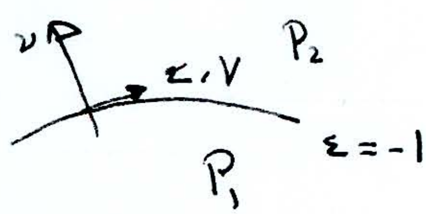
Application: Incompressible

$\Omega = 0$

$$\cancel{V\frac{dV}{d\nu}} + \frac{1}{\rho}\frac{\partial P}{\partial \nu} = \frac{V^2}{R} \cancel{+ V\frac{\partial V}{\partial \nu}}$$



$$\frac{\partial P}{\partial \nu} = \frac{\rho V^2}{R}, \quad P_2 > P_1$$



$$\zeta_b = \left(\frac{\varepsilon V}{R} - \frac{\partial V}{\partial \nu}\right)$$

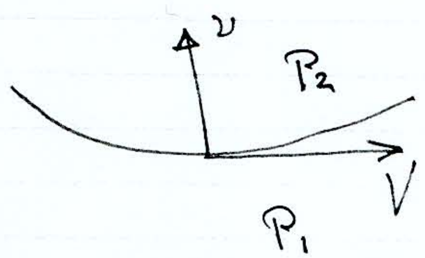
$$\frac{d}{d\nu}\left(\frac{1}{2}V^2 + \int \frac{dP}{\rho} + \Omega\right) = \frac{V^2}{R} + V\frac{dV}{d\nu}$$

neglect Ω —

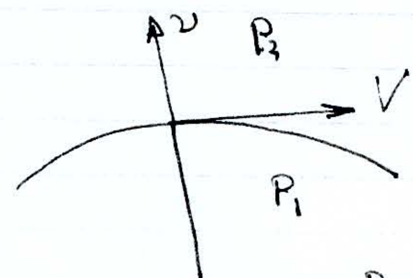
$$\frac{\partial P}{\partial \nu} = \frac{\rho V^2}{R}$$

$$\frac{\partial P}{\partial \nu} + \rho\frac{d\Omega}{\partial \nu} = \frac{\rho V^2}{R}$$

If $\frac{\partial \Omega}{\partial \nu} = 0$



$P_1 > P_2$

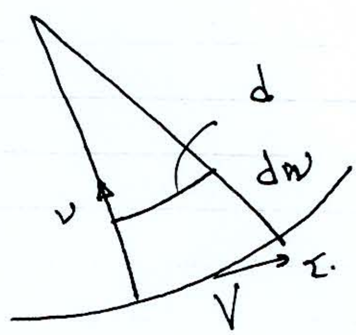


$P_2 > P_1$

$$VR\,d\theta - (V+dV)(R-dn)\,d\theta$$

$$= +V\,dn\,d\theta - R\,dV\,d\theta$$

$$= \left(+\frac{V}{R} - \frac{dV}{dn}\right)d\theta$$



$$\zeta_b = +\frac{V}{R} - \frac{\partial V}{d\nu}$$

$$\zeta_b = \varepsilon\frac{V}{R} - \frac{\partial V}{\partial \nu}$$

$\varepsilon = \begin{cases} +1 & \text{center of curvature to the left.} \\ -1 & \text{center of curvature to the right} \end{cases}$ of the moving particle

Rate of Change of Circulation

Kelvin Theorem: The circulation is constant in a circuit moving with the fluid in an inviscid barotropic fluid.

$$\vec{a} = \frac{D\vec{V}}{Dt} \quad , \quad \vec{B} = \nabla \times \vec{a}$$

$$= \frac{\partial \vec{V}}{\partial t} + \vec{\xi} \times \vec{V} + \tfrac{1}{2}\nabla V^2 \qquad \vec{B} = \frac{\partial \vec{\xi}}{\partial t} + \nabla \times (\vec{\xi} \times \vec{V})$$

$$\mathcal{C} = \int_C \vec{V} \cdot d\vec{s} \qquad\qquad = \nabla T \times \nabla S$$

$$\frac{D}{Dt} C = \int_C \frac{D}{Dt}(\vec{V} \cdot d\vec{s}) = \int_C \vec{a} \cdot d\vec{s} + \int_C \vec{V} \frac{D}{Dt} d\vec{s}$$

$$\frac{D}{Dt}(d\vec{s}) = d \frac{D}{Dt}\vec{s} = d\vec{V}. \text{ Hence } \int_C \vec{V} \cdot d\vec{V} = 0$$

$$\frac{D}{Dt} C = \int \vec{a} \cdot d\vec{s} = \int (\nabla \times \vec{a}) \cdot \vec{n} \, d\Sigma = \int_\Sigma \vec{B} \cdot \vec{n} \, d\Sigma$$

From Crocco's: $\frac{\partial \vec{u}}{\partial t} + \vec{\xi} \times \vec{u} = -\nabla h_0 + T \nabla s \xrightarrow{\Sigma} \frac{\partial \vec{\xi}}{\partial t} + \nabla(\vec{\xi} \times \vec{u}) = \nabla T \times \nabla S$

$$\vec{a} = \vec{f} - \frac{1}{\rho} \nabla P$$

$$\vec{B} = \nabla \times \vec{f} + \nabla P \times \nabla\left(\frac{1}{\rho}\right) = \frac{\partial \vec{\xi}}{\partial t} + \nabla \times (\vec{\xi} \times \vec{V})$$

If $\vec{f}$ is conservative : $\vec{f} = -\nabla \Omega \qquad \nabla \times \vec{f} = 0$

If $P = f(\rho)$, i.e. the fluid is barotropic, $\nabla P \times \nabla \frac{1}{\rho} = 0$

$$\rightarrow \vec{B} = 0 \Rightarrow \frac{D}{Dt} C = 0.$$

If we consider a symmetric airfoil at zero angle of attack. There is no lift because of the symmetry. I we suddenly, put the airfoil at an angle of attack α, a lift will ensue and a circulation around the airfoil will be created.

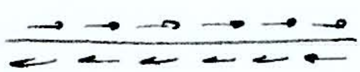
Kelvin theorem explains the phenomenon of the starting vortex, vortex shedding in unsteady flow and the time lag in establishing the lift.

Boundary conditions:

- $\vec{V}\cdot\vec{n} =$ along a surface or $(\vec{V}-\vec{V}_s)\cdot\vec{n} = 0$
- Pressure is normal to a boundary.
- Pressure is continuous in fluids.

Example: the vortex sheet.

discontinuous tangential Velocity



Continuous pressure
Continuous normal Velocity

$$\frac{D\Delta\vec{V}}{Dt} = 0$$

$\Delta\vec{V}$ = jump across the vortex sheet is purely convected.

$$\Delta\vec{V} = \Delta\vec{V}\left(t - \frac{x}{U}\right).$$