FINITE WING THEORY

Consider a wing in a uniform upstream flow, V and let the y_0-axis be the axis along the span centered at the wing root. and let $c(y_0)$ be the chord length. We define the lift per unit span, $L'(y_0)$, as that of an infinite span wing whose geometry and angle of attack to the mean flow are those of the wing at y_0. The corresponding lift coefficient is

$$c_\ell = \frac{L'(y_0)}{\frac{1}{2}\rho V^2 c(y_0)}, \tag{1}$$

where $c(y_0)$ is the wing chord length at y_0. Using the theorem of Kutta-Joukowski, $L'(y_0) = \rho V \Gamma(y_0)$, we rewrite (1) as

$$c_\ell = \frac{2\Gamma(y_0)}{Vc(y_0)}. \tag{2}$$

The expression for c_ℓ can also be written in terms of the effective angle of attack $\alpha_{eff} = \alpha - \alpha_i$,

$$c_\ell = a_0(\alpha - \alpha_{L=0} - \alpha_i), \tag{3}$$

where the induced angle of attack α_i is calculated using the Biot-Savart law,

$$\alpha_i = \frac{1}{4\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy. \tag{4}$$

a_0 is a constant. For a thin airfoil, $a_0 = 2\pi$.
At every positition y_0 along the span, we can then write

$$\alpha(y_0) - \alpha_{L=0}(y_0) - \alpha_i(y_0) = \frac{2\Gamma(y_0)}{a_0 V c(y_0)}. \tag{5}$$

Note that $\overline{\alpha}(y_0) = \alpha(y_0) - \alpha_{L=0}(y_0)$ is determined by the wing geometry and angle of attack. Substituting the expression (4) for α_i in (5) gives the fundamental equation of the finite wing theory,

$$\overline{\alpha}(y_0) = \frac{2\Gamma(y_0)}{a_0 V c(y_0)} + \frac{1}{4\pi V}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}}{y_0 - y} dy. \tag{6}$$

The integral in (6) should be understood as a Cauchy principal value.
We note that wings are symmetric, i.e., .
We introduce the transformation

$$y_0 = -\frac{b}{2} cos\theta_0 \tag{7}$$

$$y = -\frac{b}{2} cos\theta. \tag{8}$$

Equation(6) can then be rewritten as

$$\overline{\alpha}(\theta_0) = \frac{2\Gamma(\theta_0)}{a_0 V_\infty c(\theta_0)} + \frac{1}{2\pi V_\infty b}\int_0^\pi \frac{\frac{d\Gamma}{d\theta}}{cos\theta - cos\theta_0} d\theta. \tag{9}$$

We note that $\Gamma(y_0)$ vanishes at both ends of the wing. Moreover, we assume the wing to be symmetric, i.e., $\Gamma(-y_0) = \Gamma(y_0)$. This suggests the following expansion for Γ :

$$\Gamma(\theta) = 2bV \sum_1^N A_n sinn\theta \tag{10}$$

$A_1, A_2, \ldots,\ A_N$ are constants to be determined. The condition of wing symmetry, $\Gamma(\pi - \theta) = \Gamma(\theta)$, implies $A_n = 0$ for even n.

We note that

$$\int_0^\pi \frac{cosn\theta}{cos\theta - cos\theta_0} d\theta = \pi \frac{sinn\theta_0}{sin\theta_0} \tag{11}$$

Substituting (10) into (4 and 9) and using (11), we obtain the following expressions for the induced angle of attack

$$\alpha_i(\theta_0) = \sum_1^N n A_n \frac{sinn\theta_0}{sin\theta_0}, \tag{12}$$

and the fundamental equation (9) for the finite wing becomes

$$\overline{\alpha}(\theta_0) = \frac{4b}{a_0 c(\theta_0)} \sum_1^N A_n sinn\theta_0 + \sum_1^N n A_n \frac{sinn\theta_0}{sin\theta_0} \tag{13}$$

Equation (5) must be satisfied at N locations of the span. This gives N equations for determining $A_1, A_3, \ldots,\ A_N$. The expressions for the wing lift, L, and induced drag, D_i, are readily obtained in terms of Γ,

$$L = \rho V \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y_0) dy_0, \tag{14}$$

$$D_i = \rho V \int_{-\frac{b}{2}}^{\frac{b}{2}} \alpha_i \Gamma(y_0) dy_0. \tag{15}$$

We define the wing lift and induced drag coefficients as follows

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}, \tag{16}$$

$$C_{D,i} = \frac{D_i}{\frac{1}{2}\rho V^2 S}. \tag{17}$$

This gives :

$$
\begin{aligned}
C_L &= \pi \mathcal{AR} A_1, & (18)\\
C_{D,i} &= \pi \mathcal{AR} A_1^2 [1 + \sum_2^N n (\frac{A_n}{A_1})^2], & (19)\\
\text{which is commonly cast as} & & (20)\\
C_{D,i} &= \frac{C_L^2}{\pi \mathcal{AR}} (1+\delta). & (21)
\end{aligned}
$$

For a wing with no geometric twist

$$C_L = a(\alpha - \alpha_{L=0})$$

$$a = \frac{a_0}{1 + (\frac{a_0}{\pi \mathcal{AR}})(1+\tau)}$$

For a thin airfoil, $a_0 = 2\pi$.

ELLIPTIC WING

For a wing of uniform cross-section and no geometric twist, $\overline{\alpha}(\theta)$ is constant. We further assume the wing to have an elliptic planform, i.e.,

$$c = c_0 \sqrt{1 - (\frac{2y}{b})^2} \qquad or \quad c(\theta) = c_0 sin\theta$$

Substituting (11) into (5), we find the following solution

$$
\begin{aligned}
A_1 = \frac{\overline{\alpha}}{1 + \frac{4b}{a_0 c_0}} &= \frac{\overline{\alpha}}{1 + \frac{\pi \mathcal{AR}}{a_0}} \\
A_2 = A_3, \ldots, &= A_N = 0.
\end{aligned}
$$

All aerodynamic quantities can now be calculated :

$$\Gamma(\theta) = 2bV_\infty \frac{\overline{\alpha}}{1 + \frac{\pi \mathcal{AR}}{a_0}} sin\theta$$

$$\alpha_i = A_1 = \frac{\overline{\alpha}}{1 + \frac{\pi \mathcal{AR}}{a_0}}$$

$$C_L = \pi \mathcal{AR} \alpha_i = \frac{a_0 \overline{\alpha}}{1 + \frac{a_0}{\pi \mathcal{AR}}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathcal{AR}}$$

$$a = \frac{a_0}{1 + \frac{a_0}{\pi \mathcal{AR}}}$$