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1 The Buckingham Π–Theorem

Let

$$a = f(a_1, a_2, ..., a_k, a_{k+1}, ..., a_n) \tag{1}$$

where $a_1, a_2, ..., a_k$ have independent dimensions, and $a_{k+1}, a_{k+2}, ..., a_n$ are expressible in terms of the dimensions of $a_1, a_2, ..., a_k$, as follows

$$\begin{aligned} [a_{k+1}] &= [a_1]^{p_{k+1}} \cdots [a_k]^{r_{k+1}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ [a_n] &= [a_1]^{p_n} \cdots [a_k]^{r_n} \\ [a] &= [a_1]^{p} \cdots [a_k]^{r} \end{aligned} \tag{2}$$

Define

$$\begin{aligned} \Pi_1 &= \frac{a_{k+1}}{a_1^{p_{k+1}} \cdots a_k^{r_{k+1}}} \\ &\cdots\cdots\cdots\cdots\cdots \\ \Pi_{n-k} &= \frac{a_n}{a_1^{p_n} \cdots a_k^{r_n}} \\ \Pi &= \frac{a}{a_1^{p} \cdots a_k^{r}} \end{aligned} \tag{3}$$

Substituting equation (1) for a, we get

$$\Pi = \frac{f(a_1, a_2, ..., a_k, a_{k+1}, ..., a_n)}{a_1^{p} \cdots a_k^{r}}$$

Using equations (3) to express $a_{k+1}, a_{k+2}, ..., a_n$, we get

$$\Pi = \frac{f(a_1, a_2, ..., a_k, \Pi_1 a_1^{p_{k+1}} \cdots a_k^{r_{k+1}}, ..., \Pi_{n-k} a_1^{p_n} \cdots a_k^{r_n})}{a_1^{p} \cdots a_k^{r}}$$

Or,

$$\Pi = F(a_1, a_2, ..., a_k, \Pi_1, \Pi_2, \Pi_{n-k}) \tag{4}$$

If we change the system of units in which $a_1, a_2, ..., a_k$ are expressed, Π does not change. Therefore,

$$\frac{\partial \Pi}{\partial a_1} = 0, \ \frac{\partial \Pi}{\partial a_2} = 0, ..., \ \frac{\partial \Pi}{\partial a_k} = 0$$

The function Π is thus independent of $a_1, a_2, ..., a_k$, and as a result, equation (4) reduces to

$$\Pi = \Phi(\Pi_1, \Pi_2, ..., \Pi_{n-k}) \tag{5}$$

The above result can be summarized with the followin theorem.

Theorem :

Let a be a quantity dependent on n physical variables. If k is the number of independent dimensions, then the dimensionless dependent variable corresponding to a can be expressed in terms of only n-k dimensionless variables.

In *Fluid Mechanics*, there are only three independent dimensions, mass (M), length (L) and time (T). Thus, $k = 3$. In most problems, the physical variables are the density (ρ), the velocity (V), the body length (ℓ, or c), the viscosity (μ) and the speed of sound (a). Thus, $n = 5$. Dimensionless dependent variables, therefore, can be expressed in terms of only *2* dimensioless variables. The two dimensionless variables are

the Reynolds number : $Re = \frac{\rho V \ell}{\mu}$

and

the Mach number : $M = \frac{V}{a}$

The quantities used to define these variables are usually evaluated at upstream conditions denoted with the subscript ∞.
The lift, drag, normal and axial forces are nondimensionlized with respect to the dynamic head, $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$, times the area S. Hence, the lift and drag coefficients, for example, can be expressed as functions of Re and M

$$C_L = F(Re, M)$$

$$C_D = G(Re, M)$$

For flows which may be considered incompressible, $M < .3$, C_L and C_D depend only on the Reynolds number.

$$C_L = F(Re)$$

$$C_D = G(Re)$$

2 Flow Similarity

Two flows over two two different bodies are said to be dynamically similar if

1. The streamline patterns are geometrically similar.

2. The ratios $\frac{V_1}{V_2}, \frac{\rho_1}{rho_2}, \frac{p_1}{p_2}, \frac{T_1}{T_2}$, etc. are the same throughout the flowfield at geometrically similar points. The subscripts 1 and 2 denote quantities of flow 1 and 2, respectively.

Corollary: If two flows are dynamically similar their force coefficients are the same.

Criteria for Similarity:

1. Geometrically similar bodies.

2. Identical initial flow conditions.

3. Similarity parameters are the same.